The Two-Body Problem

Devendra Ghate

November 21, 2026

Table of contents

1 Where we are going **2**
1.1 The plan, and the one place where the conics part company 2

2 What every orbit shares **3**
2.1 From two bodies to one 3
2.1.1 The relative equation of motion 3
2.1.2 Recovering the two bodies 4
2.2 Angular momentum: the motion is planar 5
2.3 The eccentricity vector: the orbit does not turn 6
2.4 The orbit equation 8
2.4.1 Reading the orbit equation 9
2.5 Energy: the vis-viva integral 10
2.6 Seven constants, five independent: the Casimir relations 10
2.7 The size parameter a 12
2.7.1 The same result from the physics 12
2.8 Bounded and unbounded motion 13
2.9 Escape speed 14
2.10 Playing with the conics 15
2.11 Velocity 15
2.11.1 The rotating basis 16
2.11.2 The transverse component 16
2.11.3 The radial component 16
2.11.4 The velocity vector and the speed 17
2.12 The flight path angle 17
2.12.1 Where is the climb steepest? 18
2.13 The hodograph 19
2.13.1 Speed and flight path angle on the four conics 20
2.14 From $r(f)$ to $r(t)$ 21
2.15 Where the three conics part ways 21

3 The ellipse $(0 \leq e < 1)$ **22**
3.1 What is new 22
3.2 Why a is the semi-major axis 22
3.3 The period 23
3.4 Two bodies in three dimensions: an example 24
3.5 Speed and direction on an ellipse 25
3.5.1 Scaling out the size 25
3.5.2 Apsidal speeds 26
3.5.3 The flight path angle across a family of ellipses 27
3.5.4 Worked example: a geostationary transfer orbit 28
3.6 The eccentric anomaly and Kepler's equation 28
3.6.1 The auxiliary circle 29
3.6.2 Taming the time integral 30

3.6.3 Relating E and f . . . 31
3.6.4 Solving Kepler's equation . . . 31
3.6.5 How the three anomalies are related . . . 32
3.6.6 Worked example: propagating an eccentric geosynchronous orbit . . . 34

4 The parabola ($e = 1$) 35
4.1 The one-parameter conic . . . 35
4.2 Speed and direction on a parabola . . . 35
4.3 Time on a parabola: Barker's equation . . . 36

5 The hyperbola ($e > 1$) 37
5.1 Geometry . . . 37
5.2 Speed at infinity . . . 38
5.3 Turning angle . . . 38
5.4 The aiming radius, and what b means for a hyperbola . . . 39
5.5 The hyperbolic anomaly H . . . 40
5.6 The hyperbolic Kepler equation . . . 41

6 The conics side by side 42

7 Appendix: the rotating basis 44
7.0.1 The payoff: the equation of motion in polar form . . . 45

8 Summary 46

9 Exercises 46

1 Where we are going

The *N*-body note ended with a problem and a way out. The problem: for three or more bodies the equations of motion have no general closed-form solution. The way out: the masses in the solar system are extremely uneven, so most of the time one body moves under the pull of *one* other body, and everything else is a small disturbance. This note solves that problem, the **two-body problem**, completely.

It is worth knowing in advance how much comes out of it. From a single vector equation we will obtain Kepler's three laws, the shape of every possible orbit, the speed and direction of motion at every point, the condition for escape, the bending of a flyby, and the position of the body at any time. Almost every calculation in the rest of the course starts from a formula derived here.

As in the *N*-body note, two kinds of boxes run through the text. A blue **Try it** box asks you to do something small: compute a number, check a step, play with a figure. A green **Check yourself** box asks a short question about what you have just read. Each has a hidden answer; open it only after you have one of your own.

1.1 The plan, and the one place where the conics part company

The note is in two halves, and it pays to see the split before starting.

Section 2: what every orbit shares. The equation of motion, its three integrals (angular momentum $\mathbf{h}$, the eccentricity vector $\mathbf{e}$ and the energy ε), the shape of the orbit, the energy equation, the velocity, the flight path angle and the hodograph. None of these derivations needs to know whether the orbit is a circle, an ellipse, a parabola or a hyperbola. They hold for all four.

Section 3, Section 4 and Section 5: where the conics part company. Only one question forces us to treat the conics separately: **where is the body at time t?**

Answering it needs an integral that is evaluated differently for each conic, and a handful of quantities (the farthest point, the period, the speed at infinity) that exist for some conics and not for others.

Section 6 puts the conics side by side, and an appendix (Section 7) proves two kinematic formulas used along the way.

A first reading. The whole note takes about five hours. On a first pass, read Section 2 and the ellipse up to and including Kepler's equation (Section 3.6), then the table of Section 6. The parabola, the hyperbola and the appendix can wait until they are needed: the hyperbola before the patched-conics note, the appendix whenever a rotating basis puzzles you. The symbols are collected, with their names in Curtis's textbook, on the notation page.

When you finish, you should be able to answer the following without hand-waving.

1. Why is the relative motion of two bodies confined to a plane, and why is the orbit a conic section?
2. Why does the energy of an orbit depend only on its size and not on its shape?
3. What is the escape speed, and why is it a *speed* and not a *velocity*?
4. What are the speed and the direction of flight at any point of any orbit?
5. Given the time, how do you find the position on an ellipse, a parabola and a hyperbola, and why is this so much harder than finding the shape?
6. How much does a flyby turn a spacecraft, and what decides it?

2 What every orbit shares

2.1 From two bodies to one

2.1.1 The relative equation of motion

Take $n = 2$ in the N-body equations. With $\mathbf{r}_{12} = \mathbf{R}_2 - \mathbf{R}_1$ (the vector from body 1 to body 2) and $r_{12} = |\mathbf{r}_{12}|$, the two equations of motion, divided by the masses, are

$$\ddot{\mathbf{R}}_1 = \frac{Gm_2}{r_{12}^3}\,\mathbf{r}_{12}, \qquad \ddot{\mathbf{R}}_2 = \frac{Gm_1}{r_{12}^3}\,\mathbf{r}_{21} = -\frac{Gm_1}{r_{12}^3}\,\mathbf{r}_{12}.$$

The N-body note told us that what matters is the *relative* motion, so subtract the first from the second:

$$\ddot{\mathbf{r}}_{12} = \ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = -\frac{Gm_1}{r_{12}^3}\,\mathbf{r}_{12} - \frac{Gm_2}{r_{12}^3}\,\mathbf{r}_{12} = -\frac{G(m_1+m_2)}{r_{12}^3}\,\mathbf{r}_{12}.$$

Something good has happened: the right-hand side contains only $\mathbf{r}_{12}$. The relative position obeys an equation of its own, which no longer involves $\mathbf{R}_1$ or $\mathbf{R}_2$ separately. Dropping the subscripts, writing $\mathbf{r}$ for $\mathbf{r}_{12}$ and $r = |\mathbf{r}|$,

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\,\mathbf{r} = \mathbf{0}, \qquad \mu = G(m_1 + m_2). \tag{1}$$

This is **the two-body equation of motion**, and the rest of the note is about its solutions. The constant μ is the **gravitational parameter** of the pair. Only μ appears, not the two masses separately: to predict the relative motion we never need to know G, m_1 or m_2 on their own.

History: why astronomers quote μ. Orbits measure μ directly, through Kepler's third law, while G must be measured in a laboratory, with small masses and tiny forces. So μ is known far better than G: $GM_\oplus = 398\,600.4418 \pm 0.0008$ km^3/s^2 (two parts in 10^9), whereas G itself is known only to about two parts in 10^5. Since the Sun outweighs every planet by at least a factor of a thousand, astronomers could compute planetary orbits accurately long before anyone knew the planets' masses, or the Sun's.

Weigh the Sun. For the Earth's orbit, $a = 1$ AU $= 1.496 \times 10^8$ km and the period is one sidereal year, 365.256 days. Kepler's third law in Newton's form (proved in Section 3.3) is $T = 2\pi\sqrt{a^3/\mu}$. Solve for μ.

What you should find

$\mu = 4\pi^2 a^3/T^2 = 1.327 \times 10^{11}$ km^3/s^2. This is $G(M_\odot + M_\oplus)$, which to six figures is $GM_\odot$. To get the mass in kilograms you would still need G, and that is the weak link.

2.1.2 Recovering the two bodies

Equation 1 gives the relative motion. Where are the bodies themselves? The answer combines it with the centre of mass, which moves uniformly (the N-body note). Write $M = m_1 + m_2$ and

$$\mathbf{R}_{\mathrm{cm}} = \frac{m_1}{M}\mathbf{R}_1 + \frac{m_2}{M}\mathbf{R}_2, \qquad \mathbf{r} = -\mathbf{R}_1 + \mathbf{R}_2.$$

This is a linear change of variables, $(\mathbf{R}_1, \mathbf{R}_2) \mapsto (\mathbf{R}_{\mathrm{cm}}, \mathbf{r})$, acting in the same way on each Cartesian component:

$$\begin{pmatrix} \mathbf{R}_{\mathrm{cm}} \\ \mathbf{r} \end{pmatrix} = \underbrace{\begin{pmatrix} m_1/M & m_2/M \\ -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix}, \qquad \det A = \frac{m_1}{M} + \frac{m_2}{M} = 1.$$

Since $\det A \neq 0$ the change can be undone. Inverting the 2×2 matrix,

$$A^{-1} = \begin{pmatrix} 1 & -m_2/M \\ 1 & m_1/M \end{pmatrix},$$

that is,

$$\boxed{\begin{aligned} \mathbf{R}_1 &= \mathbf{R}_{\mathrm{cm}} - \frac{m_2}{M}\,\mathbf{r}, \\ \mathbf{R}_2 &= \mathbf{R}_{\mathrm{cm}} + \frac{m_1}{M}\,\mathbf{r}. \end{aligned}} \tag{2}$$

Differentiating gives the velocities in the same way. So the recipe is: solve Equation 1 for $\mathbf{r}(t)$, move the centre of mass uniformly, and combine them with Equation 2.

In the barycentric frame, $\mathbf{R}_{\mathrm{cm}} = \mathbf{0}$, and Equation 2 says that both bodies are fixed multiples of the same vector $\mathbf{r}(t)$: they stay on opposite sides of the barycentre, and each traces a copy of the relative orbit scaled by a constant. Figure 1 shows this.

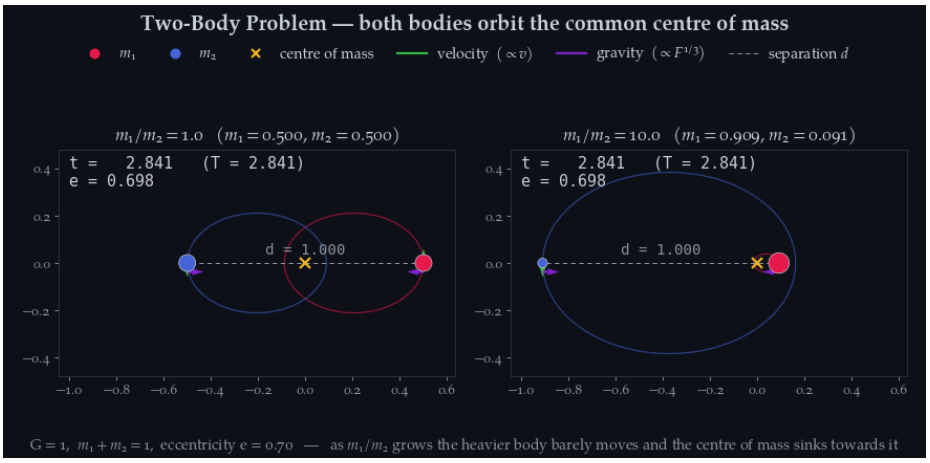

Figure 1: Two-body motion at two mass ratios. Both panels have the same total mass, the same initial separation and the same eccentricity ($e = 0.70$); only m_1/m_2 differs. Green arrows: velocities, to scale. Purple arrows: the gravitational forces, drawn on a compressed $F^{1/3}$ scale since $F \propto 1/d^2$ varies too much to draw to scale. *Script: two-body-animation.py.*

In Equation 2, body 1 is scaled by m_2/M and body 2 by m_1/M: the weights are crossed. Which body traces the smaller orbit about the barycentre? The two forces are equal and opposite; are the two accelerations?

Answer

The heavier body traces the smaller orbit: if $m_1 > m_2$ then $m_2/M < m_1/M$. The forces are equal in size, but the accelerations are not, since $|\ddot{\mathbf{R}}_1|/|\ddot{\mathbf{R}}_2| = m_2/m_1$. The heavier body responds less, and so moves less.

Pluto and its moon Charon are 19 596 km apart, and $m_{\text{Charon}}/m_{\text{Pluto}} = 0.122$. How far from the centre of Pluto is their barycentre? Pluto's radius is 1188 km.

What you should find

From Equation 2 the distance is $r\, m_C/(m_P + m_C) = 19\,596 \times 0.122/1.122 \approx 2130$ km, which is *outside* Pluto. Pluto and Charon both orbit a point in empty space between them. Compare the Earth–Moon barycentre, 4671 km from the Earth's centre, inside the Earth.

As m_1/m_2 grows, the orbit of the heavier body shrinks and the barycentre moves towards its centre. For the Sun and Jupiter, $m_1/m_2 \approx 1047$, the Sun's orbit about the barycentre is 1/1047 the size of Jupiter's (just over a solar radius). This is the quantitative content of the *restricted* two-body problem: when one mass is much larger, treat it as fixed, and let $\mathbf{r}$ be the position of the small body measured from the large one. Equation 1 does not change; only the interpretation of $\mathbf{r}$ does, and for planets and satellites μ is, to excellent accuracy, G times the mass of the central body.

2.2 Angular momentum: the motion is planar

Equation 1 is a nonlinear vector ODE of second order. Rather than attack it directly, we repeat the strategy of the N-body note: look for **integrals**, combinations of $\mathbf{r}$ and $\dot{\mathbf{r}}$ that do not change. The acceleration in Equation 1 is parallel to $\mathbf{r}$, and the cross product of parallel vectors vanishes, so the first thing to try is a cross product with $\mathbf{r}$:

$$\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\, \mathbf{r} \times \mathbf{r} = \mathbf{0}.$$

The left-hand side is an exact derivative, because

$$\frac{d}{dt}\,(\mathbf{r}\times\dot{\mathbf{r}}) = \dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{r}\times\ddot{\mathbf{r}}.$$

Therefore the **specific angular momentum** (angular momentum per unit mass)

$$\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}} \tag{3}$$

is a constant vector. Three scalar constants.

What it implies. At every instant $\mathbf{r}\cdot\mathbf{h} = \mathbf{r}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = 0$, since a triple product with a repeated vector vanishes. So $\mathbf{r}(t)$ is always perpendicular to $\mathbf{h}$. Since $\mathbf{h}$ does not change, $\mathbf{r}(t)$ lies for all time in **one fixed plane** through the origin, the plane perpendicular to $\mathbf{h}$. The relative motion is planar.

This is a strong geometric statement. A curve in three dimensions has collapsed into a curve in a plane, and from now on we can do orbital geometry in two dimensions.

Is $h = 0$ possible? If yes, what kind of motion results?

Answer

Yes. $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}} = \mathbf{0}$ exactly when $\mathbf{r}$ and $\dot{\mathbf{r}}$ are parallel, and then the motion is **rectilinear**: a straight line through the centre of attraction. Dropping a body from rest is the standard example. The argument above then fails (there is no normal to define a plane), and correctly so: every plane that contains the line will do. We will meet this case again; it is the one orbit that the formulas of this note describe badly.

Kepler's second law. In the orbital plane, use polar coordinates (r,θ) centred on the attracting body. In a short time dt the radius vector sweeps a thin triangle of area $dA = \frac{1}{2}r^2\,d\theta$, and $|\mathbf{h}| = r^2\dot{\theta}$ (we prove this in Section 2.11; the transverse component of the velocity is $r\dot{\theta}$). So

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2} = \text{constant}.$$

Equal areas in equal times: Kepler's second law is the conservation of angular momentum.

Kepler's second law holds for *any* central force, not only for the inverse square. Why?

Answer

The only property of the force used above was that $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$, so that $\mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{0}$. Any force directed along $\mathbf{r}$, whatever its dependence on r, conserves $\mathbf{h}$, and with it the areal velocity. The second law tells you the force is central; it does not tell you it is an inverse square. The first and third laws do.

2.3 The eccentricity vector: the orbit does not turn

Angular momentum fixes the *plane* of the orbit. It does not tell us whether the orbit keeps a fixed orientation *inside* that plane, or slowly turns. Is there a second vector integral, one that lies in the plane and points in a fixed direction?

Here is the idea. We want to integrate Equation 1, and the obstacle is the right-hand side $-\mu\mathbf{r}/r^3$, which is not obviously the derivative of anything. Since $\mathbf{h}$ is constant, $\ddot{\mathbf{r}}\times\mathbf{h}$ *is* an exact derivative, of $\dot{\mathbf{r}}\times\mathbf{h}$. So take the cross product of Equation 1 with $\mathbf{h}$ and hope that the right-hand side becomes a derivative too.

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\,\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\,\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})\,.$$

Expand the triple product with $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$:

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\left[(\mathbf{r} \cdot \dot{\mathbf{r}})\,\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\,\dot{\mathbf{r}}\right] = -\frac{\mu}{r^3}\left[(\mathbf{r} \cdot \dot{\mathbf{r}})\,\mathbf{r} - r^2\,\dot{\mathbf{r}}\right].$$

We need $\mathbf{r} \cdot \dot{\mathbf{r}}$. Write $\mathbf{r} = r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$. Then $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}}$ and

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\hat{\mathbf{r}} \cdot \left(\dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}}\right) = r\dot{r}\,(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) + r^2\,(\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}}) = r\dot{r},$$

because $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$ and, differentiating that, $\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}} = 0$. (Note that $\dot{r}$, the rate of change of the *distance*, is not the same as $|\dot{\mathbf{r}}|$, the *speed*.) Substituting,

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\left[r\dot{r}\,\mathbf{r} - r^2\dot{\mathbf{r}}\right] = \mu\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\,\mathbf{r}\right).$$

The bracket is exactly the derivative of the unit vector $\mathbf{r}/r$, by the quotient rule:

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\,\mathbf{r}.$$

So the hope came true: $\ddot{\mathbf{r}} \times \mathbf{h} = \mu\,\dfrac{d}{dt}\left(\dfrac{\mathbf{r}}{r}\right)$. Both sides are exact derivatives, and integrating once,

$$\dot{\mathbf{r}} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right), \qquad \text{i.e.} \qquad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}, \tag{4}$$

where $\mathbf{e}$ is a constant vector of integration. It is dimensionless (each term on the right is), and it is called the **eccentricity vector**, for reasons that will be clear in Section 2.4.

Since $\mathbf{e}$ arose as a constant of integration, its constancy is built in. As a check, differentiate Equation 4 using $\dot{\mathbf{h}} = \mathbf{0}$:

$$\dot{\mathbf{e}} = \frac{\ddot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \mathbf{0},$$

by the identity just proved. A constant vector has a constant length $e = |\mathbf{e}|$ and a constant direction. So $\mathbf{e}$ singles out **one fixed line in the orbital plane**, for all time. We will see in a moment that it points to the point of closest approach, and that e is the eccentricity of the orbit.

Show that $\mathbf{e}$ lies in the orbital plane, i.e. $\mathbf{h} \cdot \mathbf{e} = 0$. Then compute $\mathbf{e} \cdot \dot{\mathbf{r}}$. Is $\mathbf{e}$ perpendicular to the velocity?

What you should find

$\mathbf{h} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = 0$ (repeated vector in a triple product) and $\mathbf{h} \cdot \mathbf{r} = 0$, so $\mathbf{h} \cdot \mathbf{e} = 0$.

