Topocentric Frames and Observations

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November 1, 2026

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Where we are going

Everything so far was computed at the centre of the Earth. But nobody *measures* a satellite from the centre of the Earth. A radar on the ground measures how far the satellite is (**range**), and in which direction (**azimuth** and **elevation**), and an optical telescope measures only the direction (as **right ascension** and **declination**). A radio receiver measures how fast the range is changing, through the Doppler shift. All of these are quantities in a frame attached to the observer, the **topocentric frame**, and the only way to use them in the rest of astrodynamics is to know the observer's position and orientation precisely and to convert between the frame of the observer and the inertial frame of the orbit. That conversion is this note.

> What you should be able to answer at the end
>
> (1) Why do we need two kinds of latitude, and which one belongs in which formula? (2) How are the axes of the topocentric (south-east-zenith) frame written in Earth-fixed components, and how do you convert a satellite's position into azimuth, elevation and range? (3) Why does the range rate need the velocity *relative to the rotating Earth*, not the inertial velocity? (4) How do you turn a radar observation (range, angles and their rates) into an inertial state vector, and then into orbital elements? (5) How do topocentric right ascension and declination relate to azimuth and elevation, and why does a low satellite look nothing like its geocentric position? (6) What limits the accuracy of each measurement, and by how much?

The required background is the two-body problem, the orbital elements (especially the algorithm from a state vector to the elements), the active rotation matrices of the perifocal note, and the Greenwich sidereal angle and the Earth's rotation rate of the time note. The sub-satellite point and the footprint of a satellite are in the ground-tracks note; this note looks at the same geometry from the other end. The next step in the course, *preliminary orbit determination* (Gibbs, Lambert-based and angles-only methods), builds directly on the observations defined here, and this note stops where it begins. As in the other notes, a blue **Try it** box asks you to do a small calculation before reading on, and a green **Check yourself** box asks a short question; each has a hidden answer.

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes. In this note φ is the **geodetic** latitude (the one on maps and GPS receivers) and λ the east longitude of the site, φ_{gc} the geocentric latitude, A the azimuth, and ε

the **elevation** (the only use of ε in this note; it is not an energy).

1 The observer on the ellipsoid

1.1 Two kinds of latitude

The Earth is flattened by its rotation (the gravity note explains why), and the surface that surveyors and GPS use is a **reference ellipsoid**, a surface of revolution with equatorial radius a and flattening f. The standard one is WGS 84 (National Imagery and Mapping Agency 2000):

TODO: This diagram is wrong. Correct it. Define both angles at the equatorial plane.

$$a = 6378.137\,\text{km}, \qquad f = \frac{1}{298.257\,223\,563}, \qquad e^2 = f(2-f) = 0.006\,694\,380.$$

Here e is the eccentricity of the *ellipsoid* (not of an orbit). Two latitudes describe the direction of a point.

- The **geocentric latitude** φ_{gc} is the angle between the equatorial plane and the *line from the Earth's centre* to the point. It is the latitude used for $\sin\varphi_{gc} = z/r$ in the gravity and ground-track notes.
- The **geodetic latitude** φ is the angle between the equatorial plane and the **normal to the ellipsoid** at the point (Figure 1). It is the latitude of maps and of GPS, and, since the normal is what a plumb line (or a level) follows, it is the latitude that fixes which way is *up* for an observer.

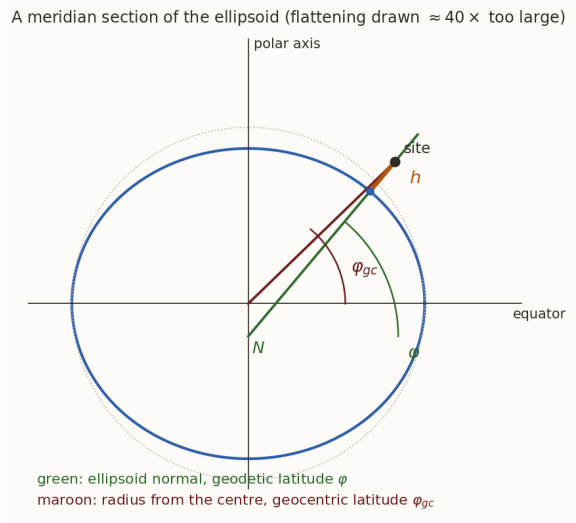


Figure 1: A meridian section of the ellipsoid (flattening drawn about 40 times too large). The normal to the ellipsoid (green) makes the geodetic latitude φ with the equatorial plane. The line from the centre (maroon) makes the geocentric latitude $\varphi_{gc} < \varphi$. The normal meets the polar axis below the centre, at distance N from the surface. The site is a height h above the surface along the normal. *Script: fig-topo-frames.py.*