For the velocity, $\dot{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = 0$ too, but $\dot{\mathbf{r}} \cdot \mathbf{r}/r = r\dot{r}/r = \dot{r}$. Hence $\mathbf{e} \cdot \dot{\mathbf{r}} = -\dot{r}$. This vanishes only where the distance is momentarily stationary (the closest and farthest points). So $\mathbf{e}$ is perpendicular to $\mathbf{h}$ always, but perpendicular to $\dot{\mathbf{r}}$ only at those two points. A common mistake is to claim the second.

Why it matters. $\mathbf{h}$ fixes the orbital plane; $\mathbf{e}$ fixes the orientation of the orbit within the plane. Its constancy is the statement that the orbit **does not precess**: the closed curve that we are about to find returns exactly on itself, revolution after revolution. That is a special property of the inverse-square law. Any departure from $1/r^2$ (the Earth's oblateness, the pull of a third body, the corrections of general relativity) makes $\dot{\mathbf{e}} \neq \mathbf{0}$, and the line through the closest and farthest points, the **line of apsides**, slowly rotates. This rotation is called **apsidal precession**. The best-known case is Mercury: after all the Newtonian perturbations of the other planets are accounted for, its line of apsides still turns by an extra 43″ per century, which general relativity explains exactly.

2.4 The orbit equation

We now have two constant vectors, $\mathbf{h}$ (normal to the plane) and $\mathbf{e}$ (in the plane). That is enough to find the shape of the orbit without solving any differential equation. The trick is to take a dot product that turns Equation 4 into a scalar relation involving only r and one angle.

Take the dot product of $\mathbf{r}$ with the first form of Equation 4, $\dot{\mathbf{r}} \times \mathbf{h} = \mu(\hat{\mathbf{r}} + \mathbf{e})$. On the left, cycle the triple product:

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{h} \cdot \mathbf{h} = h^2.$$

On the right, define the **true anomaly** f as the angle from $\mathbf{e}$ to $\mathbf{r}$, measured in the direction of motion. Then $\mathbf{r} \cdot \mathbf{e} = re\cos f$ and

$$\mathbf{r} \cdot \mu\,(\hat{\mathbf{r}} + \mathbf{e}) = \mu\,(r + re\cos f)\,.$$

Equating the two and solving for r,

$$\boxed{r = \frac{p}{1 + e\cos f}, \qquad p = \frac{h^2}{\mu}.} \tag{5}$$

This is the polar equation of a **conic section** with the origin at a focus.

Why is this a conic? (four lines)

Put Cartesian axes at the origin with x along $f = 0$, so $x = r\cos f$, $y = r\sin f$. The orbit equation says $r + e\,r\cos f = p$, that is $r = p - ex$. Squaring, $x^2 + y^2 = (p - ex)^2$, or

$$(1 - e^2)\,x^2 + 2pe\,x + y^2 = p^2.$$

For $e = 1$ the x^2 term drops out and $y^2 = p^2 - 2px$ is a **parabola**. Otherwise, write $A = p/(1 - e^2)$ and complete the square in x:

$$\frac{(x + Ae)^2}{A^2} + \frac{y^2}{A^2(1 - e^2)} = 1.$$

For $e < 1$ both denominators are positive: an **ellipse** with semi-axes $|A|$ and $|A|\sqrt{1 - e^2} = \sqrt{|A|p}$, centred at $x = -Ae$, so the origin is a distance $|A|e$ from the centre. For $e > 1$ the second denominator is negative: a **hyperbola**. The length A is the a of Equation 10, and the second semi-axis is the b of Section 3 and Section 5, with $b^2 = |a|p$ in both cases.

The length p is called the **semi-latus rectum**; it is the value of r at $f = \pm 90°$. The single number $e = |\mathbf{e}|$ decides the type of conic:

History: many names for one vector. The vector $\mu\mathbf{e}$ is often called the *Laplace–Runge–Lenz vector*. Laplace used it in his *Traité de mécanique céleste* (1799), but it appears already in a 1710 exchange between Jakob Hermann and Johann Bernoulli. Carl Runge put it in a textbook on vector analysis, Wilhelm Lenz used it in 1924 for the old quantum theory of the hydrogen atom, and Wolfgang Pauli then used it to derive the hydrogen spectrum, which made it famous among physicists. (H. Goldstein, *Am. J. Phys.* 1975 and 1976, traces the history.)

e	conic	
0	circle	closed
$0 < e < 1$	ellipse	closed
1	parabola	open
> 1	hyperbola	open

This is Kepler's first law, and more: Kepler found ellipses, but the inverse-square law also allows parabolas and hyperbolas.

The direction of e. For $e > 0$, r in Equation 5 is smallest where $1 + e\cos f$ is largest, at $f = 0$. That point of closest approach is the **periapsis**. Since f is measured from **e**, we have shown:

> **e** points from the focus towards periapsis.

(For a closed orbit, r is largest at $f = 180°$, the **apoapsis**. With the Earth as the central body these are called perigee and apogee; with the Sun, perihelion and aphelion.)

Two quick questions. (a) What does the orbit look like when $\mathbf{e} = \mathbf{0}$, and why is the direction of **e** then meaningless? (b) In Equation 5, what are p and e for an orbit with $h = 0$?

Answer

(a) $r = p$ for every f: a circle. There is no closest point, so there is nothing for **e** to point at, and the true anomaly f has no natural origin.

(b) $p = h^2/\mu = 0$. From Equation 4 with $\mathbf{h} = \mathbf{0}$, $\mathbf{e} = -\hat{\mathbf{r}}$, so $e = 1$ whatever the energy. The "conic" degenerates into a straight segment, and (p, e) no longer tells you its size. This is the rectilinear orbit of the $h = 0$ question in Section 2.2.

2.4.1 Reading the orbit equation

Before any more dynamics, Equation 5 already says which values of f are *allowed*. Since $r > 0$ and $p > 0$, we need

$$1 + e\cos f > 0.$$

- $e < 1$**.** Then $1 + e\cos f \geq 1 - e > 0$ for every f, so f runs over the whole circle and the curve **closes**. Both apsides exist, at $r_p = p/(1+e)$ and $r_a = p/(1-e)$.
- $e = 1$**.** The denominator vanishes only as $f \to \pm 180°$, where $r \to \infty$. The curve is open and there is no apoapsis. Periapsis is at $r_p = p/2$.
- $e > 1$**.** The denominator vanishes at a *finite* angle f_∞, given by

$$\cos f_\infty = -\frac{1}{e}, \tag{6}$$

and the body exists only on the arc $|f| < f_\infty$. As $f \to \pm f_\infty$, $r \to \infty$: the two directions $f = \pm f_\infty$ are the directions of the **asymptotes** of the hyperbola. The asymptotes are *parallel* to these two rays from the focus, but they are not the rays themselves: they cross at the centre of the hyperbola and pass the focus at a distance (Section 5).

So the split between the conics is visible already in the geometry, and it is entirely a matter of the denominator.

As $e \to 1^+$, what happens to f_∞? And as $e \to \infty$? Describe the trajectory in each limit.

Answer

$f_\infty = \arccos(-1/e)$ decreases from 180° (at $e = 1$) towards 90° (as $e \to \infty$). As $e \to 1^+$, the asymptotes fold back towards the line of apsides and the hyperbola approaches a parabola, which "reaches infinity" only at $f = 180°$. As $e \to \infty$, the asymptotes become perpendicular to the line of apsides and the trajectory is nearly a straight line, barely bent by the central body: a very fast flyby.

2.5 Energy: the vis-viva integral

We crossed Equation 1 with $\mathbf{r}$ and then with $\mathbf{h}$. The remaining natural product is the dot product with the velocity, which in mechanics always produces an energy equation:

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\, \mathbf{r} \cdot \dot{\mathbf{r}}.$$

With $v = |\dot{\mathbf{r}}|$, the left side is $\frac{d}{dt}\left(\frac{1}{2}v^2\right)$. On the right, $\mathbf{r}\cdot\dot{\mathbf{r}} = r\dot{r}$ (shown in Section 2.3), so

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = -\frac{\mu \dot{r}}{r^2} = \frac{d}{dt}\left(\frac{\mu}{r}\right).$$

Integrating,

$$\boxed{\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon} \tag{7}$$

where the constant ε is the **specific mechanical energy**: kinetic energy per unit mass ($v^2/2$) plus potential energy per unit mass ($-\mu/r$). This is the energy integral, or the **vis-viva** equation (Latin for "living force", an old name for kinetic energy). It is arguably the single most useful formula in mission design: it connects speed and distance on *any* orbit with one constant.

2.6 Seven constants, five independent: the Casimir relations

We have squeezed Equation 1 in three ways and obtained three integrals:

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}, \qquad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}, \qquad \varepsilon = \frac{v^2}{2} - \frac{\mu}{r}.$$

That is $3 + 3 + 1 = 7$ scalar constants. But the state $(\mathbf{r}, \dot{\mathbf{r}})$ has only 6 components, and the constants must not pin the state down completely (otherwise the body could not move). A system of order 6 has at most 5 independent integrals that do not involve time. So the seven constants must be tied together by at least two relations. Here they are:

$$\boxed{\mathbf{h} \cdot \mathbf{e} = 0} \qquad \boxed{\mu^2\left(e^2 - 1\right) = 2\varepsilon h^2} \tag{8}$$

These are sometimes called the **Casimir relations**. Neither is a new integral; they are identities satisfied by the integrals we have, on every orbit. The first you proved in the Try-it box of Section 2.3. We prove the second.

Take the squared length of Equation 4, written as $\mu\mathbf{e} = \dot{\mathbf{r}} \times \mathbf{h} - \mu\hat{\mathbf{r}}$:

The name. "Casimir" comes from physics, not astrodynamics: Curtis and Battin derive the two identities without naming them. The bound Kepler problem has a hidden symmetry, rotations in four dimensions, and the two combinations of $\mathbf{h}$ and $\mathbf{e}$ above are the Casimir invariants of that symmetry group (named after the physicist Hendrik Casimir). The same hidden symmetry is why bound Kepler orbits close.

$$\mu^2 e^2 = \underbrace{|\dot{\mathbf{r}} \times \mathbf{h}|^2}_{\text{(i)}} - 2\mu \underbrace{\hat{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \mathbf{h})}_{\text{(ii)}} + \mu^2 \underbrace{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}}_{=1}.$$

Term (i). $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to $\dot{\mathbf{r}}$, so the cross product has magnitude vh, with no sine factor: $|\dot{\mathbf{r}} \times \mathbf{h}|^2 = v^2 h^2$.

Term (ii). Cycle the triple product so that $\mathbf{h}$ meets $\mathbf{r} \times \dot{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \frac{1}{r}\, \mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{h^2}{r}.$$

Substituting both,

$$\mu^2 e^2 = v^2 h^2 - \frac{2\mu h^2}{r} + \mu^2 = h^2 \left(v^2 - \frac{2\mu}{r} \right) + \mu^2 = 2\varepsilon h^2 + \mu^2,$$

since the bracket is 2ε by Equation 7. Rearranged, this is Equation 8, or, solved for the eccentricity,

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}. \tag{9}$$

Since $h^2 \geq 0$, the **sign of the energy alone decides the type of conic**:

energy	eccentricity	orbit
$\varepsilon < 0$	$e < 1$	ellipse (or circle)
$\varepsilon = 0$	$e = 1$	parabola
$\varepsilon > 0$	$e > 1$	hyperbola

This is why ε is the natural mission-design variable: one number, computed from one position and one speed, tells you at a glance whether the spacecraft is captured or escaping.

Counting. Seven constants minus two relations leaves **five independent constants**. Five numbers fix the orbit as a curve in space: two for the orientation of the plane, one for the orientation of the line of apsides in the plane, and two for the size and shape (for example h and e). What they cannot tell us is *where on the curve* the body is at a given time. A sixth constant does that, for example the time t_p at which the body passes periapsis:

$$5 \text{ (geometry)} + 1 \text{ (timing, e.g. } t_p) = 6,$$

which is the order of Equation 1.

The first five come from the integrals; the sixth will cost us the hardest work in the note (Section 2.14 onwards).

A spacecraft is at $r = 7000$ km from the Earth's centre, moving at $v = 8.0$ km/s ($\mu = 398\,600$ km^3/s^2). Without knowing the direction of the velocity, can you tell whether it will escape? Can you tell its eccentricity?

Answer

$\varepsilon = 8.0^2/2 - 398\,600/7000 = 32.0 - 56.9 = -24.9$ km^2/s^2 < 0, so it is bound and cannot escape, whatever the direction. The eccentricity needs $h = rv_\perp$, with $v_\perp$ the part of the velocity perpendicular to the radius (written $v\cos\gamma$ once the flight-path angle γ is introduced in Section 2.12). That depends on the direction of the velocity, so e cannot be found from r and v alone.

2.7 The size parameter a

The second Casimir relation, with $h^2 = \mu p$, already expresses the energy through the *shape* parameters alone:

$$\mu^2 \left(e^2 - 1\right) = 2\varepsilon\, \mu p \qquad \Longrightarrow \qquad \varepsilon = \frac{\mu \left(e^2 - 1\right)}{2p}.$$

This is exact for every conic, but awkward: engineers quote neither p nor ε, but a length. So we *define* one,

$$\boxed{a \equiv \frac{p}{1 - e^2}} \qquad \Longleftrightarrow \qquad \boxed{p = a\left(1 - e^2\right),} \tag{10}$$

and the energy collapses to a single, famous line:

$$\boxed{\varepsilon = -\frac{\mu}{2a}.} \tag{11}$$

One orbit slips through the definition Equation 10: the straight-line (rectilinear) orbit, with $h = 0$, so $p = 0$ and $e = 1$, where $p/(1 - e^2)$ is $0/0$. For it, and in general whenever $\varepsilon \neq 0$, read Equation 11 the other way round, as the definition $a = -\mu/(2\varepsilon)$. The two definitions agree for every orbit with $h \neq 0$, and the second one is what the rectilinear exercises below use.

Since $p > 0$, the sign of a is the sign of $1 - e^2$, and the conic can be read off it:

e	$1 - e^2$	a	$\varepsilon = -\mu/2a$
< 1	> 0	$a > 0$, finite	< 0
$= 1$	0	$a \to \infty$: **useless**, use p or r_p	0
> 1	< 0	$a < 0$	> 0

Two consequences we will use constantly, valid for **all** conics:

$$h^2 = \mu p = \mu a\left(1 - e^2\right), \qquad r_p = \frac{p}{1 + e} = a(1 - e). \tag{12}$$

At this stage a is only a number, defined by Equation 10. That it is the **semi-major axis** of an ellipse (half the distance from periapsis to apoapsis) is a theorem about $e < 1$, proved in Section 3. For a hyperbola $a < 0$; for a parabola a does not exist. In particular, never write $r_a = a(1 + e)$ unless $e < 1$.

2.7.1 The same result from the physics

Equation 11 came from the Casimir identity, which is pure algebra. As a check, let us get it from the energy directly. Since ε is constant we may evaluate it at any point of the orbit, and periapsis is the cheapest: there the distance is momentarily stationary ($\dot{r} = 0$), the velocity is perpendicular to $\mathbf{r}$, and $h = r_p v_p$. So $v_p = h/r_p$ and

$$\varepsilon = \frac{v_p^2}{2} - \frac{\mu}{r_p} = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p}.$$

Substitute Equation 12 into this expression and simplify. You should get $-\mu/(2a)$.

What you should find

$$\varepsilon = \frac{\mu a\,(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} = \frac{\mu(1+e)}{2a(1-e)} - \frac{\mu}{a(1-e)}$$
$$= \frac{\mu}{a(1-e)}\left[\frac{1+e}{2} - 1\right] = \frac{\mu}{a(1-e)} \cdot \frac{e-1}{2} = -\frac{\mu}{2a}.$$

Combining with Equation 7 gives the vis-viva equation in its working form:

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).} \tag{13}$$

Three readings of it.

- **Energy is blind to shape.** Two orbits with the same a but very different e have the same energy (and, as we will see, for $e < 1$ the same period).
- **Circular speed.** For a circle, $r = a$ always, and $v_c = \sqrt{\mu/r}$.
- **Escape speed.** For $\varepsilon = 0$ ($a \to \infty$), $v_{\text{esc}} = \sqrt{2\mu/r} = \sqrt{2}\,v_c$.

The space station orbits at about 420 km altitude; take $R_\oplus = 6378$ km and $\mu = 398\,600$ km^3/s^2. Compute its circular speed and the escape speed at that radius.

What you should find

$r = 6798$ km, $v_c = \sqrt{398\,600/6798} = 7.66$ km/s and $v_{\text{esc}} = 10.83$ km/s. Going from a low circular orbit to escape needs only 3.2 km/s more, much less than the 7.7 km/s needed to get into orbit in the first place (and that before counting drag and gravity losses during launch).

Two spacecraft are on orbits with the same a, one with $e = 0.1$ and one with $e = 0.9$. Which moves faster at periapsis, and which has the larger h? (Use vis-viva at $r_p = a(1-e)$ and Equation 12.)

Answer

Vis-viva at periapsis gives $v_p = \sqrt{\mu/a}\,\sqrt{(1+e)/(1-e)}$, and $h = \sqrt{\mu a(1-e^2)}$.

Faster at periapsis: $e = 0.9$, by a wide margin: $\sqrt{1.9/0.1} = 4.36$ against $\sqrt{1.1/0.9} = 1.11$, a factor of 3.9.

Larger h: the *other* one. $\sqrt{1-e^2}$ is 0.995 for $e = 0.1$ but only 0.436 for $e = 0.9$.

The eccentric orbit is much faster at periapsis yet has less than half the angular momentum, because it is so close in ($r_p = 0.1a$) when it is fast, and $h = r_p v_p$. Equal a fixes the energy; it says nothing about h.

2.8 Bounded and unbounded motion

The Casimir relation tied the sign of ε to the type of conic. Here is a second argument, independent of the conic and of **e**, that uses only the energy equation and the obvious fact $v^2 \geq 0$. Rearranging Equation 7,

$$v^2 = 2\left(\varepsilon + \frac{\mu}{r}\right) \geq 0. \tag{14}$$

This one inequality decides whether r can grow without limit.

Negative energy, $\varepsilon < 0$. Write $\varepsilon = -|\varepsilon|$. Then Equation 14 requires $\mu/r \geq |\varepsilon|$, i.e.

$$r \leq \frac{\mu}{|\varepsilon|}.$$

There is a finite largest distance: **the motion is bounded.**

Show that $r \le 2a$ on any orbit with $\varepsilon < 0$. When is the bound reached?

Answer

With $|\varepsilon| = \mu/(2a)$, the bound above reads $r \le \mu/|\varepsilon| = 2a$. Equality needs $v = 0$. A genuine orbit never stops, so for $h \neq 0$ the bound is not reached and the actual largest distance is $r_a = a(1+e) < 2a$. The case that does reach it is the rectilinear orbit ($h = 0, e = 1$): it rises, comes to rest exactly at $r = 2a$, and falls back.

Positive energy, $\varepsilon > 0$. Both terms in Equation 14 are positive at every finite r, so $v^2 > 0$: the body never comes to rest, and nothing stops r growing. As $r \to \infty$, $\mu/r \to 0$ and

$$v \to v_\infty = \sqrt{2\varepsilon} > 0. \tag{15}$$

The body reaches infinity with speed to spare. This speed is the **hyperbolic excess speed v_∞. The motion is unbounded.**

Zero energy, $\varepsilon = 0$. Then $v^2 = 2\mu/r > 0$ at every finite r, so again there is no turning point, but now $v \to 0$ as $r \to \infty$: the body reaches infinity with nothing left. **The motion is marginally unbounded**: the parabola, the boundary between capture and escape.

conic	energy	behaviour at large r	a
ellipse	$\varepsilon < 0$	$r \le \mu/\lvert\varepsilon\rvert = 2a$	$a > 0$
parabola	$\varepsilon = 0$	$r \to \infty, v \to 0$	$a \to \infty$
hyperbola	$\varepsilon > 0$	$r \to \infty, v \to \sqrt{2\varepsilon}$	$a < 0$

2.9 Escape speed

The **escape speed** $v_{\text{esc}}(r)$ at distance r from a body of gravitational parameter μ is the least speed for which the motion is unbounded. By Section 2.8 the motion is unbounded exactly when $\varepsilon \ge 0$, and the threshold $\varepsilon = 0$ gives

$$\boxed{v_{\text{esc}}(r) = \sqrt{\frac{2\mu}{r}} = \sqrt{2}\, v_c(r),} \tag{16}$$

on the parabolic orbit. Four remarks, each a common source of confusion.

- **It is a speed, not a velocity.** The energy depends only on $|\dot{\mathbf{r}}|$, so the escape condition does not care about direction: straight up, horizontally, or at 45°, the speed needed is the same, provided the path does not hit the planet or its atmosphere. Direction changes *where* you go, not *whether* you leave. "Escape velocity" is a historical misnomer.
- **It depends on where you are.** "The escape velocity of the Earth is 11.2 km/s" means *at the Earth's surface* ($r = 6378$ km gives 11.18 km/s). From $r = 6700$ km it is 10.9 km/s; from geostationary radius (42 164 km) only 4.35 km/s.
- **It is escape from one body.** Leaving the Earth is not leaving the Sun. To escape the solar system from the Earth's distance takes $\sqrt{2\mu_\odot/1\ \text{AU}} = 42.1$ km/s relative to the Sun, of which the Earth's own orbital speed already supplies 29.8 km/s if you leave in the right direction.
- **What you buy with excess speed.** Above escape, the orbit is a hyperbola, and the speed left at infinity is

$$v_\infty^2 = 2\varepsilon = v^2 - v_{\text{esc}}^2, \qquad C_3 \equiv v_\infty^2 = -\frac{\mu}{a}. \tag{17}$$

C_3, the **characteristic energy**, is the number on every launch vehicle's performance chart. Notice that v_∞^2 is a *difference of squares*: a little speed above escape buys a lot of v_∞, because $v^2 - v_{\text{esc}}^2 = (v - v_{\text{esc}})(v + v_{\text{esc}})$.

Two escape calculations. (a) Compute the escape speed from the surface of the Moon ($\mu = 4903$ km^3/s^2, $R = 1737$ km). (b) A spacecraft leaves a 300 km parking orbit about the Earth at 11.0 km/s. The local escape speed there is 10.926 km/s. What is its v_∞? Compare with the 0.074 km/s by which it exceeds escape.

What you should find

(a) $\sqrt{2 \times 4903/1737} = 2.38$ km/s, about a fifth of the Earth's.

(b) $v_\infty = \sqrt{11.0^2 - 10.926^2} = \sqrt{1.62} = 1.27$ km/s. Seventy-four metres per second above escape at periapsis leaves more than a kilometre per second at infinity. This leverage is why departure burns are made deep in the gravity well, close to the planet.

2.10 Playing with the conics

We now have several ways to describe the same orbit: by (h, e), by (p, e), by (a, e), or by (ε, e). The explorer below lets you see how they are linked. Its units are non-dimensional, with $\mu = 1$; the attracting body sits at the amber focus and periapsis is to the right. The read-out also shows the semi-minor axis b and the period T, which we derive in Section 3.

Use the explorer to answer the four questions below.

1. Choose "drive the size with h" and sweep e from 0 to 2.5. What stays fixed, and what happens to ε as e passes 1?
2. Now drive with a and sweep e (tick the box to see the whole family). What stays fixed this time? Where does the focus go?
3. Compare the a and ε buttons. Why do they behave identically?
4. With a held fixed, try to push e through 1. What happens, and why?

What you should find

1. The semi-latus rectum $p = h^2/\mu$ stays fixed: the vertical blue segment does not move. The conic unrolls from circle through ellipse and parabola to hyperbola, and ε changes sign exactly at $e = 1$, as Equation 8 demands.
2. The size and the energy stay fixed (and, on the ellipse branch, the period). Every ellipse in the family has the same major axis, and the focus slides from the centre towards periapsis as e grows. **Energy is blind to shape.**
3. $\varepsilon = -\mu/2a$ is a one-to-one map, so (a, e) and (ε, e) are the *same* description; only (h, e) is genuinely different.
4. The slider stops. An ellipse cannot become a hyperbola without a passing through infinity, which is the Casimir relation $\mu^2(e^2 - 1) = 2\varepsilon h^2$ forbidding $\operatorname{sign}\varepsilon \neq \operatorname{sign}(e - 1)$. With h held fixed instead, the passage is smooth.

2.11 Velocity

We know the shape, $r(f)$. The next question a mission designer asks is: **how fast is the body moving at each point, and in which direction?** It turns out that we can answer this as a function of f, without knowing *when* the body gets there. We need only two facts already in hand.

1. **The orbit equation** Equation 5, $r = p/(1 + e\cos f)$, with f measured from periapsis.

2. **Angular momentum**, which tells us how fast f changes.

2.11.1 The rotating basis

In the orbital plane, attach to the moving body two unit vectors: $\hat{\mathbf{r}}$, radially outward, and $\hat{\theta}$, perpendicular to it in the direction of increasing f (Figure 14 in the appendix). Since $\mathbf{e}$ is fixed, the direction of periapsis does not move, and f is simply the polar angle of the body measured from a fixed line. The two unit vectors turn with the body, and their rates of change are

$$\dot{\hat{\mathbf{r}}} = \dot{f}\,\hat{\theta}, \qquad \dot{\hat{\theta}} = -\dot{f}\,\hat{\mathbf{r}}. \tag{18}$$

(Both are proved in Section 7. If you have not seen them before, try to guess them first from a sketch: a unit vector can only turn, so its derivative is perpendicular to it.)

Differentiating $\mathbf{r} = r\hat{\mathbf{r}}$ and using Equation 18,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{f}\,\hat{\theta} \equiv v_r\,\hat{\mathbf{r}} + v_\perp\,\hat{\theta}. \tag{19}$$

So there are only two numbers to find: the **radial** speed $v_r = \dot{r}$, which changes the *distance*, and the **transverse** speed $v_\perp = r\dot{f}$, which changes the *direction*. Only the transverse part carries angular momentum:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{r}} \times \left(v_r\hat{\mathbf{r}} + v_\perp\hat{\theta}\right) = rv_\perp\,\hat{\mathbf{r}} \times \hat{\theta}, \qquad h = rv_\perp = r^2\dot{f}. \tag{20}$$

The relation $h = r^2\dot{f}$ is the tool that converts a rate with respect to f into a rate in time: $\dfrac{d}{dt} = \dot{f}\dfrac{d}{df} = \dfrac{h}{r^2}\dfrac{d}{df}$.

2.11.2 The transverse component

Angular momentum gives it with no work at all:

$$v_\perp = \frac{h}{r} = \frac{h}{p}\,(1 + e\cos f)\,.$$

Since $p = h^2/\mu$, the prefactor is $h/p = \mu/h$, and

$$\boxed{v_\perp = \frac{\mu}{h}\,(1 + e\cos f)\,.} \tag{21}$$

It is positive everywhere the conic exists (the body never swings backwards), largest at periapsis and smallest at apoapsis, exactly as $h = rv_\perp$ = constant requires.

2.11.3 The radial component

Differentiate the orbit equation. It is cleanest in reciprocal form:

$$\frac{1}{r} = \frac{1 + e\cos f}{p} \qquad \Longrightarrow \qquad -\frac{\dot{r}}{r^2} = -\frac{e\sin f}{p}\,\dot{f}.$$

Multiplying by $-r^2$ and using $r^2\dot{f} = h$,

$$\dot{r} = \frac{r^2\dot{f}}{p}\, e\sin f = \frac{h}{p}\, e\sin f \qquad \Longrightarrow \qquad \boxed{v_r = \frac{\mu}{h}\, e\sin f.} \tag{22}$$

Both components carry the same prefactor $\mu/h = \sqrt{\mu/p}$: the circular speed at radius p, the natural unit of speed for the orbit.

2.11.4 The velocity vector and the speed

Together,

$$\boxed{\mathbf{v}(f) = \frac{\mu}{h}\left[e\sin f\,\hat{\mathbf{r}} + (1 + e\cos f)\,\hat{\theta}\right].} \tag{23}$$

Reading it:

- $v_r = 0$ at $f = 0$ and $f = 180°$. The apsides are exactly the points where the distance stops growing or shrinking, and **there the velocity is perpendicular to r**.
- $v_r > 0$ on the outbound half, $0 < f < 180°$, and $v_r < 0$ on the inbound half.
- $e = 0$ kills v_r everywhere: circular motion at the constant speed $\sqrt{\mu/p}$, always perpendicular to the radius.

The speed follows from $v^2 = v_r^2 + v_\perp^2$:

$$v^2 = \frac{\mu^2}{h^2}\left[e^2\sin^2 f + (1 + e\cos f)^2\right] = \frac{\mu^2}{h^2}\left[1 + 2e\cos f + e^2\right],$$

since $e^2\sin^2 f + e^2\cos^2 f = e^2$. With $\mu^2/h^2 = \mu/p$,

$$\boxed{v(f) = \sqrt{\frac{\mu}{p}}\,\sqrt{1 + 2e\cos f + e^2}.} \tag{24}$$

The speed depends on f only through $\cos f$, so it is symmetric about the line of apsides: the body passes a given distance on the way out and on the way back with the same speed.

Two checks on the speed formula. (a) Show that Equation 24 agrees with vis-viva Equation 13. (Write $2/r$ using the orbit equation and $1/a = (1 - e^2)/p$.) (b) The Earth's orbit has $e = 0.0167$ and $\sqrt{\mu_\odot/p} \approx 29.8$ km/s. What is the largest radial speed of the Earth, towards or away from the Sun?

What you should find

(a) $\mu\left(\dfrac{2}{r} - \dfrac{1}{a}\right) = \dfrac{\mu}{p}\left[2(1 + e\cos f) - (1 - e^2)\right] = \dfrac{\mu}{p}\left(1 + 2e\cos f + e^2\right).$

(b) $|v_r|$ is largest at $f = \pm 90°$, where it equals $(\mu/h)e = 0.0167 \times 29.8 = 0.50$ km/s. The Earth's distance from the Sun changes by up to half a kilometre every second, a tiny fraction of its orbital speed.

2.12 The flight path angle

The direction of the velocity is described by one angle. The **flight path angle** γ is the angle between the velocity and the **local horizontal**, the direction perpendicular to $\mathbf{r}$ (that is, $\hat{\theta}$). It is positive when the body is climbing (moving away from the central body). From Equation 19,

$$\tan\gamma = \frac{v_r}{v_\perp}, \qquad \cos\gamma = \frac{v_\perp}{v} = \frac{h}{rv}, \qquad \sin\gamma = \frac{v_r}{v}.$$

Substituting Equation 21 and Equation 22, the common factor μ/h cancels:

$$\boxed{\tan\gamma(f) = \frac{e\sin f}{1 + e\cos f}.} \tag{25}$$

Only e and f survive. Make the orbit bigger, or put it round a different planet: the *direction* of flight at a given true anomaly does not change.

The relation $h = rv\cos\gamma$ is the practical form of the angular momentum. A tracking station does not measure r and γ directly: it measures range, two pointing angles and their rates, relative to itself. Once those are turned into the spacecraft's position and velocity relative to the Earth's centre, r, v and γ follow, and with them h and the orbit.

Where is $\gamma = 0$? Is the velocity on an eccentric orbit ever perpendicular to $\mathbf{r}$ anywhere else?

Answer

$\gamma = 0$ where $\sin f = 0$, i.e. only at the two apsides (for $e > 0$). Everywhere else the body is climbing or descending. On a circle, $\gamma \equiv 0$.

2.12.1 Where is the climb steepest?

Differentiate Equation 25 with respect to f (quotient rule):

$$\begin{aligned}\frac{d}{df}\left(\frac{e\sin f}{1 + e\cos f}\right) &= \frac{e\cos f\,(1 + e\cos f) + e^2\sin^2 f}{(1 + e\cos f)^2} \\ &= \frac{e\,(\cos f + e)}{(1 + e\cos f)^2}.\end{aligned} \tag{26}$$

This vanishes where $\cos f = -e$. **That equation has a solution only if** $e \le 1$: this is the first result in the note that cares which conic we are on.

Ellipse ($e < 1$)**.** At $\cos f = -e$ we have $\sin f = \sqrt{1 - e^2}$ and $1 + e\cos f = 1 - e^2$, so

$$\tan\gamma_{\max} = \frac{e\sqrt{1 - e^2}}{1 - e^2} = \frac{e}{\sqrt{1 - e^2}} \qquad \Longrightarrow \qquad \boxed{\gamma_{\max} = \arcsin e.} \tag{27}$$

The point $\cos f = -e$ has $r = p/(1 - e^2) = a$. We will see in Section 3 that it is the end of the minor axis.

Hyperbola ($e > 1$)**.** Now $\cos f + e > 0$ on the whole arc $|f| < f_\infty$, so $\tan\gamma$ increases monotonically with f. As $f \to f_\infty$, the denominator of Equation 25 tends to 0^+ while the numerator stays positive, so $\tan\gamma \to +\infty$ and $\gamma \to 90°$. Far out on a hyperbola the body moves radially, straight away from (or towards) the central body. The same is true of the parabola, as we will see.

How steeply does the Earth's orbit ($e = 0.0167$) ever climb? And an orbit with $e = 0.8$? For small e, show that $\gamma \approx e\sin f$.

What you should find

$\arcsin 0.0167 = 0.96°$, and $\arcsin 0.8 = 53.1°$. For small e the denominator of Equation 25 is $1 + O(e)$ and $\tan\gamma \approx \gamma$, so $\gamma \approx e\sin f$. A nearly circular orbit flies nearly horizontally everywhere, which is why "circular" is such a good first approximation for the planets and for most operational satellites.

2.13 The hodograph

The component form of the velocity hides something remarkable. Plot the tip of the velocity vector, drawn from a fixed origin in *velocity space*, as the body goes round its orbit. The curve traced is called the **hodograph**. For the inverse-square law it is a circle, on every orbit.

To see why, go back to Equation 4, $\dot{\mathbf{r}} \times \mathbf{h} = \mu(\hat{\mathbf{r}} + \mathbf{e})$, and take the cross product with $\mathbf{h}$ from the left. Using $\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = h^2\mathbf{v} - \mathbf{h}(\mathbf{h} \cdot \mathbf{v})$ and $\mathbf{h} \cdot \mathbf{v} = 0$,

$$h^2\mathbf{v} = \mu\,\mathbf{h} \times (\hat{\mathbf{r}} + \mathbf{e}) \qquad \Longrightarrow \qquad \boxed{\mathbf{v} = \frac{\mu}{h}\,\hat{\mathbf{h}} \times (\hat{\mathbf{r}} + \mathbf{e})\,.} \tag{28}$$

Now read it. The operation $\hat{\mathbf{h}}\times$ simply rotates a vector in the orbital plane by 90°. So $\mathbf{v}$ is the sum of

$$\underbrace{\frac{\mu}{h}\,\hat{\mathbf{h}} \times \hat{\mathbf{r}}}_{\text{length } \mu/h,\ \text{turning with } f} \;+\; \underbrace{\frac{\mu}{h}\,\hat{\mathbf{h}} \times \mathbf{e}}_{\text{fixed vector}} :$$

a vector of constant length μ/h that rotates, added to a constant vector. The tip therefore moves on a **circle of radius μ/h, whose centre is at distance $\mu e/h$ from the origin**, in the direction perpendicular to $\mathbf{e}$.

In the **perifocal frame**, with unit vector $\hat{\mathbf{p}}$ towards periapsis and $\hat{\mathbf{q}}$ at 90° to it in the direction of motion, $\hat{\mathbf{r}} = \cos f\,\hat{\mathbf{p}} + \sin f\,\hat{\mathbf{q}}$ and the components of Equation 28 are

$$\boxed{v_P = -\frac{\mu}{h}\sin f, \qquad v_Q = \frac{\mu}{h}\,(e + \cos f)\,,} \tag{29}$$

so that

$$v_P^2 + \left(v_Q - \frac{\mu e}{h}\right)^2 = \left(\frac{\mu}{h}\right)^2.$$

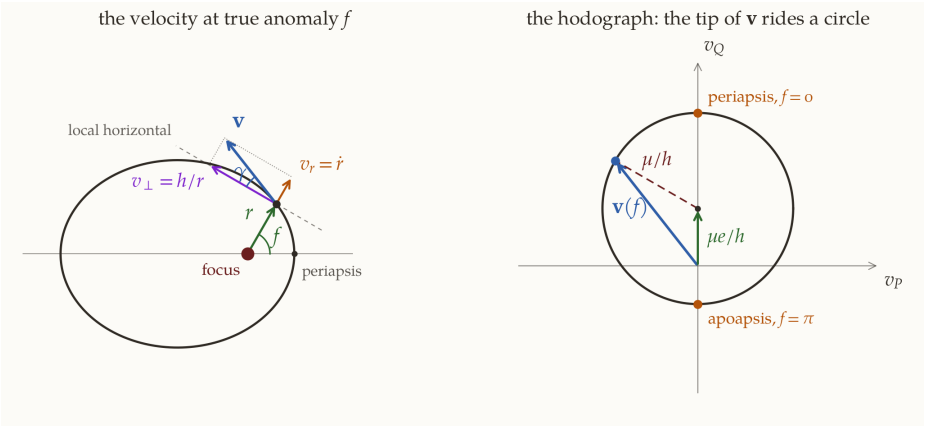

Figure 2: **Left:** the velocity splits into $v_r = (\mu/h)e\sin f$ along $\hat{\mathbf{r}}$ and $v_\perp = (\mu/h)(1 + e\cos f)$ along the local horizontal; γ is the tilt of $\mathbf{v}$ above that horizontal. **Right:** the same velocity in velocity space, with perifocal components (v_P, v_Q). As f runs round the orbit the tip of $\mathbf{v}$ traces a circle of radius μ/h whose centre is displaced by $\mu e/h$ perpendicular to the line of apsides. The top and bottom of the circle are periapsis and apoapsis. *Script: fig-velocity-hodograph.py*.

Every Kepler orbit, ellipse, parabola or hyperbola, has a circular hodograph. What differs is where the origin of velocity space sits, and how much of the circle is used.

- $e < 1$: the origin ($\mu e/h$ from the centre) is *inside* the circle (radius μ/h). The whole circle is traversed, and the velocity points in every direction once per orbit.
- $e = 1$: the origin is *on* the circle. The speed $v = 0$ is reached, but only in the limit at infinity.
- $e > 1$: the origin is *outside* the circle. Only an arc is traversed, and the velocity is confined to a wedge of directions.

For $e > 1$, the velocity directions available are those of lines from the origin of velocity space to points of the arc. The extreme directions are along the two tangents from the origin to the circle. What is the half-angle of that wedge? (You will meet this angle again in Section 5.)

Answer

The tangent from a point at distance $d = \mu e/h$ to a circle of radius $\rho = \mu/h$ makes an angle α with the line to the centre, where $\sin\alpha = \rho/d = 1/e$. The two tangent directions are the directions of the velocity at infinity, incoming and outgoing. So the flyby turns the velocity through an angle δ with $\sin(\delta/2) = 1/e$; this is the *turning angle* of Section 5.

History: the hodograph. That the velocity of a Kepler orbit moves on a circle was first shown by August Ferdinand Möbius in *Die Elemente der Mechanik des Himmels* (1843). William Rowan Hamilton found it independently, coined the word *hodograph* and used it to give a geometric derivation of the Kepler orbit (Hamilton 1847); he later acknowledged Möbius's priority. Maxwell used it as a teaching device. It is perhaps the cleanest single statement of what is special about the inverse-square law.