They agree at the equator and at the poles and differ in between, by at most

$$\varphi - \varphi_{gc} \leq 0.192^\circ = 11.5' \qquad \text{at } \varphi \approx 45.1^\circ,$$

which is 21 km along the ground. That sounds small, but we will see that using the wrong one in the wrong place tilts the local vertical by that amount, and an elevation error of 0.19° at 1000 km range is 3.4 km.

If I put a spirit level normal to the line joining the a point on an ellipsoidal earth (with constant gravity) then the bubble will be in the center. True/False? What happens for real Earth? In other words, we are asking the question if the gravitational field be locally perpendicular to the Earth surface at point?

1.2 The site vector

Take a point on the surface at geodetic latitude φ. In the meridian plane, let its distance from the polar axis be ϱ and its height above the equatorial plane z. The ellipse is $\varrho^2/a^2 + z^2/b^2 = 1$ with $b^2 = a^2(1-e^2)$. Its normal at the point points along the gradient $(\varrho/a^2,\ z/b^2)$, and that direction is $(\cos\varphi, \sin\varphi)$ by definition of geodetic latitude, so

$$\frac{z}{\varrho} = \frac{b^2}{a^2}\tan\varphi = (1-e^2)\tan\varphi.$$

Putting this into the ellipse equation,

$$\varrho^2\left[\frac{1}{a^2} + \frac{(1-e^2)^2\tan^2\varphi}{a^2(1-e^2)}\right] = 1 \;\Longrightarrow\; \varrho = \frac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}} \equiv N\cos\varphi, \qquad z = N(1-e^2)\sin\varphi,$$

where

$$N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \tag{1}$$

is the length of the normal from the surface to the polar axis (the *prime vertical radius of curvature*). A point a height h above the surface *along the normal* is $h\,(\cos\varphi\cos\lambda,\ \cos\varphi\sin\lambda,\ \sin\varphi)$ further out, so in Earth-fixed (ECEF) components

$$\boxed{\mathbf{r}_{\text{site}} = \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ (N(1-e^2)+h)\sin\varphi \end{bmatrix}} \tag{2}$$

and the geocentric latitude of the site is $\varphi_{gc} = \operatorname{atan2}(z_{\text{site}}, \sqrt{x_{\text{site}}^2 + y_{\text{site}}^2})$.

A station near Bengaluru is at geodetic latitude 12.97°N, longitude 77.59°E, height 0.92 km. Find N, the site vector in ECEF, and the geocentric latitude.

What you should find

$N = 6379.2127$ km. $\mathbf{r}_{\text{site}} = (1336.147,\ 6072.091,\ 1422.378)$ km, with $|\mathbf{r}_{\text{site}}| = 6377.99$ km. The geocentric latitude is 12.886°, which is 5.03′ less than the geodetic one.

Where on the Earth is $\mathbf{r}_{\text{site}}$ simply $(a+h)$ times a unit vector? Where is the flattening felt most?

Answer

At the equator, $N = a$, $z = 0$ and $\mathbf{r}_{\text{site}} = (a+h)(\cos\lambda, \sin\lambda, 0)$; at the poles, $N = a/\sqrt{1-e^2} = a^2/b$ and $z = (a^2/b)(1-e^2)+h = b+h$, so $\mathbf{r}_{\text{site}} = (0, 0, b+h)$, the shorter polar radius $b = 6356.752$ km. The *direction* of the vertical differs from the direction of the radius most at mid-latitudes (45°), by the 0.192° above.

History: reference ellipsoids. WGS 84 is the reference ellipsoid of GPS. Its two defining numbers are

2 The topocentric frame

An observer needs three directions: up, east and north. The **topocentric horizon frame**, or **SEZ frame**, has its origin at the site and its axes along

- $\hat{\mathbf{Z}}$, the local vertical (the outward normal to the ellipsoid, the **zenith** direction),
- $\hat{\mathbf{E}}$, due **east** (the direction in which the longitude increases),
- $\hat{\mathbf{S}}$, due **south**, chosen so that $(\hat{\mathbf{S}}, \hat{\mathbf{E}}, \hat{\mathbf{Z}})$ is right-handed, $\hat{\mathbf{S}} \times \hat{\mathbf{E}} = \hat{\mathbf{Z}}$.