2.13.1 Speed and flight path angle on the four conics

Figure 3 plots Equation 24 and Equation 25 for one orbit of each type.

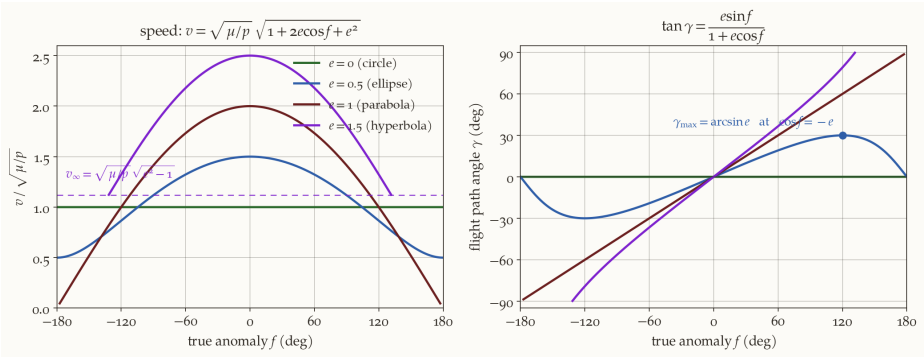

Figure 3: Speed (left, in units of $\sqrt{\mu/p}$) and flight path angle (right) against true anomaly. Circle: constant speed, $\gamma \equiv 0$. Ellipse: fastest at periapsis, slowest at apoapsis, γ peaking at $\arcsin e$ where $\cos f = -e$. Parabola: speed falls to zero as $f \to \pm 180°$ and $\gamma = f/2$ exactly. Hyperbola: the curve stops at the asymptote $\cos f = -1/e$, where the speed has fallen not to zero but to $v_\infty = \sqrt{\mu/p}\sqrt{e^2-1}$, and $\gamma \to \pm 90°$. *Script: fig-speed-fpa.py.*

orbit	e	$v(f)$	γ	hodograph
circle	0	$\sqrt{\mu/a}$, constant	0	circle centred on the origin
ellipse	$0 < e < 1$	$\sqrt{\mu/p}\sqrt{1+2e\cos f+e^2}$	$\pm\arcsin e$	origin inside
parabola	1	$\sqrt{2\mu/r} = 2\sqrt{\mu/p}\,\lvert\cos\frac{f}{2}\rvert$	$f/2$	origin on the circle
hyperbola	> 1	$\sqrt{\mu/p}\sqrt{1+2e\cos f+e^2}$ $\to v_\infty$	$\pm 90°$ at the asymptotes	origin outside

For the hyperbola, at $\cos f_\infty = -1/e$ Equation 24 gives

$$v_\infty^2 = \frac{\mu}{p}\left(1 - 2 + e^2\right) = \frac{\mu}{p}\left(e^2 - 1\right) = \frac{\mu}{|a|} = C_3,$$

the characteristic energy of Equation 17, recovered from the velocity side.

2.14 From $r(f)$ to $r(t)$

Everything so far gives the orbit and the velocity as functions of the true anomaly f. None of it says *when* the body is at a given f. That is the last question, and the hardest.

From Equation 20, $h = r^2\dot{f}$, so

$$dt = \frac{r^2}{h}\,df.$$

Substituting the orbit equation and integrating from periapsis ($f = 0$ at $t = t_p$, the time of periapsis passage),

$$t - t_p = \frac{p^2}{h}\int_0^f \frac{df'}{\left(1 + e\cos f'\right)^2}. \tag{30}$$

This is the whole problem of timing. The integral is elementary, but it is ugly, and the result cannot be solved for f as a function of t in closed form.

Try it on the circle first: set $e = 0$ in Equation 30 and find $f(t)$. Then try $e = 0.5$ by any means you like (a table of integrals, or a computer algebra system). Can you solve the result for f?

What you should find

For $e = 0$: $t - t_p = (p^2/h)f$, so $f = (h/p^2)(t - t_p)$, uniform motion. For $e \neq 0$ the integral is a combination of $\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right)$ and $\frac{\sin f}{1 + e\cos f}$, and no algebra will turn "time equals arctan plus a rational function of sines" into "f equals something".

The classical remedy is not to fight the integral but to **change the variable**. And this is where the conics finally separate: the substitution that tames $(1+e\cos f)^{-2}$ is different for $e < 1$, $e = 1$ and $e > 1$.

2.15 Where the three conics part ways

Everything up to here holds on every orbit, with no case distinction:

$$r = \frac{p}{1 + e\cos f}, \quad h = r^2\dot{f}, \quad \varepsilon = -\frac{\mu}{2a}, \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

$$v_r = \frac{\mu}{h}e\sin f, \quad v_\perp = \frac{\mu}{h}\left(1 + e\cos f\right), \quad \tan\gamma = \frac{e\sin f}{1 + e\cos f}.$$

What is left is genuinely different for each conic:

question	$e < 1$	$e = 1$	$e > 1$
an **auxiliary variable** that tames Equation 30	eccentric anomaly E	$D = \tan\frac{f}{2}$	hyperbolic anomaly H
the resulting **Kepler equation**	$M = E - e\sin E$	$M_p = D + \frac{D^3}{3}$	$M_h = e\sinh H - H$
can it be **inverted in closed form**?	no	**yes** (a cubic)	no

together with a few quantities that exist only for some conics: the apoapsis r_a and the period T (ellipse only), and v_∞, C_3 and the turning angle (hyperbola only). The next three parts take the conics one at a time; nothing already derived is derived again.

3 The ellipse $(0 \le e < 1)$

3.1 What is new

Bound motion. Everything in Section 2 still holds word for word. What the restriction $e < 1$ adds is:

- $a > 0$ becomes a genuine **length on the figure**, the semi-major axis;
- an **apoapsis** exists, $r_a = a(1+e)$, so the curve is closed and encloses a finite area;
- a finite area swept at a constant rate gives a **period** T;
- the closed curve has an **auxiliary circle**, and with it the eccentric anomaly E, which tames the time integral.

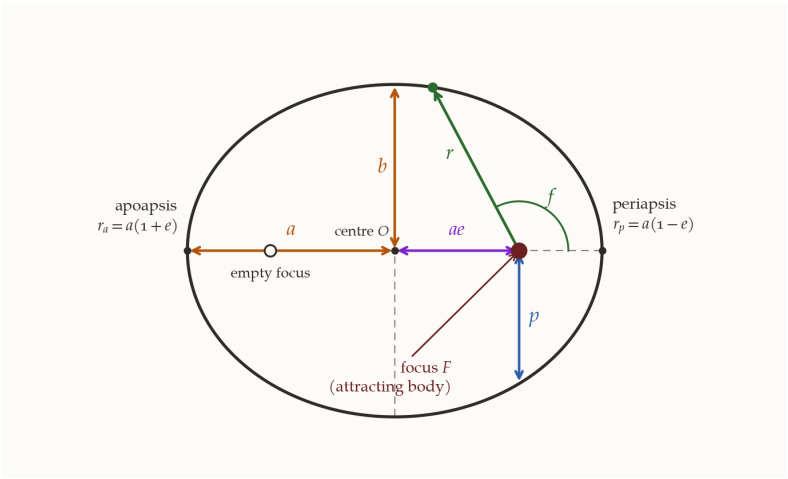

Figure 4: Geometry of the elliptic orbit ($e = 0.6$). The attracting body is at the occupied focus F; the other focus is empty. The centre-to-focus distance is ae, so e is the fractional offset of the focus from the centre. The semi-latus rectum p is the radius at $f = 90°$, and f is measured at the focus from the direction of periapsis. *Script: fig-conic-geometry.py.*

3.2 Why a is the semi-major axis

In Section 2.7 we *defined* a by $p = a(1-e^2)$ and claimed nothing about its geometry. For $e < 1$ we can now show that it is exactly half the distance from periapsis to apoapsis.

Put $f = 0$ and $f = 180^\circ$ in the orbit equation. The apoapsis distance is finite only because $e < 1$:

$$r_p = \frac{p}{1+e}, \qquad r_a = \frac{p}{1-e}.$$

The major axis runs from apoapsis to periapsis straight through the focus, so its length is $r_p + r_a$, and half of it is

$$\frac{r_p + r_a}{2} = \frac{p}{2}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{p}{2} \cdot \frac{(1-e)+(1+e)}{1-e^2} = \frac{p}{1-e^2} = a.$$

So the *defined* a and the *drawn* semi-major axis are the same number, for bound orbits. Three corollaries we use constantly:

$$r_p = a(1-e), \qquad r_a = a(1+e), \qquad h^2 = \mu a\left(1-e^2\right). \tag{31}$$

Given only r_p and r_a, how do you find a and e? Apply it to a transfer orbit from 300 km altitude to geostationary radius: $r_p = 6678$ km, $r_a = 42\,164$ km.

Answer

$a = (r_p + r_a)/2$ and $e = (r_a - r_p)/(r_a + r_p)$, from adding and subtracting the first two of Equation 31. Here $a = 24\,421$ km and $e = 0.7265$. We will use this orbit, a geostationary transfer orbit (GTO), as a running example.

3.3 The period

A closed orbit repeats, and the time per revolution is its **period** T. We have a quick way to find it: area is swept at a constant rate, so T is the total area divided by that rate.

Step 1: the rate. In Section 2.2 we found $dA/dt = h/2$ (Kepler's second law), using $h = r^2\dot{f}$ from Equation 20.

Step 2: the area. An ellipse of semi-axes a and b has area πab, so we need the semi-minor axis b. It follows from the orbit equation alone. Put $\cos f = -e$ in $r = a(1-e^2)/(1+e\cos f)$:

$$r = \frac{a\left(1-e^2\right)}{1-e^2} = a, \qquad r\cos f = -ae.$$

This point lies a distance ae to the left of the focus (the side away from periapsis). But the centre is also ae to the left of the focus (Figure 4), so the point sits directly above the centre: it is the end of the minor axis. Its distance from the centre is b and from the focus is a, so by Pythagoras

$$b^2 = a^2 - (ae)^2 \qquad \Longrightarrow \qquad b = a\sqrt{1-e^2} = \sqrt{ap}. \tag{32}$$

(This is also the point of steepest climb found in Section 2.12, where $r = a$.)

Step 3: divide.

$$T = \frac{\pi ab}{h/2} = \frac{2\pi\, a \cdot a\sqrt{1-e^2}}{\sqrt{\mu a\left(1-e^2\right)}} = \frac{2\pi a^2}{\sqrt{\mu a}},$$

$$\boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}, \qquad n \equiv \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}.} \tag{33}$$

The quantity n is the **mean motion**, the average angular rate; we need it for Kepler's equation.

Once again e has cancelled: the period, like the energy, depends only on a. Indeed T and $\varepsilon = -\mu/2a$ are two readings of the same number. This is **Kepler's third law**, with the constant now explained: $T^2/a^3 = 4\pi^2/\mu$.

History: Newton's correction to Kepler's third law. Kepler stated $T^2 \propto a^3$ with the *same* constant for every planet. Newton's version, $T^2 = 4\pi^2 a^3/(G(m_1+m_2))$, shows that the constant depends on the planet's mass too. For Jupiter, the heaviest planet, T^2/a^3 is smaller than for a massless planet by the factor $1/(1+1/1047) = 0.99905$, a part in a thousand. Kepler's data could not show this, and neither can the four-figure table in the N-body note.

Two period calculations. (a) Find the period of the space station's orbit ($r = 6798$ km). (b) A geostationary satellite must complete one revolution per *sidereal* day, 86 164.1 s, the Earth's rotation period relative to the stars. Find the radius of its circular orbit.

What you should find

(a) $T = 2\pi\sqrt{6798^3/398\,600} = 5578$ s $= 93.0$ min, about 15.5 orbits a day.

(b) $a = \left(\mu\,(T/2\pi)^2\right)^{1/3} = 42\,164$ km, i.e. 35 786 km above the equator.

In the explorer of Section 2.10, drive with a and sweep e. Does the period change?

Answer

No. T depends on a alone. A circle of radius a and a needle-thin ellipse of semi-major axis a take exactly the same time to go round.

3.4 Two bodies in three dimensions: an example

Here is a full example that uses most of what we have so far. Take units with $G = 1$ and two equal masses, $m_1 = m_2 = \frac{1}{2}$, so $\mu = G(m_1+m_2) = 1$. At $t = 0$:

$$\mathbf{R}_1(0) = \left(+\tfrac{1}{2}, 0, 0\right), \qquad \mathbf{V}_1(0) = \tfrac{1}{2}\,\hat{\mathbf{y}},$$
$$\mathbf{R}_2(0) = \left(-\tfrac{1}{2}, 0, 0\right), \qquad \mathbf{V}_2(0) = \tfrac{1}{2}\,\hat{\mathbf{z}}.$$

The bodies sit on the x-axis; one is launched along y, the other along z. Before reading on, try to predict the motion.

Work out the orbit from the initial data alone. (a) Find the relative state $\mathbf{r}(0) = \mathbf{R}_2 - \mathbf{R}_1$ and $\dot{\mathbf{r}}(0) = \mathbf{V}_2 - \mathbf{V}_1$. (b) From it compute $\mathbf{h}$, ε, then a, e and T, without integrating anything. (c) Is the starting point periapsis, apoapsis, or neither?

What you should find

(a) $\mathbf{r}(0) = -\hat{\mathbf{x}}$ and $\dot{\mathbf{r}}(0) = \frac{1}{2}(\hat{\mathbf{z}} - \hat{\mathbf{y}})$.

(b) $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}} = \frac{1}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}})$, so $h = 1/\sqrt{2}$. $\varepsilon = v^2/2 - \mu/r = \frac{1}{4} - 1 = -\frac{3}{4}$. Then $a = -\mu/2\varepsilon = \frac{2}{3}$, $e = \sqrt{1 + 2\varepsilon h^2/\mu^2} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$ and $T = 2\pi\sqrt{a^3/\mu} = 3.4201$.

(c) $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$, so $\dot{r} = 0$ and the start is an apsis. Since $r(0) = 1 = a(1+e)$, it is the apoapsis.

There is one more feature of *this* initial state. The total momentum is $\mathbf{P} = m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = \frac{1}{4}(\hat{\mathbf{y}} + \hat{\mathbf{z}}) \neq \mathbf{0}$, so the barycentre drifts, at $\mathbf{V}_{\mathrm{cm}} = \mathbf{P}/M = \frac{1}{4}(\hat{\mathbf{y}} + \hat{\mathbf{z}})$. This is *parallel* to $\mathbf{h}$: the barycentre moves along the normal to the orbital plane. The plane of the relative orbit is carried through space face-first, like a sheet pushed along its normal, and neither body ever returns to a point it has visited. In the inertial frame each body traces a **helix**, even though the relative orbit is a closed ellipse of period 3.4201 (Figure 5).

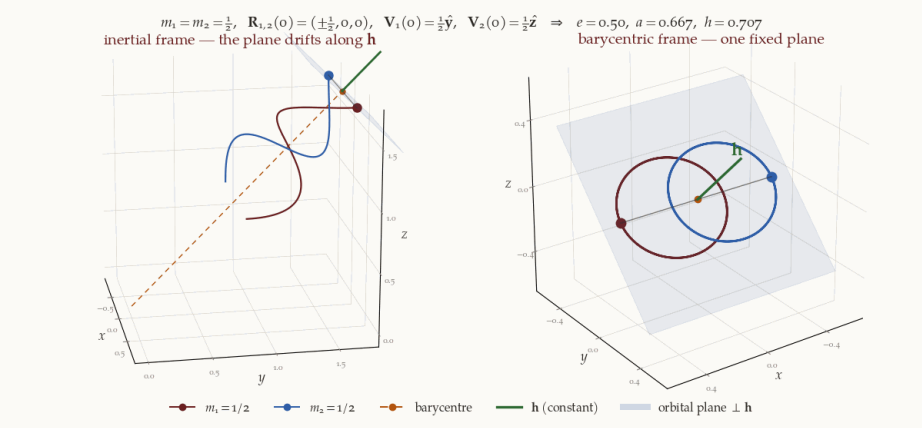

Figure 5: **Left:** the inertial frame. The barycentre (amber) moves along a straight line, carrying the orbital plane with it, and each body traces a helix. **Right:** the barycentric frame. With the drift subtracted, the two bodies run on congruent ellipses (equal masses) in one plane that never moves. Shaded patch: the plane perpendicular to **h**. Integrated with an adaptive Runge–Kutta–Fehlberg 7(8) method. *Script: two-body-3d-plane.py.*

What must be true of the initial data for both bodies to move in a single fixed plane *in the inertial frame* as well?

Answer

In the barycentric frame both bodies lie in the plane through the barycentre perpendicular to **h** (by Equation 2, both are multiples of **r**). In the inertial frame that plane is carried along by $\mathbf{V}_{\text{cm}}t$. It stays the *same* plane only if the drift lies in it: $\mathbf{V}_{\text{cm}} \cdot \mathbf{h} = 0$, i.e. $(m_1\mathbf{V}_1 + m_2\mathbf{V}_2) \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = 0$ at $t = 0$. Here the drift is along **h**, the worst case.

3.5 Speed and direction on an ellipse

3.5.1 Scaling out the size

Divide the velocity components Equation 22, Equation 21 and the speed Equation 24 by $\sqrt{\mu/a}$, the circular speed at radius a. With $\mu/h = \sqrt{\mu/p} = \sqrt{\mu/a}/\sqrt{1-e^2}$,

$$\frac{v_r}{\sqrt{\mu/a}} = \frac{e\sin f}{\sqrt{1-e^2}}, \qquad \frac{v_\perp}{\sqrt{\mu/a}} = \frac{1+e\cos f}{\sqrt{1-e^2}},$$

$$\frac{v}{\sqrt{\mu/a}} = \frac{\sqrt{1+2e\cos f+e^2}}{\sqrt{1-e^2}}.$$

The semi-major axis has disappeared: e **fixes the shape of all three curves, and** a **only sets the scale.**

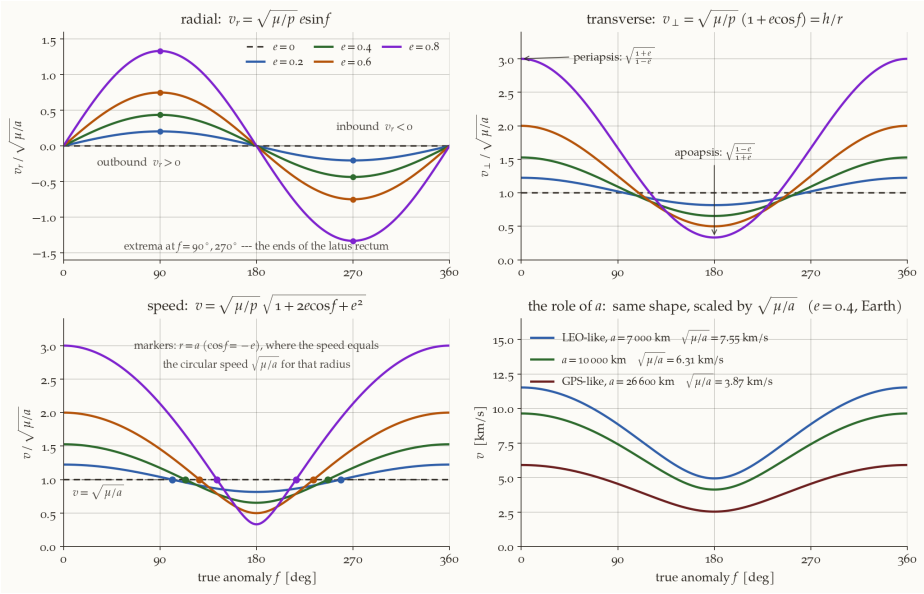

Figure 6: Radial, transverse and total speed against true anomaly for several eccentricities, in units of $\sqrt{\mu/a}$. The fourth panel repeats one eccentricity at three sizes of orbit: the same curve, stretched vertically by $\sqrt{\mu/a}$. *Script: fig-velocity-components.py*.

What the figure shows:

- v_r **is odd about the line of apsides.** It vanishes at both apsides and is largest in size at $f = \pm 90°$, where $|v_r| = (\mu/h)e$. Outbound is positive, inbound negative.
- $v_\perp$ **never changes sign.** It is h/r, largest at periapsis and smallest at apoapsis.
- **The speed equals $\sqrt{\mu/a}$ exactly where** $r = a$, at $\cos f = -e$ (the ends of the minor axis). Vis-viva says so at once: $v^2 = \mu(2/r - 1/a) = \mu/a$ when $r = a$. Inside that radius the body is faster than its "average" speed $\sqrt{\mu/a}$; outside, slower.
- **Eccentric orbits are lopsided in time as well as speed.** The body hurries through periapsis and dawdles at apoapsis, which is Kepler's second law seen from the velocity side.

3.5.2 Apsidal speeds

Put $f = 0$ and $f = 180°$ in Equation 24:

$$\begin{aligned} v_p &= \sqrt{\frac{\mu}{p}}(1+e) = \sqrt{\frac{\mu}{a}\,\frac{1+e}{1-e}}, \\ v_a &= \sqrt{\frac{\mu}{p}}(1-e) = \sqrt{\frac{\mu}{a}\,\frac{1-e}{1+e}}. \end{aligned} \tag{34}$$

From Equation 34 find v_p/v_a and $v_p v_a$. Interpret both.

What you should find

$$\frac{v_p}{v_a} = \frac{1+e}{1-e} = \frac{r_a}{r_p}, \qquad v_p v_a = \frac{\mu}{p}\,(1-e^2) = \frac{\mu}{a}.$$

The first is $h = rv$ at the two apsides (the velocity is perpendicular to $\mathbf{r}$ there, so $h = r_p v_p = r_a v_a$). The second says that the *geometric mean* of the two apsidal speeds is the circular speed at radius a.

3.5.3 The flight path angle across a family of ellipses

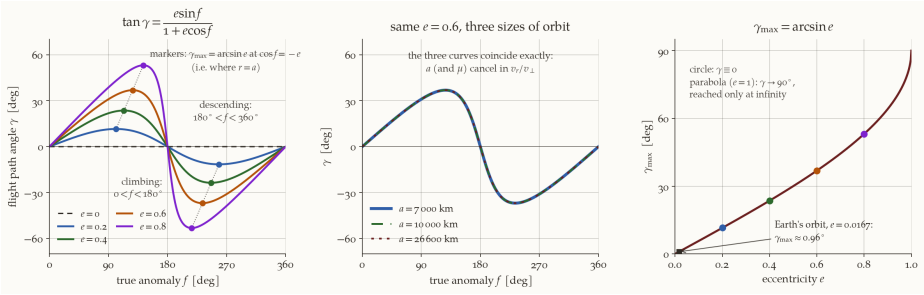

Figure 7: **Left:** γ against true anomaly for several eccentricities; markers at $\cos f = -e$, where $\gamma = \arcsin e$. **Centre:** one eccentricity flown at three semi-major axes; the curves lie exactly on top of each other. **Right:** the envelope $\gamma_{\max} = \arcsin e$. *Script: fig-flight-path-angle.py.*

Figure 7 confirms what Equation 25 said: a does nothing to γ; e does everything. $\gamma_{\max} = \arcsin e$ grows slowly at first ($e = 0.2$ gives $11.5°$) and then quickly ($e = 0.8$ gives $53°$), approaching $90°$ as $e \to 1$.

The flight path angle is easy to draw with the wrong sign. Figure 8 draws it at one point in each quadrant of an ellipse.

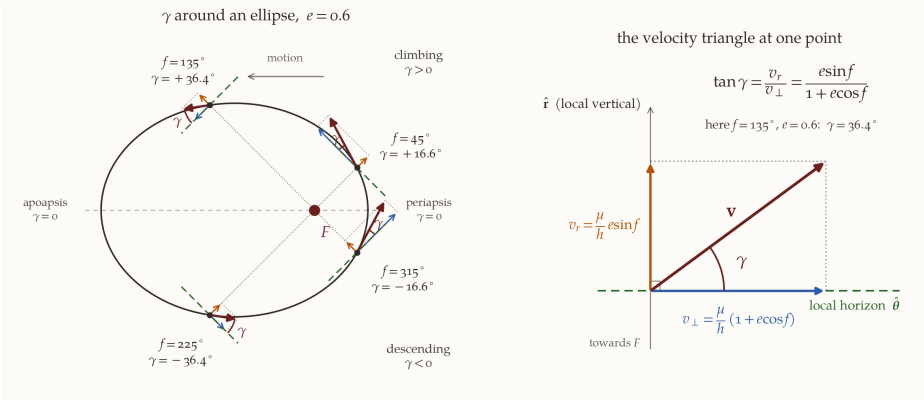

Figure 8: **Left:** one point in each quadrant of an ellipse with $e = 0.6$. At each, the dotted line is the radius from the focus F, the green dashes are the local horizontal (perpendicular to $\mathbf{r}$), the blue and amber arrows are $v_\perp$ and v_r, and the maroon arrow is $\mathbf{v}$; γ is marked between the horizontal and $\mathbf{v}$. **Right:** the construction at $f = 135°$ alone, rotated so that the local horizontal is flat. *Script: fig-fpa-quadrants.py.*

Look at Figure 8. (a) Why is the "horizontal" in the figure not horizontal on the page? (b) Which arrow keeps its sense all the way round, and which reverses? (c) The figure gives $\gamma = 16.6°$ at $f = 45°$ and $36.4°$ at $f = 135°$. What are the values at $f = 315°$ and $f = 225°$?

Answer

(a) The local horizontal is perpendicular to $\mathbf{r}$, so it turns with the body. γ is measured from a reference that itself rotates.

(b) The blue $v_\perp = h/r$ always points the same way round the orbit; it carries all of the angular momentum. The amber v_r reverses between the second and third quadrants, and that reversal *is* the sign of γ: climbing on the way out, descending on the way back.

(c) $-16.6°$ and $-36.4°$: same radius, same speed, velocity reflected in the line of apsides. (With $e = 0.6$ the steepest climb is $\arcsin 0.6 = 36.87°$ at $f = 126.87°$, so the $135°$ point is within half a degree of it.)

An orbit is raised by a small burn δv applied along the velocity at periapsis. Show that the periapsis distance does not change, and find how much the apoapsis distance changes. Where does the energy go?

Answer

After a burn along the velocity at periapsis, the velocity is still perpendicular to $\mathbf{r}$ there, so the burn point is still an apsis of the new orbit, and since the speed went up it is still the periapsis: $\delta r_p = 0$. All of the change happens on the far side. The energy changes by $\delta\varepsilon = v_p\,\delta v$ (to first order), and from $\varepsilon = -\mu/2a$, $\delta a = (2a^2/\mu)\,\delta\varepsilon$. With $r_a = 2a - r_p$,

$$\delta r_a = 2\,\delta a = \frac{4a^2 v_p}{\mu}\,\delta v.$$

For a circular orbit 400 km up, $4a^2 v_p/\mu \approx 3500$ km per km/s: each metre per second raises the far side by about 3.5 km. The energy goes into potential energy on the far side. The lever arm $4a^2 v_p/\mu$ grows with v_p, which is why burns are most effective where the speed is high (the *Oberth effect*).

3.5.4 Worked example: a geostationary transfer orbit

Take the GTO of the earlier Check box: $r_p = 6678$ km, $r_a = 42\,164$ km, so $a = 24\,421$ km and $e = 0.7265$, with $\mu = 398\,600$ km^3/s^2.

Compute the perigee and apogee speeds, check $v_p/v_a = r_a/r_p$, and find where the climb is steepest: the true anomaly, γ, the radius and the speed there.

What you should find

From Equation 34, $v_p = 10.15$ km/s and $v_a = 1.608$ km/s; $v_p/v_a = 6.31 = r_a/r_p$. The steepest climb is at $\cos f = -e$, i.e. $f = 136.6°$, where $\gamma = \arcsin 0.7265 = 46.6°$, $r = a = 24\,421$ km and $v = \sqrt{\mu/a} = 4.04$ km/s.

The spacecraft starts at 10.15 km/s and arrives at geostationary height at only 1.61 km/s, whereas a geostationary satellite moves at 3.07 km/s. The difference is what the apogee engine has to supply.

At which true anomaly is a body on an ellipse moving at exactly the local circular speed $\sqrt{\mu/r}$? Is it a coincidence that this is also where γ is largest?

Answer

Vis-viva gives $\mu(2/r - 1/a) = \mu/r$, i.e. $r = a$, i.e. $1 + e\cos f = p/a = 1 - e^2$, so $\cos f = -e$: the ends of the minor axis. Not a coincidence. From Equation 26, γ is stationary exactly where $\cos f = -e$. Both are the single fact $r = a$: inside that radius the body is faster than a circular orbit there, outside it is slower, and at the crossing its velocity leans furthest from the horizontal.

3.6 The eccentric anomaly and Kepler's equation

We return to the timing problem Equation 30. For the ellipse the trick is geometric: describe the position not by the angle f at the focus, but by an angle at the **centre**

of the ellipse.

3.6.1 The auxiliary circle

Draw the **auxiliary circle**, of radius a, centred at the centre O of the ellipse (Figure 9). From the body P, drop a perpendicular to the major axis and extend it to meet the circle at Q. The **eccentric anomaly** E is the angle at the centre from the direction of periapsis to Q.

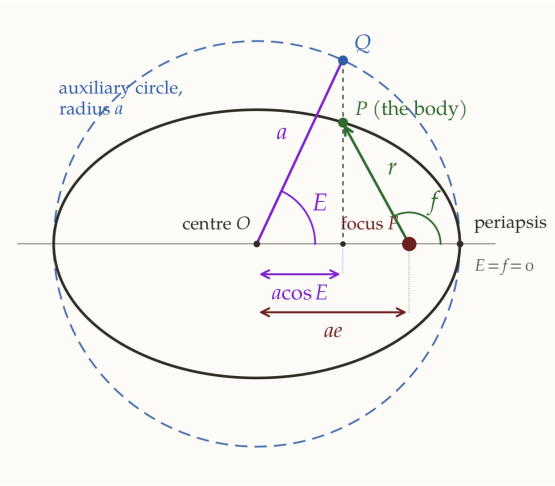

Figure 9: The two anomalies for $e = 0.75$. The true anomaly f is the physical angle of the body, measured at the focus F. The eccentric anomaly E is measured at the centre O, to the point Q on the auxiliary circle directly above (or below) the body P. P and Q share the abscissa $a\cos E$. The two angles agree only at periapsis and apoapsis. *Script: fig-anomalies.py.*

With the origin at O and the x-axis towards periapsis, $Q = (a\cos E,\ a\sin E)$, and by construction $x_P = x_Q = a\cos E$. To find y_P use the equation of the ellipse about its centre:

$$\frac{x_P^2}{a^2} + \frac{y_P^2}{b^2} = 1 \quad \xrightarrow{x_P = a\cos E} \quad y_P^2 = b^2\left(1 - \cos^2 E\right) = b^2 \sin^2 E,$$

so $P = (a\cos E,\ b\sin E)$: the ellipse is the auxiliary circle squashed vertically by b/a. The focus is at $(ae, 0)$ relative to O, so measured **from the focus**,

$$x = a\left(\cos E - e\right), \qquad y = b\sin E = a\sqrt{1-e^2}\,\sin E. \tag{35}$$

Now compute the distance from the focus:

$$\begin{aligned} r^2 = x^2 + y^2 &= a^2\left(\cos E - e\right)^2 + a^2\left(1-e^2\right)\sin^2 E \\ &= a^2\left[\cos^2 E - 2e\cos E + e^2 + \sin^2 E - e^2\sin^2 E\right] \\ &= a^2\left[1 - 2e\cos E + e^2\cos^2 E\right] = a^2\left(1 - e\cos E\right)^2, \end{aligned}$$

$$\boxed{r = a\left(1 - e\cos E\right).} \tag{36}$$

Compare with $r = a(1-e^2)/(1+e\cos f)$: the radius is now **linear** in $\cos E$ instead of reciprocal in $\cos f$. That is the first sign that E is the better variable.

Check Equation 36 at $E = 0$ and $E = 180°$. Why are these the natural points to check?

Answer

$E = 0$ gives $a(1 - e) = r_p$ and $E = 180°$ gives $a(1 + e) = r_a$. They are the natural checks because the apsides are the only points where E and f agree: both are measured from the direction of periapsis, and on the major axis the vertical projection from the ellipse to the circle does nothing.

3.6.2 Taming the time integral

We need $\int r^2\, df$. In polar coordinates about the focus, $x = r\cos f$ and $y = r\sin f$, and a short calculation (try it) shows that

$$x\, dy - y\, dx = r^2\, df.$$

So we can compute $r^2 df$ from Equation 35, with $dx = -a\sin E\, dE$ and $dy = a\sqrt{1-e^2}\cos E\, dE$:

$$\begin{aligned} r^2\, df &= a\,(\cos E - e)\cdot a\sqrt{1-e^2}\cos E\, dE \\ &\quad - a\sqrt{1-e^2}\sin E\cdot(-a\sin E\, dE) \\ &= a^2\sqrt{1-e^2}\left[\cos^2 E - e\cos E + \sin^2 E\right] dE \\ &= a^2\sqrt{1-e^2}\,(1 - e\cos E)\, dE. \end{aligned}$$

The awkward $(1 + e\cos f)^{-2}$ has become a *linear* factor. Using $h = \sqrt{\mu a(1-e^2)}$,

$$dt = \frac{r^2\, df}{h} = \frac{a^2\sqrt{1-e^2}}{\sqrt{\mu a}\sqrt{1-e^2}}\,(1 - e\cos E)\, dE = \sqrt{\frac{a^3}{\mu}}\,(1 - e\cos E)\, dE.$$

Integrating from periapsis ($E = 0$ at $t = t_p$),

$$t - t_p = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)\,.$$

The prefactor is $1/n$, with n the mean motion of Equation 33. Defining the **mean anomaly**

$$\boxed{M \equiv n\left(t - t_p\right) = \frac{2\pi}{T}\left(t - t_p\right),} \tag{37}$$

the result is **Kepler's equation**:

$$\boxed{M = E - e\sin E.} \tag{38}$$

Two quick checks: $e = 0$ gives $M = E = f$ (a circle, swept uniformly); and $E = 2\pi$ gives $M = 2\pi$, so $t - t_p = T$, one full revolution.

What M is. Of the three anomalies, M is the only one that is not an angle you can point at on the figure. It is the angle of a fictitious body that goes round a circle of radius a at the constant rate n, passing periapsis together with the real one. Equivalently, by Kepler's second law, it is the swept area turned into an angle: $M = 2\pi \times$ (area swept since periapsis)$/(\pi ab)$.

anomaly	symbol	measured at	how it changes with time
true	f	the focus	the physical angle; non-uniform
eccentric	E	the centre	a computational intermediary
mean	M	(nowhere)	**linear in** t, by construction

3.6.3 Relating E and f

To use Kepler's equation we must convert between E and f. From Equation 35 and Equation 36, the polar coordinates of the body about the focus give

$$\cos f = \frac{x}{r} = \frac{\cos E - e}{1 - e\cos E}, \qquad \sin f = \frac{y}{r} = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}. \tag{39}$$

A single formula, valid in every quadrant, is better. Use $\tan\frac{f}{2} = \dfrac{\sin f}{1+\cos f}$. The denominator is

$$1 + \cos f = 1 + \frac{\cos E - e}{1 - e\cos E} = \frac{(1-e)(1+\cos E)}{1 - e\cos E}, \tag{40}$$

so

$$\tan\frac{f}{2} = \frac{\sqrt{1-e^2}\sin E}{(1-e)(1+\cos E)} = \sqrt{\frac{1+e}{1-e}}\,\frac{\sin E}{1+\cos E},$$

and since $\sin E/(1+\cos E) = \tan\frac{E}{2}$,

$$\boxed{\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}.} \tag{41}$$

$\frac{f}{2}$ and $\frac{E}{2}$ are always in the same quadrant, so this never confuses quadrants (in a program, use the two-argument arctangent near $E = 180°$, where both tangents blow up).

3.6.4 Solving Kepler's equation

The three anomalies give the standard way to find the position at time t, given a, e and t_p:

$$t \;\longrightarrow\; M = n(t - t_p) \;\longrightarrow\; E \;\longrightarrow\; f \;\longrightarrow\; r,\ \mathbf{v}.$$

- $t \to M$ is trivial; that is the whole point of M.
- $M \to E$ is the hard step. Equation 38 is *transcendental*: it has no solution in elementary functions. The standard tool is Newton's method. With $g(E) = E - e\sin E - M$,

$$E_{k+1} = E_k - \frac{E_k - e\sin E_k - M}{1 - e\cos E_k}, \tag{42}$$

started from $E_0 = M$ (or $E_0 = \pi$ when e is close to 1). The denominator is $1 - e\cos E = r/a > 0$, so the iteration is well defined for every $e < 1$, and near the root it converges quadratically. - $E \to f$ uses Equation 41. - $\to r$: in fact $r = a(1 - e\cos E)$ needs no f at all; f is needed only to orient $\mathbf{r}$ in the plane, and for the velocity.

Solve $M = E - e\sin E$ for $e = 0.5$ and $M = 1$ rad by Newton's method from $E_0 = 1$. Keep nine decimals. How many correct digits do you gain per step? Then find f and r/a.

What you should find

$E_1 = 1.576469353$, $E_2 = 1.500208269$, $E_3 = 1.498701721$, $E_4 = 1.498701134$. The error goes roughly $5\times10^{-1} \to 8\times10^{-2} \to 2\times10^{-3} \to 6\times10^{-7} \to 10^{-13}$: the number of correct digits roughly doubles each step (quadratic convergence). Then $f = 116.36°$ from Equation 41, and $r/a = 1 - 0.5\cos E = 0.964$.

The sixth constant. Rather than t_p, catalogues usually quote the **mean anomaly at epoch**, $M_0 = n(t_0 - t_p)$, at a stated epoch t_0; then $M(t) = M_0 + n(t - t_0)$. It is the same information, given at an epoch close to the observations rather than at a periapsis passage that may be long past. It does not cure the circular orbit: M_0 is still counted from periapsis, so as $e \to 0$ it becomes undefined together with t_p. Near-circular orbits are timed from a fixed direction instead, with the argument of latitude or the mean longitude of the orbital-elements note. This is the timing constant promised in Section 2.6: five constants fix the curve, and M_0 fixes where on it the body is.

3.6.5 How the three anomalies are related

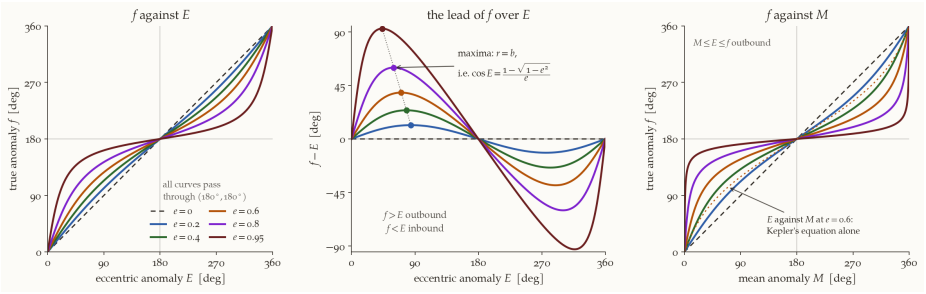

Figure 10: The three anomalies against one another. **Left:** $E \mapsto f$, monotone and pinned at 0°, 180° and 360° for every e; $e = 0$ is the diagonal. **Centre:** the lead $f - E$, odd about $E = 180°$; its maximum grows from about 12° at $e = 0.2$ to about 93° at $e = 0.95$, and always occurs where $r = b$. **Right:** $M \mapsto f$; the dotted curve is the intermediate step $M \mapsto E$ at $e = 0.6$. *Script: fig-anomaly-map.py.*

The left and right panels of Figure 10 look alike, but they answer different questions. $E \leftrightarrow f$ is about a *picture*: freeze the ellipse, and the two angles describe the same point seen from two places, the centre and the focus. No clock is involved. $M \leftrightarrow f$ is about *time*: M is Kepler's second law turned into an angle.