(Many books use the equivalent *east-north-up*, ENU, frame, with $\hat{\mathbf{N}} = -\hat{\mathbf{S}}$ in place of $\hat{\mathbf{S}}$. The two differ only by the sign of one axis, and by the order of the rows of the matrix below.) Figure 2 draws the frame.

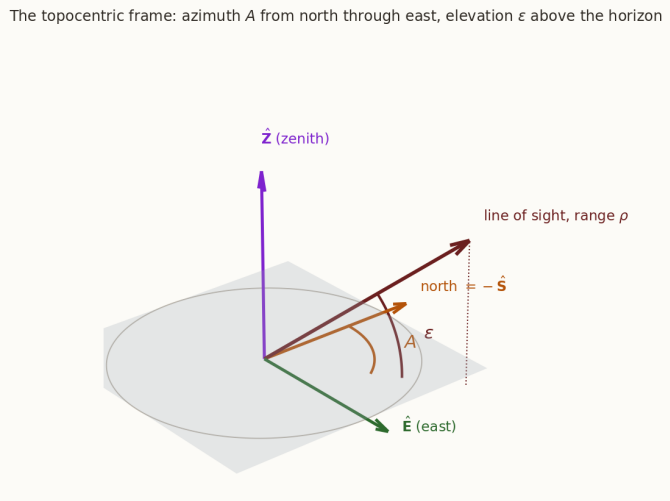


Figure 2: The topocentric frame at a site. The horizon plane contains the east and north directions ($\hat{\mathbf{S}} = -$ north); the zenith is perpendicular to it. A satellite is seen along the line of sight at distance ρ (the range), at azimuth A measured from north through east, and elevation ε above the horizon. *Script: fig-topo-frames.py*.

In ECEF components, $\hat{\mathbf{Z}}$ is the outward ellipsoid normal $(\cos\varphi\cos\lambda,\ \cos\varphi\sin\lambda,\ \sin\varphi)$, $\hat{\mathbf{E}} = (-\sin\lambda,\ \cos\lambda,\ 0)$, and $\hat{\mathbf{S}} = \hat{\mathbf{E}} \times \hat{\mathbf{Z}}$. The matrix that takes ECEF components to SEZ components has these unit vectors as rows:

$$\mathbf{M} = \begin{bmatrix} \hat{\mathbf{S}}^{\mathsf{T}} \\ \hat{\mathbf{E}}^{\mathsf{T}} \\ \hat{\mathbf{Z}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\lambda & \sin\varphi\sin\lambda & -\cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \tag{3}$$

Check that $\hat{\mathbf{S}} \times \hat{\mathbf{E}} = \hat{\mathbf{Z}}$ from the components, and that $\mathbf{M}\mathbf{M}^{\mathsf{T}} = \mathbf{1}$ with

$\det \mathbf{M} = +1$. Then evaluate $\mathbf{M}$ for the Bengaluru site.

What you should find

The cross product is $(\sin\varphi\sin\lambda\cdot 0 - (-\cos\varphi)\cos\lambda,\ (-\cos\varphi)(-\sin\lambda) - \sin\varphi\cos\lambda\cdot 0,\ \sin\varphi\cos^2\lambda + \sin\varphi\sin^2\lambda) = (\cos\varphi\cos\lambda,\ \cos\varphi\sin\lambda,\ \sin\varphi) = \hat{\mathbf{Z}}$. The three rows are mutually perpendicular unit vectors, so $\mathbf{M}$ is a rotation. For $\varphi = 12.97°, \lambda = 77.59°$:

$$\mathbf{M} = \begin{bmatrix} 0.04823 & 0.21920 & -0.97449 \\ -0.97663 & 0.21491 & 0 \\ 0.20942 & 0.95172 & 0.22444 \end{bmatrix}.$$

$\mathbf{M}$ is also a product of two elementary rotations, as the geometry suggests: turn the frame by λ about the polar axis (so that the new x-axis lies in the site's meridian), then by $90° - \varphi$ about the new y-axis (so that the new z-axis lies along the normal). In the passive form used by some books, $\mathbf{M} = \mathbf{R}_y(90° - \varphi)\,\mathbf{R}_z(\lambda)$, with $\mathbf{R}_z(\lambda) = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$. (The verify script checks this against Equation 3.)