Why the curves bend. Differentiate Equation 41:

$$\tfrac{1}{2}\sec^2\tfrac{f}{2}\,df = \sqrt{\frac{1+e}{1-e}}\;\tfrac{1}{2}\sec^2\tfrac{E}{2}\,dE,$$

so

$$\frac{df}{dE} = \sqrt{\frac{1+e}{1-e}}\,\frac{\cos^2\frac{f}{2}}{\cos^2\frac{E}{2}} = \sqrt{\frac{1+e}{1-e}}\,\frac{1+\cos f}{1+\cos E}.$$

Using Equation 40 for $1 + \cos f$,

History: Kepler's problem. Kepler met this equation in *Astronomia Nova* (1609) and could not solve it. He wrote that he was satisfied it could not be solved *a priori*, because the arc and the sine are quantities of different kinds, and promised that whoever showed him how would be "the great Apollonius" to him. He was right: $E(M)$ is not an elementary function. Expanding E in powers of e (Lagrange's series) converges only for $e < 0.6627$, the *Laplace limit*. Bessel's expansion in the functions now named after him, $E = M + \sum_{k\ge1}\frac{2}{k}J_k(ke)\sin kM$, converges for every $e < 1$; Bessel introduced these functions for this very problem. Hundreds of papers on solving Kepler's equation have appeared since.

$$\frac{df}{dE} = \sqrt{\frac{1+e}{1-e}}\,\frac{1-e}{1-e\cos E} = \frac{\sqrt{1-e^2}}{1-e\cos E} = \frac{b/a}{r/a} \qquad \Longrightarrow \qquad \boxed{\frac{df}{dE} = \frac{b}{r}.}$$

That one formula reads off the whole left panel:

- $r < b$ **(near periapsis):** f **runs ahead of** E**.** The focus is close to periapsis, so a small displacement there subtends a large angle at the focus but a smaller one at the centre.
- $r > b$ **(near apoapsis):** f **falls behind** E**.**
- $r = b$**: they run level**, and the lead $f - E$ is largest (centre panel).
- $E = 0$ **and** $180°$ **are fixed points** for every e: there the centre, the focus and the body are in line.

For M, differentiate Kepler's equation and use the chain rule:

$$\frac{dM}{dE} = 1 - e\cos E = \frac{r}{a}, \qquad \frac{dM}{df} = \frac{dM}{dE}\,\frac{dE}{df} = \frac{r}{a}\cdot\frac{r}{b},$$

$$\boxed{\frac{dM}{df} = \frac{r^2}{ab}.}$$

This is not a new fact: it is Kepler's second law in another form, since $dA = \frac{1}{2}r^2 df$ and $M = 2A/(ab)$. Each pair of anomalies runs level at its own radius:

relation	derivative	runs level where
M against E	$dM/dE = r/a$	$r = a$
f against E	$df/dE = b/r$	$r = b$
f against M	$df/dM = ab/r^2$	$r = \sqrt{ab}$

and since $b \le \sqrt{ab} \le a$, the three crossings occur in that order as the body climbs from periapsis. On the outbound half, $M \le E \le f$: the real body runs *ahead* of its fictitious uniformly moving twin after periapsis, because that is where it is fastest.

For nearly circular orbits, show that

$$\begin{aligned} f &= E + e\sin E + O(e^2), \\ E &= M + e\sin M + O(e^2), \\ \text{hence}\quad f &= M + 2e\sin M + O(e^2). \end{aligned}$$

How far ahead of its mean position does the Earth ($e = 0.0167$) get?

Answer

From $df/dE = \sqrt{1-e^2}/(1-e\cos E) = 1 + e\cos E + O(e^2)$, integrating from 0 gives $f = E + e\sin E + O(e^2)$. From Kepler's equation $E = M + e\sin E$, and replacing E by M inside the sine costs only $O(e^2)$, so $E = M + e\sin M + O(e^2)$. Adding the two leads, $f = M + 2e\sin M + O(e^2)$.

For the Earth, the largest lead is $2e = 0.0334$ rad $= 1.9°$. The Sun's apparent position in the sky runs up to about $1.9°$ ahead of or behind its average position. Since the Sun moves about $1°$ per day, that is almost two days, and it is a major part of the *equation of time* (the difference between sundial time and clock time).

3.6.6 Worked example: propagating an eccentric geosynchronous orbit

Not every geosynchronous orbit is geostationary. Take an eccentric, *Tundra*-type orbit: the period is one sidereal day, but the orbit is far from circular. With $\mu = 398\,600.4\ \mathrm{km}^3/\mathrm{s}^2$,

$$a = 42\,164\ \mathrm{km}, \quad e = 0.25,$$

so that $r_p = 31\,623$ km, $r_a = 52\,705$ km, $b = 40\,825$ km, and

$$n = \sqrt{\frac{\mu}{a^3}} = 7.29216 \times 10^{-5}\ \mathrm{rad/s}, \qquad T = \frac{2\pi}{n} = 86\,163.6\ \mathrm{s} = 23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}}.$$

At t_0 the spacecraft is at $f_0 = 30°$. Where is it six hours later?

Before reading the solution, plan it: which way round the pipeline must you go first?

Step 0: run the pipeline backwards to find the clock. We are given f_0, not M_0, so first $f \to E \to M$:

$$\tan\frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{f_0}{2} = \sqrt{\frac{0.75}{1.25}}\,\tan 15°,$$

so $E_0 = 23.4508° = 0.409294$ rad, and

$$M_0 = E_0 - e\sin E_0 = 0.409294 - 0.25\sin(0.409294) = 0.309804\ \mathrm{rad},$$

i.e. $M_0 = 17.7504°$.

So $t_0 - t_p = M_0/n = 4248.5$ s $= 1.180$ h: the spacecraft passed perigee 1.18 hours earlier. Its radius is $r_0 = a(1 - e\cos E_0) = 32\,494$ km.

Step 1: $t \to M$. Six hours is 21 600 s:

$$M_1 = M_0 + n\,\Delta t = 0.309804 + 1.575107 = 1.884910\ \mathrm{rad} = 107.997°.$$

Step 2: $M \to E$. Newton's method on $g(E) = E - e\sin E - M_1$, from $E = M_1$:

k	E_k (rad)	$g(E_k)$	ΔE_k
0	1.884910	-2.378×10^{-1}	$+2.207 \times 10^{-1}$
1	2.105629	$+5.630 \times 10^{-3}$	-4.994×10^{-3}
2	2.100635	$+2.685 \times 10^{-6}$	-2.383×10^{-6}
3	2.100633	$+6.1 \times 10^{-13}$	—

$E_1 = 2.100633$ rad $= 120.3574°$. The residual goes $10^{-1} \to 10^{-3} \to 10^{-6} \to 10^{-13}$, the signature of quadratic convergence. The derivative $1 - e\cos E = r/a$ was about 1.13, far from zero: a well-conditioned problem.

Step 3: $E \to f$.

$$\tan\frac{f_1}{2} = \sqrt{\frac{1.25}{0.75}}\,\tan(60.179°) \quad \Longrightarrow \quad f_1 = 132.118°.$$

Step 4: the state. Two independent routes to r must agree:

$$r_1 = a\,(1 - e\cos E_1) = 47\,491 \text{ km},$$
$$r_1 = \frac{p}{1 + e\cos f_1} = \frac{39\,529 \text{ km}}{1 + 0.25\cos 132.118^\circ} = 47\,491 \text{ km}. \checkmark$$

Then $v_1 = \sqrt{\mu(2/r_1 - 1/a)} = 2.708$ km/s and, from Equation 25, $\gamma_1 = 12.56^\circ$: still climbing towards apogee.

Read the three anomalies at the final instant, $M_1 = 108.00^\circ$, $E_1 = 120.36^\circ$ and $f_1 = 132.12^\circ$. Is their order what you expect? Explain the 24° difference between f_1 and M_1 in one sentence.

Answer

Yes: on the outbound half, $M \le E \le f$. The spacecraft is 24° ahead of its fictitious uniformly moving twin because it has just spent seven hours near perigee, where it moves fastest.

4 The parabola ($e = 1$)

4.1 The one-parameter conic

Set $e = 1$ in the orbit equation and use $1 + \cos f = 2\cos^2\frac{f}{2}$:

$$r = \frac{p}{1 + \cos f} = \frac{p}{2}\sec^2\frac{f}{2}, \qquad r_p = \frac{p}{2} \quad (f = 0).$$

There is **no second parameter**. An ellipse needs two numbers for its size and shape, (a, e); a parabola is fixed by one, p, or equivalently r_p or $h = \sqrt{\mu p}$. Its eccentricity is not a free number but the defining condition.

The formulas of Section 2 specialise by putting $e = 1$:

$$\varepsilon = \frac{\mu\,(e^2 - 1)}{2p} = 0, \qquad a = \frac{p}{1 - e^2} \to \infty.$$

So **never use a for a parabola**. Vis-viva becomes

$$\boxed{v^2 = \frac{2\mu}{r},}$$

the escape speed of Equation 16: a body on a parabola moves at exactly the local escape speed at every point. It reaches infinity, and arrives there with nothing left.

4.2 Speed and direction on a parabola

Put $e = 1$ in the speed Equation 24 and the flight path angle Equation 25, and simplify with half-angle formulas. What is γ as a function of f?

What you should find

$$v = \sqrt{\frac{\mu}{p}}\sqrt{2 + 2\cos f} = 2\sqrt{\frac{\mu}{p}}\left|\cos\frac{f}{2}\right|,$$

$$\tan\gamma = \frac{\sin f}{1 + \cos f} = \frac{2\sin\frac{f}{2}\cos\frac{f}{2}}{2\cos^2\frac{f}{2}} = \tan\frac{f}{2},$$

so $\boxed{\gamma = f/2}$. The speed agrees with $\sqrt{2\mu/r}$, as it must.

The result $\gamma = f/2$ says that the direction of motion turns at exactly **half** the rate of the position vector. As f runs from -180° to 180°, γ runs from -90° to 90°: the velocity swings through half a turn, from radially inbound at infinity to radially outbound at infinity. The hodograph is still the circle of radius μ/h, but now the origin of velocity space lies **on** it: $v = 0$ is reached, once, at infinity.

Why are there no parabolic orbits in nature?

Answer

$\varepsilon = 0$ is a single value in a continuum: initial conditions with *exactly* zero energy are infinitely improbable, and any disturbance (a third body, outgassing, radiation pressure) moves ε off zero. Long-period comets arrive on orbits with $e = 0.999\ldots$, ellipses with periods of hundreds of thousands of years or more. Astronomers often fit them as parabolas because the short arc observed near perihelion cannot distinguish $e = 1$ from $e = 1 \pm 10^{-3}$.

The parabola matters because it is the **boundary** between capture and escape. "Escape" means crossing it.

4.3 Time on a parabola: Barker's equation

The time integral Equation 30 with $e = 1$ is

$$t - t_p = \frac{p^2}{h}\int_0^f \frac{df'}{(1+\cos f')^2} = \frac{p^2}{4h}\int_0^f \sec^4\frac{f'}{2}\,df'.$$

Here the right substitution is suggested by the integrand itself: $\sec^4$ is $\sec^2 \times \sec^2$, one $\sec^2$ is the derivative of tan, and the other is $1 + \tan^2$. So put

$$D = \tan\frac{f}{2}, \qquad dD = \tfrac{1}{2}\sec^2\frac{f}{2}\,df, \qquad \sec^2\frac{f}{2} = 1 + D^2.$$

D is the **parabolic anomaly**. Like E, it is not an angle you can point at, but it turns the integrand into something simple:

$$\frac{p^2}{4h}\sec^4\frac{f}{2}\,df = \frac{p^2}{4h}\left(1+D^2\right)\cdot 2\,dD = \frac{p^2}{2h}\left(1+D^2\right)dD.$$

The integral is now a *polynomial*. With $h = \sqrt{\mu p}$ and $p = 2r_p$,

$$t - t_p = \frac{p^2}{2h}\left(D + \frac{D^3}{3}\right) = \sqrt{\frac{2r_p^3}{\mu}}\left(D + \frac{D^3}{3}\right).$$

Defining the parabolic mean anomaly $M_p \equiv \sqrt{\mu/(2r_p^3)}\,(t - t_p)$,

$$\boxed{M_p = D + \frac{D^3}{3}, \qquad D = \tan\frac{f}{2}.} \tag{43}$$

This is **Barker's equation**. (Curtis defines the parabolic mean anomaly as $\mu^2(t - t_p)/h^3$, which is exactly half of ours, since $h^2 = 2\mu r_p$; his Barker equation reads $\frac{1}{2}D + \frac{1}{6}D^3$. The two differ only in scale.)

History: Barker's equation. It is named after Thomas Barker, who in 1757 published tables for the motion of comets on parabolic orbits based on it. Comets were the main application: in the eighteenth and nineteenth centuries most newly discovered comets were fitted with parabolic orbits first.

Unlike Kepler's equation, Equation 43 is a **cubic** in D, so it can be inverted in closed form. Since $dM_p/dD = 1 + D^2 > 0$, M_p increases strictly with D and there is exactly one real root. Writing $D^3 + 3D - 3M_p = 0$, Cardano's formula gives

$$D = \sqrt[3]{\frac{3M_p}{2} + \sqrt{\frac{9M_p^2}{4} + 1}} + \sqrt[3]{\frac{3M_p}{2} - \sqrt{\frac{9M_p^2}{4} + 1}}, \tag{44}$$

and then $f = 2\arctan D$.

Apart from the trivial circle, the parabola is the only conic whose timing problem has an elementary solution.

A comet moves on a parabola with perihelion at $r_p = 0.5$ AU ($\mu_\odot = 1.327 \times 10^{11}$ km^3/s^2, 1 AU $= 1.496 \times 10^8$ km).

(a) How long does it take to go from perihelion out to the Earth's distance, 1 AU? (Find f from the orbit equation first.) (b) Where is it 20 days after perihelion? Use Equation 44. (c) How fast is it moving when it crosses the Earth's orbit?

What you should find

(a) $p = 2r_p = 1$ AU, so $r = 1$ AU needs $1 + \cos f = 1$, $f = 90°$, $D = 1$ and $M_p = 4/3$. With $\sqrt{2r_p^3/\mu} = 2.512 \times 10^6$ s, $t - t_p = 3.35 \times 10^6$ s $= 38.8$ days.

(b) $M_p = 20 \times 86\,400/2.512 \times 10^6 = 0.688$, and Equation 44 gives $D = 0.612$, so $f = 62.9°$ and $r = p/(1 + \cos f) = 0.687$ AU.

(c) $\sqrt{2\mu_\odot/1\text{ AU}} = 42.1$ km/s, the local escape speed from the Sun. Compare the Earth's 29.8 km/s.

5 The hyperbola ($e > 1$)

Hyperbolic orbits are the orbits of spacecraft leaving a planet for another, and of spacecraft flying past a planet. Two questions dominate: how fast is the spacecraft when it is far away, and how much does the planet bend its path?

5.1 Geometry

With $e > 1$, Equation 10 gives a **negative** a. Writing $|a| = p/(e^2 - 1)$, every formula reads naturally:

$$r = \frac{|a|\left(e^2 - 1\right)}{1 + e\cos f}, \qquad r_p = \frac{p}{1+e} = |a|(e-1),$$

$$\varepsilon = -\frac{\mu}{2a} = +\frac{\mu}{2|a|} > 0.$$

The body exists only on $|f| < f_\infty$, with $\cos f_\infty = -1/e$ (Equation 6). Geometrically (Figure 11):

- $|a|$ is the distance from the **centre** of the hyperbola to its vertex (periapsis). The focus is a further $|a|e$ from the centre, so $r_p = |a|e - |a| = |a|(e-1)$.
- The asymptotes make the angle $\arccos(1/e)$ with the line of apsides, measured at the centre.
- The *other* branch of the hyperbola is the path a body would follow under a *repulsive* inverse-square force with the same h and ε. It is not physical here.

Vis-viva is best written with the sign of a made explicit:

$$\boxed{v^2 = \mu\left(\frac{2}{r} + \frac{1}{|a|}\right),} \tag{45}$$

larger than the local escape speed $\sqrt{2\mu/r}$ everywhere, by a margin that does not vanish as $r \to \infty$.

5.2 Speed at infinity

Letting $r \to \infty$ in Equation 45,

$$\boxed{v_\infty = \sqrt{\frac{\mu}{|a|}} = \sqrt{2\varepsilon}, \qquad C_3 = v_\infty^2 = \frac{\mu}{|a|} = -\frac{\mu}{a}.}$$

For a departing interplanetary spacecraft, v_∞ is the only thing that survives to interplanetary distances: it is the spacecraft's velocity relative to the planet once it has left the planet's neighbourhood.

5.3 Turning angle

The body arrives along the asymptote parallel to the direction $f = -f_\infty$ and leaves along the one parallel to $f = +f_\infty$. The direction of motion therefore turns through the **turning angle**

$$\delta = 2f_\infty - \pi.$$

(At infinity the velocity is along the asymptote, i.e. radial; its direction changes from $-f_\infty + \pi$ on arrival to $+f_\infty$ on departure.) Then $\sin\frac{\delta}{2} = \sin\left(f_\infty - \frac{\pi}{2}\right) = -\cos f_\infty$, and with Equation 6,

$$\boxed{\sin\frac{\delta}{2} = \frac{1}{e}.} \tag{46}$$

This is the angle you found from the hodograph in Section 2.13. Combining $r_p = |a|(e-1)$ with $|a| = \mu/v_\infty^2$ expresses e through the two quantities a mission designer controls:

$$\boxed{e = 1 + \frac{r_p v_\infty^2}{\mu}.} \tag{47}$$

So a flyby is steered by how *deep* the spacecraft goes and how *fast* it arrives. Slow and deep turns hard ($e \to 1$, $\delta \to 180^\circ$); fast and shallow barely bends ($e \to \infty$, $\delta \to 0$).

A spacecraft passes Jupiter ($\mu = 1.267 \times 10^8$ km^3/s^2) with $v_\infty = 6$ km/s and periapsis radius $r_p = 10^6$ km. Through what angle is it turned? Does its speed relative to Jupiter change?

Answer

$e = 1 + 10^6 \times 36/(1.267 \times 10^8) = 1.284$ and $\delta = 2\arcsin(1/1.284) = 102.3^\circ$.

Relative to Jupiter the *speed* does not change: v_∞ in equals v_∞ out, by energy conservation. Only the direction changes, by more than a right angle. But relative

to the Sun, that rotated $\mathbf{v}_\infty$ is added to Jupiter's own orbital velocity of 13.1 km/s, so the spacecraft's speed about the Sun can change by up to $2v_\infty \sin(\delta/2)$ (here 9.3 km/s). That is a gravity assist; the momentum comes from Jupiter, whose orbit changes by an immeasurably small amount.

5.4 The aiming radius, and what b means for a hyperbola

The perpendicular distance from the focus to the incoming asymptote is the **aiming radius** (or impact parameter) Δ. It is the distance by which the spacecraft would miss the planet if gravity were switched off. At infinity the lever arm of the velocity about the focus *is* Δ, so

$$h = \Delta\, v_\infty .$$

For the ellipse, b was a length we could measure: the semi-minor axis, from the centre to the curve. A hyperbola never crosses the perpendicular to the line of apsides through its centre (it needs $|x| \geq |a|$ there), so for $e > 1$ the symbol b must be **defined**. Eliminating f from the orbit equation and moving the origin to the centre gives

$$\frac{x^2}{|a|^2} - \frac{y^2}{b^2} = 1, \qquad b^2 \equiv c^2 - |a|^2 = |a|^2\left(e^2 - 1\right),$$

where $c = |a|e$ is the centre-to-focus distance (for the ellipse, $b^2 = a^2 - c^2$; the two terms have swapped). Three equivalent ways to see this number:

- **Asymptote slope.** The asymptotes are $y = \pm(b/|a|)x$, the diagonals of the rectangle $|x| \leq |a|$, $|y| \leq b$.
- **The same p relation as the ellipse:** $b^2 = |a|p$.
- **The distance from the focus to an asymptote** is exactly b. So the aiming radius is $\Delta = b = |a|\sqrt{e^2 - 1}$.

> **Common mistake: asymptotes through the focus**
>
> The asymptotes point in the directions $f = \pm f_\infty$, but they do not start at the focus. They cross at the centre of the hyperbola, a distance $|a|e$ from the focus along the apse line, beyond periapsis, and each passes the focus at the distance $\Delta = b$. If the asymptotes went through the planet, the aiming radius would be zero and $h = \Delta v_\infty$ would vanish: every hyperbola would be a straight fall onto the planet.

Check that $h = \Delta v_\infty$ agrees with $h^2 = \mu p$, using $\Delta = |a|\sqrt{e^2 - 1}$, $v_\infty^2 = \mu/|a|$ and $p = |a|(e^2 - 1)$.

What you should find

$\Delta^2 v_\infty^2 = |a|^2(e^2 - 1) \cdot \mu/|a| = \mu|a|(e^2 - 1) = \mu p = h^2$. ✓

Note that b is larger than $|a|$ when $e > \sqrt{2}$, so "semi-minor axis" is a misnomer for a hyperbola. Figure 11 is drawn at $e = 1.6$, where $b > |a|$.

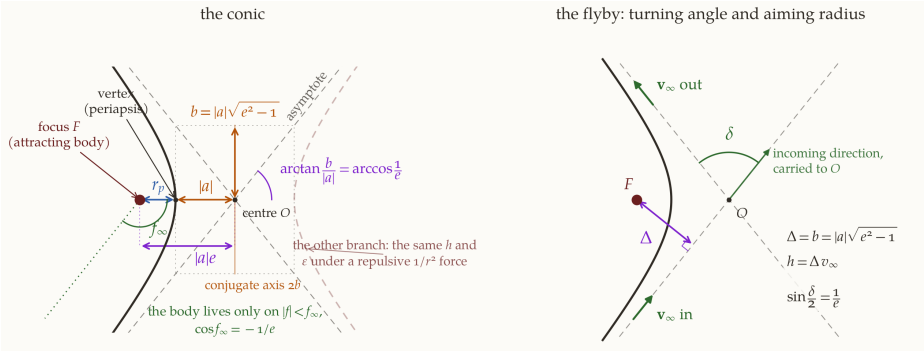

Figure 11: **Left: the conic.** The attracting body is at the focus F, at distance $|a|e$ from the centre O; the vertex is $|a|$ from the centre, so $r_p = |a|(e-1)$. The dotted rectangle (half-sides $|a|$ and $b = |a|\sqrt{e^2-1}$) has the asymptotes as diagonals, each at $\arccos(1/e)$ to the line of apsides. The faint mirror branch belongs to a repulsive force. **Right: the same orbit as a flyby.** The body arrives along the lower asymptote with $\mathbf{v}_\infty$ and leaves along the upper one at the same speed; the velocity has turned through $\delta = 2f_\infty - \pi$, with $\sin(\delta/2) = 1/e$. The aiming radius Δ is the offset of the incoming asymptote from the focus, and equals b. Drawn for $e = 1.6$. *Script: fig-hyperbola-geometry.py.*

5.5 The hyperbolic anomaly H

For timing we copy the elliptic trick. There, the auxiliary curve was a circle and $x = a(\cos E - e)$. For the hyperbola the auxiliary curve is a *rectangular hyperbola*, and the circular functions become **hyperbolic functions**, $\cosh H = \frac{1}{2}(\exp H + \exp(-H))$ and $\sinh H = \frac{1}{2}(\exp H - \exp(-H))$ (written with exp so that e stays the eccentricity), which satisfy $\cosh^2 H - \sinh^2 H = 1$ (Figure 12). Measured from the focus,

$$x = |a|\,(e - \cosh H)\,, \qquad y = |a|\sqrt{e^2-1}\,\sinh H. \tag{48}$$

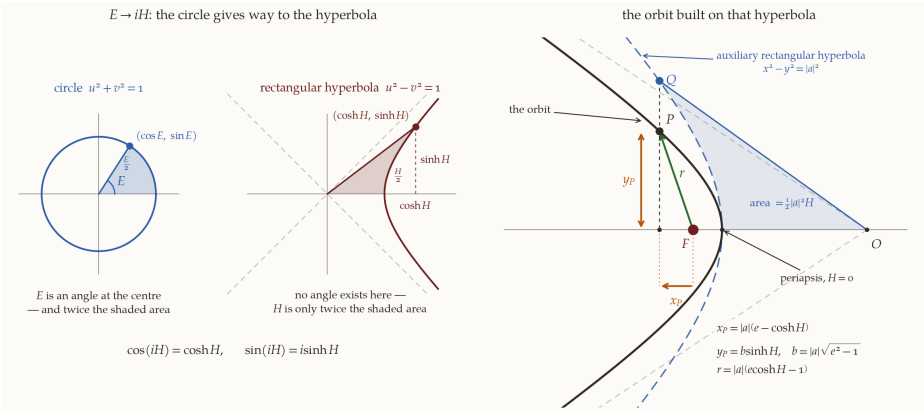

Figure 12: **Left:** why hyperbolic functions appear. On the unit circle the parameter E is twice the shaded sector area, and also an angle. On the rectangular hyperbola $u^2 - v^2 = 1$ the point is $(\cosh H, \sinh H)$, and H is twice the shaded area but no longer an angle. **Right:** the construction for an orbit with $e = 1.2$. The auxiliary curve is the rectangular hyperbola $x^2 - y^2 = |a|^2$ centred at O; the orbit is that curve stretched by $b/|a| = \sqrt{e^2-1}$ across the axis, so the body P and the auxiliary point Q share an abscissa. *Script: fig-hyperbolic-anomaly.py.*

Repeat the calculation that led to Equation 36, with Equation 48, to show that $r = |a|(e\cosh H - 1)$. You need only $\cosh^2 H - \sinh^2 H = 1$.

What you should find

$$\begin{aligned} r^2 &= a^2\,(e - \cosh H)^2 + a^2\,(e^2 - 1)\sinh^2 H \\ &= a^2\left[e^2 - 2e\cosh H + \cosh^2 H + (e^2 - 1)\left(\cosh^2 H - 1\right)\right] \\ &= a^2\left[e^2\cosh^2 H - 2e\cosh H + 1\right] = a^2\,(e\cosh H - 1)^2\,. \end{aligned}$$

So

$$\boxed{r = |a|\,(e\cosh H - 1)\,,} \tag{49}$$

the elliptic $r = a(1 - e\cos E)$ with cos → cosh and the sign flipped, and again **linear** in the new variable. H runs over $(-\infty, \infty)$, with $H = 0$ at periapsis and $H \to \pm\infty$ at the asymptotes. The same half-angle manipulation as for the ellipse gives

$$\boxed{\tanh\frac{H}{2} = \sqrt{\frac{e-1}{e+1}}\,\tan\frac{f}{2}.} \tag{50}$$

tanh never exceeds 1, and the right-hand side reaches 1 exactly when $\tan\frac{f_\infty}{2} = \sqrt{(e+1)/(e-1)}$, the asymptote: the whole infinite range of H is squeezed into the finite window $|f| < f_\infty$.

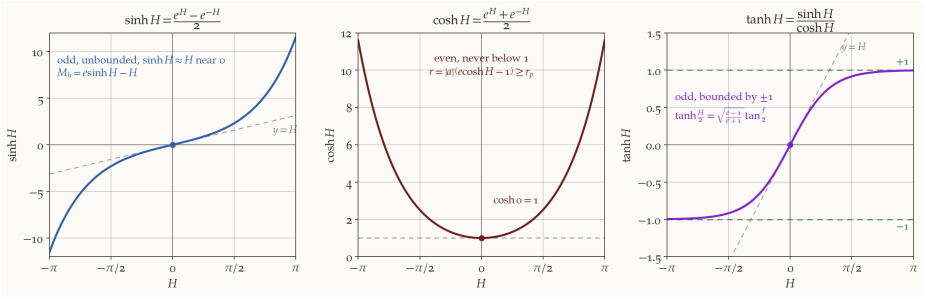

Figure 13: The three hyperbolic functions. **Left:** $\sinh H$ is odd and unbounded, close to H near the origin; it carries the time, through $M_h = e\sinh H - H$. **Centre:** $\cosh H$ is even with minimum 1 at $H = 0$, which is why $r = |a|(e\cosh H - 1)$ is smallest at periapsis. **Right:** $\tanh H$ is odd and confined between ± 1; that ceiling maps the infinite H-line onto $|f| < f_\infty$. *Script: fig-hyperbolic-functions.py.*

5.6 The hyperbolic Kepler equation

From Equation 48, with $dx = -|a|\sinh H\,dH$ and $dy = |a|\sqrt{e^2-1}\cosh H\,dH$,

$$\begin{aligned} r^2\,df = x\,dy - y\,dx &= |a|^2\sqrt{e^2-1}\left[(e - \cosh H)\cosh H + \sinh^2 H\right]dH \\ &= |a|^2\sqrt{e^2-1}\,(e\cosh H - 1)\,dH. \end{aligned}$$

With $h = \sqrt{\mu|a|(e^2-1)}$,

$$dt = \frac{r^2\,df}{h} = \sqrt{\frac{|a|^3}{\mu}}\,(e\cosh H - 1)\,dH.$$

Integrating from periapsis ($H = 0$ at $t = t_p$) and defining

$$n_h = \sqrt{\frac{\mu}{|a|^3}}, \qquad M_h = n_h\left(t - t_p\right),$$

we get the **hyperbolic Kepler equation**

$$\boxed{M_h = e\sinh H - H.} \tag{51}$$

Side by side with its elliptic twin:

	ellipse	hyperbola
radius	$r = a(1 - e\cos E)$	$r = \lvert a\rvert(e\cosh H - 1)$
Kepler equation	$M = E - e\sin E$	$M_h = e\sinh H - H$
derivative	$dM/dE = 1 - e\cos E = r/a$	$dM_h/dH = e\cosh H - 1 = r/\lvert a\rvert$
range	$E \in [0, 2\pi)$, periodic	$H \in (-\infty, \infty)$, never repeats

Both derivatives equal $r/|a| > 0$, so both equations have exactly one root, and Newton's method,

$$H_{k+1} = H_k - \frac{e\sinh H_k - H_k - M_h}{e\cosh H_k - 1},$$

is well defined. The practical difference is the starting guess: sinh grows exponentially, so $H_0 = M_h$ is poor for large M_h; a common choice is $H_0 = \ln(2M_h/e + 1.8)$. There is no period and no repetition: everything about a hyperbolic pass happens once.

A spacecraft leaves a 300 km parking orbit ($r_p = 6678$ km) on a hyperbola with $v_\infty = 3$ km/s. How long does it take to reach 925 000 km, roughly the edge of the Earth's sphere of influence? Plan: find $|a|$ and e; get H from Equation 49; get M_h from Equation 51; divide by n_h. (Here no iteration is needed: we know r and want t.)

What you should find

$|a| = \mu/v_\infty^2 = 44\,289$ km and $e = 1 + r_p v_\infty^2/\mu = 1.1508$. At $r = 925\,000$ km, $\cosh H = (r/|a| + 1)/e = 19.02$, so $H = 3.638$ and $M_h = 1.1508\sinh(3.638) - 3.638 = 18.22$. With $n_h = \sqrt{\mu/|a|^3} = 6.77 \times 10^{-5}$ s^{-1}, $t - t_p = 2.69 \times 10^5$ s $= 3.11$ days.

For comparison, the periapsis speed is $\sqrt{v_\infty^2 + 2\mu/r_p} = 11.33$ km/s, and the turning angle of this hyperbola is $2\arcsin(1/1.1508) = 120.7°$.

6 The conics side by side

	circle	ellipse	parabola	hyperbola
e	0	$0 < e < 1$	1	> 1
ε	< 0	< 0	0	> 0
$a = p/(1-e^2)$	$a = r$	$a > 0$	$\to \infty$	$a < 0$
range of f	all	all	$\lvert f\rvert < 180°$	$\lvert f\rvert <$ $\arccos(-1/e)$
r_p	a	$a(1-e)$	$p/2$	$\lvert a\rvert(e-1)$
r_a	a	$a(1+e)$	—	—
period	$2\pi\sqrt{a^3/\mu}$	$2\pi\sqrt{a^3/\mu}$	—	—
speed at ∞	—	—	0	$\sqrt{\mu/\lvert a\rvert}$
largest $\lvert\gamma\rvert$	0	$\arcsin e$	$90°$ (limit)	$90°$ (limit)
auxiliary variable	f itself	E	$D = \tan\frac{f}{2}$	H
Kepler equation	$M = f$	$M = E - e\sin E$	$M_p = D + \frac{D^3}{3}$	$M_h = e\sinh H - H$
closed-form inverse	trivial	no	**yes** (cubic)	no
hodograph origin	centre	inside	on the circle	outside

The first nine rows and the last are specialisations of formulas from Section 2. The three rows about time (the auxiliary variable, the Kepler equation and its inverse) are genuinely separate work for each conic.

Common mistakes

- **Asymptotes through the focus.** They are parallel to $f = \pm f_\infty$ but cross at the centre, and pass the focus at the aiming radius $\Delta = b$ (Section 5).
- **Using a on a parabola.** It is infinite there. Use p or $r_p = p/2$.
- **Using $r_a = a(1+e)$ on a hyperbola.** With $a < 0$ and $e > 1$ it returns a negative number: a hyperbola has no apoapsis.
- **Trusting** arccos **for the true anomaly.** $\cos f = \mathbf{e}\cdot\mathbf{r}/(er)$ gives two angles; the sign of $\mathbf{r}\cdot\mathbf{v}$ (climbing or descending) picks one.
- **Confusing $\dot r$ with $\lvert\dot{\mathbf{r}}\rvert$.** The first is the radial speed v_r, the second the whole speed v.
- **Thinking $\mathbf{e}\perp\mathbf{v}$.** $\mathbf{e}$ is perpendicular to $\mathbf{h}$, but $\mathbf{e}\cdot\mathbf{v} = -\dot r$ is not zero.
- **Degrees in Kepler's equation.** In $M = E - e\sin E$ the angles must be in radians: $e\sin E$ is a pure number.

Three formulas are worth committing to memory because they never need a case distinction:

$$r = \frac{p}{1+e\cos f}, \qquad v = \sqrt{\frac{\mu}{p}}\sqrt{1+2e\cos f+e^2}, \qquad \tan\gamma = \frac{e\sin f}{1+e\cos f}.$$

Given (p, e, f) they return the position, speed and direction on any conic whatsoever.

Universal variables: one Kepler equation for all conics

The unification can be pushed through the time equation as well. Introduce a new variable χ by $d\chi = \sqrt{\mu}\,dt/r$, and the *Stumpff functions*

$$C(z) = \frac{1-\cos\sqrt{z}}{z}, \qquad S(z) = \frac{\sqrt{z}-\sin\sqrt{z}}{z^{3/2}}, \qquad z = \frac{\chi^2}{a},$$

which are smooth for all real z (for $z < 0$ the trigonometric functions become hyperbolic). Then the three Kepler equations become one,

$$\sqrt{\mu}\left(t - t_p\right) = r_p\chi + \left(1 - \frac{r_p}{a}\right)\chi^3 S(z),$$

with $z > 0$ on an ellipse, $z = 0$ on a parabola and $z < 0$ on a hyperbola, and nothing singular at the parabolic boundary. (For an ellipse $\chi = \sqrt{a}\,E$.) Production propagators and Lambert solvers are written this way; see the universal-variables note.

7 Appendix: the rotating basis

The velocity derivation of Section 2.11 rested on two formulas,

$$\dot{\hat{\mathbf{r}}} = \dot{f}\,\hat{\theta}, \qquad \dot{\hat{\theta}} = -\dot{f}\,\hat{\mathbf{r}}.$$

We prove them twice, once with components and once without.

Setup. Let $\hat{\mathbf{p}}$ point from the focus towards periapsis and $\hat{\mathbf{q}} = \hat{\mathbf{h}} \times \hat{\mathbf{p}}$ complete a right-handed pair in the orbital plane (the *perifocal* basis). Both are built from the constants $\mathbf{e}$ and $\mathbf{h}$, so both are **fixed in space**. Since f is measured from periapsis,

$$\hat{\mathbf{r}} = \cos f\,\hat{\mathbf{p}} + \sin f\,\hat{\mathbf{q}}, \qquad \hat{\theta} = -\sin f\,\hat{\mathbf{p}} + \cos f\,\hat{\mathbf{q}}.$$

Proof 1: differentiate the components. Only f depends on time:

$$\dot{\hat{\mathbf{r}}} = \dot{f}\,(-\sin f\,\hat{\mathbf{p}} + \cos f\,\hat{\mathbf{q}}) = \dot{f}\,\hat{\theta}, \qquad \dot{\hat{\theta}} = \dot{f}\,(-\cos f\,\hat{\mathbf{p}} - \sin f\,\hat{\mathbf{q}}) = -\dot{f}\,\hat{\mathbf{r}}.$$

That is the whole proof, but it hides *why* it is true.

Proof 2: the unit circle. Two facts suffice.

1. **The derivative of a unit vector is perpendicular to it.** Differentiating $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1$ gives $2\hat{\mathbf{u}} \cdot \dot{\hat{\mathbf{u}}} = 0$. A unit vector can only turn; it cannot stretch.
2. **It turns at the rate of the angle.** In a short time the tip of $\hat{\mathbf{r}}$ moves along the unit circle by a chord of length $|\Delta\hat{\mathbf{r}}| = 2\sin(\Delta f/2) \to \Delta f$, and the chord's direction tends to the tangent, which is $\hat{\theta}$.

Dividing by Δt gives $\dot{\hat{\mathbf{r}}} = \dot{f}\,\hat{\theta}$. The same argument for $\hat{\theta}$, whose tip moves towards $-\hat{\mathbf{r}}$, gives the second formula.

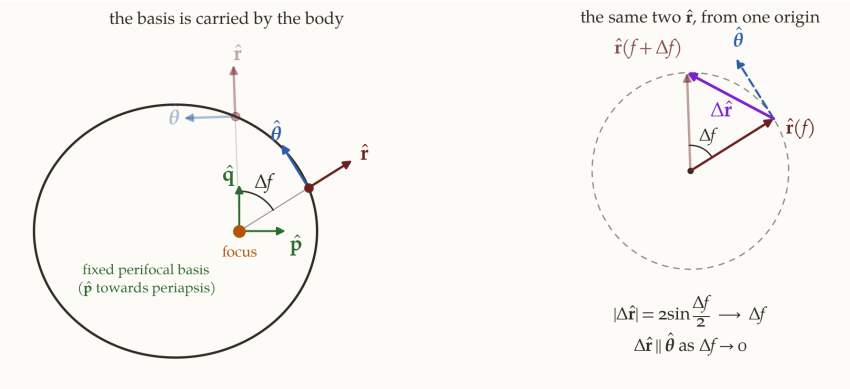

Figure 14: **Left:** the pair $(\hat{\mathbf{r}}, \hat{\theta})$ carried by the body at two nearby true anomalies, with the fixed perifocal pair $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$. **Right:** the two $\hat{\mathbf{r}}$'s from a common origin. Their tips lie on the unit circle, so $\Delta\hat{\mathbf{r}}$ is a chord of length $2\sin(\Delta f/2) \to \Delta f$, in a direction that tends to $\hat{\theta}$. *Script: fig-rotating-basis.py.*

Both formulas as one statement. The pair turns about $\hat{\mathbf{h}}$ with angular velocity $\omega = \dot{f}\,\hat{\mathbf{h}}$, and any vector fixed in a rotating frame changes at the rate $\omega \times (\cdot)$. With the right-handed triad $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\mathbf{h}}$,

$$\omega \times \hat{\mathbf{r}} = \dot{f}\,\hat{\mathbf{h}} \times \hat{\mathbf{r}} = \dot{f}\,\hat{\theta}, \qquad \omega \times \hat{\theta} = \dot{f}\,\hat{\mathbf{h}} \times \hat{\theta} = -\dot{f}\,\hat{\mathbf{r}}.$$

This is the same rule that produces the Coriolis and centrifugal terms in a rotating frame; here the frame's spin rate $\dot{f} = h/r^2$ is handed to us by a conservation law.

7.0.1 The payoff: the equation of motion in polar form

Differentiate $\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{f}\,\hat{\theta}$ once more, using the two formulas at every step:

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{f}^2\right)\hat{\mathbf{r}} + \left(r\ddot{f} + 2\dot{r}\dot{f}\right)\hat{\theta}.$$

Equating to $-\mu\hat{\mathbf{r}}/r^2$ splits Equation 1 into two scalar equations:

$$\ddot{r} - r\dot{f}^2 = -\frac{\mu}{r^2}, \qquad r\ddot{f} + 2\dot{r}\dot{f} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{f}\right) = 0.$$

The transverse equation is the conservation of angular momentum, $r^2\dot{f} = h$, found again. Substituting it into the radial equation leaves one equation for $r(t)$:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2}. \tag{52}$$

Derive the orbit equation a second way, from Equation 52. Put $u = 1/r$ and use f instead of t as the independent variable ($\frac{d}{dt} = \frac{h}{r^2}\frac{d}{df} = hu^2\frac{d}{df}$). Show that $\dot{r} = -h\,du/df$ and $\ddot{r} = -h^2u^2\,d^2u/df^2$, and hence that Equation 52 becomes

$$\frac{d^2u}{df^2} + u = \frac{\mu}{h^2}.$$

Solve it. (This is the *Binet* route; it works for any central force, with μu^2 replaced by the force law.)

What you should find

$\dot r = \frac{d}{dt}(1/u) = -\frac{1}{u^2} h u^2 \frac{du}{df} = -h\frac{du}{df}$, and differentiating again, $\ddot r = -h\,hu^2\frac{d^2u}{df^2}$. Substituting into Equation 52 and dividing by $-h^2u^2$ gives the equation above: a harmonic oscillator with a constant forcing. Its general solution is $u = \mu/h^2 + C\cos(f-\omega)$; choosing the origin of f at periapsis ($\omega = 0$) and writing $C = e\mu/h^2$,

$$u = \frac{\mu}{h^2}\,(1 + e\cos f) \quad\Longleftrightarrow\quad r = \frac{p}{1+e\cos f}.$$

The same conic, with e now appearing as a constant of integration of an ODE rather than as the length of a conserved vector.

8 Summary

1. The relative motion of two bodies obeys $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3$, $\mu = G(m_1+m_2)$; the individual motions follow from Equation 2.
2. Three integrals: $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}}$ (the motion is planar, and sweeps equal areas in equal times), $\mathbf{e}$ (points to periapsis; the orbit does not precess) and $\varepsilon = v^2/2 - \mu/r$. Two Casimir relations leave five independent constants; the sixth is a time, t_p or M_0.
3. The orbit is a conic, $r = p/(1+e\cos f)$ with $p = h^2/\mu$. The sign of ε decides the conic.
4. $\varepsilon = -\mu/2a$ and $v^2 = \mu(2/r - 1/a)$ on every conic: energy depends on size, not shape. Escape speed $\sqrt{2\mu/r}$; $C_3 = v_\infty^2 = -\mu/a$.
5. $v_r = (\mu/h)e\sin f$, $v_\perp = (\mu/h)(1+e\cos f)$, $\tan\gamma = e\sin f/(1+e\cos f)$; the hodograph is a circle of radius μ/h.
6. Ellipse: a is the semi-major axis, $T = 2\pi\sqrt{a^3/\mu}$, $r = a(1-e\cos E)$, $M = n(t-t_p) = E - e\sin E$, solved by Newton's method; $\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$.
7. Parabola: $v = \sqrt{2\mu/r}$, $\gamma = f/2$, Barker's cubic $M_p = D + D^3/3$, solved in closed form.
8. Hyperbola: $r = |a|(e\cosh H - 1)$, $M_h = e\sinh H - H$; $v_\infty = \sqrt{\mu/|a|}$; turning angle $\sin(\delta/2) = 1/e$ with $e = 1 + r_p v_\infty^2/\mu$; aiming radius $\Delta = b$, $h = \Delta v_\infty$.

9 Exercises

1. **The Sun's wobble.** Jupiter ($m_J/M_\odot = 1/1047.6$, $a = 5.2026$ AU, orbital speed 13.06 km/s) makes the Sun circle the Sun–Jupiter barycentre. Find the radius of the Sun's circle, in solar radii ($R_\odot = 6.96\times10^5$ km), and its speed. That speed is the radial-velocity signal an astronomer on another star would need to detect a Jupiter.

Solution. The radius. The barycentre divides the line between the two bodies in the inverse ratio of their masses (Section 2.1). The Sun's distance from it is therefore the fraction $m_J/(M_\odot + m_J) = 1/1048.6$ of the separation,

$$r_\odot = \frac{5.2026\times 1.495979\times10^8\ \text{km}}{1048.6} = 742\,200\ \text{km} = 1.07\,R_\odot.$$

The barycentre lies just outside the Sun's surface.

The speed. Both bodies go round the barycentre in the same period, so their speeds are in the ratio of their distances. The relative speed of Jupiter and the Sun is 13.06 km/s, and the Sun's share of it is the same fraction,

$$v_\odot = \frac{13.06\ \text{km/s}}{1048.6} = 12.5\ \text{m/s}.$$

Reading the result. An observer far away, looking along the plane of Jupiter's orbit, would see the Sun's light Doppler-shifted back and forth by ± 12.5 m/s with a period of 11.9 years. That is the speed of a fast cyclist, measured on a star. Spectrographs reached this precision in the 1990s, and the first planets found around Sun-like stars were found in exactly this way. The Earth moves the Sun by only 0.09 m/s, far harder to detect.

2. **State to elements.** At some instant a satellite has $\mathbf{r} = (7000,\, 0,\, 0)$ km and $\mathbf{v} = (1.0,\, 7.5,\, 1.0)$ km/s. Compute $\mathbf{h}$, $\mathbf{e}$, ε, a, e, p and f. Check both Casimir relations numerically. Is the satellite climbing or descending? Take $\mu = 398\,600.4418$ km^3/s^2.

Solution. The integrals. We have $r = 7000$ km, $v^2 = 58.25$ km^2/s^2, $\mu/r = 56.9429$ km^2/s^2 and $\mathbf{r}\cdot\mathbf{v} = 7000$ km^2/s. Then

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = (0,\, -7000,\, 52\,500)\ \text{km}^2/\text{s}, \qquad h = 52\,964.6\ \text{km}^2/\text{s},$$

$$\mathbf{e} = \frac{1}{\mu}\Big[\Big(v^2 - \frac{\mu}{r}\Big)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v}\Big] = (0.005393,\, -0.131711,\, -0.017561), \qquad e = 0.13299,$$

and $\varepsilon = v^2/2 - \mu/r = -27.8179$ km^2/s^2.

The size and shape. $a = -\mu/(2\varepsilon) = 7164.45$ km and $p = h^2/\mu = 7037.75$ km. As a check, $p/(1-e^2)$ gives the same a.

The true anomaly. $\cos f = \mathbf{e}\cdot\mathbf{r}/(er) = 0.04055$, so $f = 87.68°$ or $272.32°$. The radial speed is $\mathbf{r}\cdot\mathbf{v}/r = +1.0$ km/s, so the satellite is moving away from the centre, and $f = 87.68°$.

The Casimir relations. $\mathbf{h}\cdot\mathbf{e} = 0\times 0.005393 - 7000\times(-0.131711) + 52\,500\times(-0.017561) = 0$, to rounding. And $\mu^2(e^2-1) = -1.5607\times 10^{11}$, while $2\varepsilon h^2 = -1.5607\times 10^{11}$. Both hold.

Climbing or descending? Climbing, because $\dot r = v_r > 0$. The satellite is a little before the end of the latus rectum, $f = 90°$, where the radial speed $(\mu/h)\,e\sin f$ is largest.

Reading the result. The two Casimir checks are worth doing every time. A sign slip in $\mathbf{h}$ or $\mathbf{e}$ almost always breaks one of them, and they cost two lines.

3. **Same speed, different directions.** A projectile leaves the surface of an airless Earth at $v = 8$ km/s. Compare the orbits for launch straight up, at 45°, and horizontally: which have the same a? the same e? Which ones hit the ground again? Take $R_\oplus = 6378.137$ km and $\mu = 398\,600.4418$ km^3/s^2.

Solution. The same a. The energy depends only on the speed and the radius, $\varepsilon = v^2/2 - \mu/R_\oplus = 32 - 62.495 = -30.495$ km^2/s^2, so all three orbits have the same energy and the same

$$a = -\frac{\mu}{2\varepsilon} = 6535.5\ \text{km}.$$

Different e. The angular momentum depends on the direction, $h = R_\oplus v\cos\gamma$, with γ the angle above the horizontal. With $p = h^2/\mu$ and $e = \sqrt{1 - p/a}$,

launch	h (km^2/s)	p (km)	e	r_p (km)	farthest point (km)
straight up	0	0	1	0	13 071

launch	h (km^2/s)	p (km)	e	r_p (km)	farthest point (km)
45°	36 080	3265.9	0.7073	1912.9	11 158
horizontal	51 025	6531.8	0.0241	6378.1	6693.0

The vertical shot is the rectilinear orbit of Section 2.7, with $e = 1$, which rises to $2a$ and falls back. No two of the three share an eccentricity.

Which come down? The vertical shot falls straight back. The 45° shot has its periapsis 1913 km from the centre, deep inside the Earth, so it hits the ground before reaching it. The horizontal shot starts at its own periapsis, because $v = 8$ km/s is a little above the circular speed 7.905 km/s. It rises to 315 km above the far side and returns to the launch point one period later, skimming the ground there. On an airless, smooth, non-rotating Earth it would just graze it. Any hill would stop it.

Reading the result. The speed fixes the size of the orbit, the direction fixes its shape. This is the whole content of Equation 11 and of the second Casimir relation. Newton's cannon is the third case, and it needs very little speed above circular to clear the ground.

4. **Apsidal speeds.** Prove $v_p v_a = \mu/a$ directly from conservation of energy and angular momentum, without using the velocity formulas.

Solution. What we use. At both apsides the velocity is perpendicular to the radius, so the angular momentum is simply $h = r_p v_p = r_a v_a$. The energy is the same at both, $\tfrac{1}{2}v_p^2 - \mu/r_p = \tfrac{1}{2}v_a^2 - \mu/r_a$.

The plan. The product $v_p v_a = h^2/(r_p r_a)$, so we need h^2 in terms of r_p and r_a. The energy equation will give it.

The algebra. Rearranging the energy equation,

$$v_p^2 - v_a^2 = 2\mu\Big(\frac{1}{r_p} - \frac{1}{r_a}\Big) = \frac{2\mu\,(r_a - r_p)}{r_p r_a}.$$

With $v_p = h/r_p$ and $v_a = h/r_a$ the left side is

$$h^2\Big(\frac{1}{r_p^2} - \frac{1}{r_a^2}\Big) = \frac{h^2\,(r_a - r_p)(r_a + r_p)}{r_p^2 r_a^2}.$$

Equating the two and cancelling $(r_a - r_p)/(r_p r_a)$, which is not zero for a non-circular orbit,

$$\frac{h^2\,(r_a + r_p)}{r_p r_a} = 2\mu \qquad \Longrightarrow \qquad h^2 = \frac{2\mu\, r_p r_a}{r_p + r_a}.$$

Since $r_p + r_a = 2a$, this is $h^2 = \mu r_p r_a/a$, and so

$$v_p v_a = \frac{h^2}{r_p r_a} = \frac{\mu}{a}. \qquad \blacksquare$$

Reading the result. The product of the apsidal speeds equals the square of the circular speed at the distance a, $v_c^2 = \mu/a$. The circular speed at the mean distance is the geometric mean of the fastest and the slowest speeds on the orbit. For a circle, $v_p = v_a = v_c$, and the relation is trivial.

5. **Time to fall.** Using the rectilinear limit, estimate how long a body released from rest at the Earth's distance from the Sun would take to fall into the Sun. (Hint: it is half the period of a degenerate ellipse with $2a = 1$ AU.)

Solution. The idea. A body released from rest has $h = 0$, so it moves on the rectilinear orbit of Section 2.7. Its energy is $\varepsilon = -\mu_\odot/(1\text{ AU})$, so $a = -\mu_\odot/(2\varepsilon) = 0.5$ AU. A rectilinear orbit is the limit of ellipses with $e \to 1$ and this fixed a, which become ever thinner. The release point is the apoapsis, at $2a = 1$ AU, and the Sun is at the periapsis end. The fall from apoapsis to periapsis is half a period.

The period. Kepler's third law depends only on a. An orbit with $a = 0.5$ AU has the period $0.5^{3/2} = 0.35355$ year, so the fall takes half of that,

$$t = \tfrac{1}{2} \times 0.35355 \times 365.25\text{ d} = 64.6\text{ days}.$$

Reading the result. If the Earth stopped in its orbit it would hit the Sun in about nine weeks. The fall is very uneven. Most of the time is spent near the start, where the body is still slow, and the last part is very fast. A further point is that $64.6/365.25 = 0.177$ is $1/(4\sqrt{2})$, the ratio for any circular orbit, whatever the body or the planet.

6. **Kepler's equation for a highly eccentric orbit.** For $e = 0.99$ and $M = 0.05$ rad, run Newton's method from $E_0 = M$ and from $E_0 = \pi$. Which converges faster, and why?

Solution. The two runs. With Equation 42, the iterates are

k	from $E_0 = M$	from $E_0 = \pi$
0	0.050000	3.141593
1	4.453160	1.588029
2	0.177995	1.049047
3	2.022329	0.771172
4	1.267049	0.663733
5	0.880102	0.646329
6	0.698660	0.645892
7	0.649478	0.645891
8	0.645910	0.645891

Both reach $E = 0.645891$ rad. The start at π settles in about seven steps and never overshoots. The start at M first jumps to 4.45, beyond π, then wanders, and needs about nine steps.

Why. Newton's method follows the tangent of $g(E) = E - e\sin E - M$. At $E_0 = M = 0.05$ the slope is $g' = 1 - e\cos E_0 = 0.0112$, almost flat, because e is close to 1 and E_0 close to 0. A nearly flat tangent meets the axis far away, so the first step overshoots wildly. At $E_0 = \pi$ the slope is $1 + e = 1.99$, as steep as it ever gets. There is more. On $0 < E < \pi$ the curvature is $g'' = e\sin E > 0$, so the graph bends upwards. Newton's method started to the right of the root on such a curve stays to the right and approaches the root monotonically, each tangent landing between the root and the previous point.

Reading the result. This is why the note recommends $E_0 = \pi$ when e is close to 1. A program that propagates comets, or orbits just short of escape, must not start from M. Near periapsis of a very eccentric orbit Kepler's equation is at its least friendly, because the body sweeps a large angle in a short time.

7. **Comets.** A parabolic comet has $r_p = 0.1$ AU. How long does it spend inside the Earth's orbit ($r < 1$ AU)? Compare with a parabola of $r_p = 0.9$ AU.

Solution. The crossing point. On a parabola $p = 2r_p$, and the orbit equation $r = 2r_p/(1+\cos f)$ gives $r = 1$ AU at $\cos f = 2r_p - 1$ (with r_p in AU). The comet is inside the Earth's orbit from $-f$ to $+f$, so by symmetry the time is twice the time from perihelion to f. By Barker's equation Equation 43, with $D = \tan(f/2)$,

$$t = 2\sqrt{\frac{2r_p^3}{\mu_\odot}}\Big(D + \frac{D^3}{3}\Big).$$

$r_p = 0.1$ **AU.** $\cos f = -0.8$, so $f = 143.13°$ and $D = \tan 71.57° = 3$. Then $D + D^3/3 = 12$, and with $\sqrt{2r_p^3/\mu_\odot} = 2.247 \times 10^5$ s the time is 5.39×10^6 s $= 62.4$ days.

$r_p = 0.9$ **AU.** $\cos f = 0.8$, so $f = 36.87°$ and $D = \tan 18.43° = 1/3$. Then $D + D^3/3 = 0.3457$, and with $\sqrt{2r_p^3/\mu_\odot} = 6.066 \times 10^6$ s the time is 4.19×10^6 s $= 48.5$ days.

Reading the result. It is perhaps surprising that the comet that comes much closer to the Sun spends **longer** inside the Earth's orbit, 62 days against 49. Close to the Sun it moves very fast, but it also has much farther to travel. It sweeps 286° round the Sun inside 1 AU, against 74° for the comet that only dips in. Each comet crosses the Earth's orbit at the local escape speed of 42.1 km/s, so the two differ only in what they do inside.

8. **Flyby design.** A spacecraft approaches Mars ($\mu = 42\,828$ km^3/s^2, radius 3390 km) with $v_\infty = 2.6$ km/s. What is the largest turning angle it can get without going below 200 km altitude? What aiming radius does that require?

Solution. The largest turn. By Equation 47 and Equation 46, a smaller periapsis gives a smaller e and a larger turn. So the largest turn is at the lowest allowed periapsis, $r_p = 3390 + 200 = 3590$ km,

$$e = 1 + \frac{r_p v_\infty^2}{\mu} = 1 + \frac{3590 \times 6.76}{42\,828} = 1.5666, \qquad \delta = 2\arcsin\frac{1}{e} = 79.3°.$$

The aiming radius. It is $\Delta = b = |a|\sqrt{e^2 - 1}$, with $|a| = \mu/v_\infty^2 = 6335.5$ km, so

$$\Delta = 6335.5 \times \sqrt{1.5666^2 - 1} = 7640 \text{ km}.$$

An equivalent route uses $h = r_p v_p = \Delta v_\infty$ with $v_p = \sqrt{v_\infty^2 + 2\mu/r_p}$, which gives the same 7640 km.

Reading the result. The spacecraft must be aimed 7640 km from the centre of Mars, more than twice the planet's radius, to pass only 200 km above the surface. Gravity bends the approach inwards, so the aim is wide of the planet. The navigation tolerance is tight. Because Δ grows with r_p, an aiming error of a few kilometres changes the periapsis altitude by a similar amount, and 200 km leaves little margin. With only 79° of turn available, a single Mars flyby cannot reverse a trajectory, which is why missions that use Mars for a gravity assist plan for a moderate bend.

All numbers in this note, the worked examples and the task answers, and the identities used in the derivations, are checked by two-body-verify.py.

Hamilton, William Rowan. 1847. "The Hodograph, or a New Method of Expressing in Symbolical Language the Newtonian Law of Attraction." *Proceedings of the Royal Irish Academy* 3: 344–53.