Common mistakes with the frame

- **Geocentric latitude in M.** The zenith is the ellipsoid normal, so $\mathbf{M}$ takes the *geodetic* latitude. With the geocentric one, the whole frame is tilted by up to 0.19°.
- **$\hat{\mathbf{S}}$ for north.** The first row of $\mathbf{M}$ is **south**. North is $-\hat{\mathbf{S}}$, and azimuth is measured from north.
- **Inertial components.** $\mathbf{M}$ acts on *Earth-fixed* components. The satellite is first rotated into the ECEF frame (Equation 4 below).

3 Azimuth, elevation and range

3.1 From a position to three angles

Let the satellite have inertial position $\mathbf{r}$ at a time when the Greenwich sidereal angle is θ_G. The Earth-fixed position is $\mathbf{r}_{\mathrm{ECEF}} = \mathbf{R}_3(-\theta_G)\,\mathbf{r}$ (the rotation about the polar axis of step 5 in the ground-tracks note), and the vector from the site to the satellite, in SEZ components, is

$$\rho_{\mathrm{SEZ}} = \mathbf{M}\,(\mathbf{R}_3(-\theta_G)\,\mathbf{r} - \mathbf{r}_{\mathrm{site}}) \equiv (\rho_S,\ \rho_E,\ \rho_Z)^{\mathsf{T}}. \tag{4}$$

Its length is the **range** ρ. Its direction is described by two angles (Figure 2): the **elevation** ε above the horizon plane, and the **azimuth** A of the projection on the horizon plane, measured from north, clockwise (through east). With north $= -\hat{\mathbf{S}}$,

$$\boxed{\rho = \|\rho_{\text{SEZ}}\|, \qquad \varepsilon = \arcsin\frac{\rho_Z}{\rho}, \qquad A = \operatorname{atan2}(\rho_E,\, -\rho_S) \pmod{360^\circ}} \tag{5}$$

The satellite is **above the horizon** exactly when $\rho_Z > 0$, i.e. $\varepsilon > 0$. The inverse, from (ρ, A, ε) to a vector, is

$$\rho_{\text{SEZ}} = \rho \begin{bmatrix} -\cos\varepsilon\cos A \\ \cos\varepsilon\sin A \\ \sin\varepsilon \end{bmatrix}, \qquad \hat{\mathbf{u}} \equiv \frac{\rho_{\text{SEZ}}}{\rho}. \tag{6}$$

Use atan2 with the **east** component first and the **north** component second. The elevation arcsin needs no quadrant care, because $-90^\circ \le \varepsilon \le 90^\circ$ is the range of the arcsine; the azimuth does, and it is the one that most often goes wrong by 180° or by a sign.

3.2 A worked observation

A circular orbit at 420 km altitude has $i = 51.64^\circ$, $\Omega = 166^\circ$, and at the epoch ($t = 0$) the argument of latitude is 0 and $\theta_G = 100^\circ$ (the Earth turns at $\omega_\oplus$ afterwards). At $t = 174.009$ s the satellite is at $\mathbf{r} = (-6668.678,\ 815.911,\ 1038.120)$ km with velocity $\mathbf{v} = (0.3193,\ -4.8834,\ 5.8893)$ km/s, and $\theta_G = 100.727^\circ$. Find the azimuth, elevation and range from the Bengaluru station.

What you should find

ρ_{SEZ} comes out to a range of $\rho = 868.82$ km, with $\varepsilon = 25.51^\circ$ and $A = 232.21^\circ$: the satellite is 25.5° above the horizon, towards the south-west (azimuth measured from north through east, so 232° is 52° west of south).

4 Rates, and what a Doppler shift measures

4.1 The velocity that an observer sees

The station rotates with the Earth, and the SEZ axes rotate with it. The **relative position** is $\rho = \mathbf{r}_{\text{ECEF}} - \mathbf{r}_{\text{site}}$, and since the site is fixed in the Earth-fixed frame, its rate of change as seen by the station is the velocity of the

satellite *relative to the rotating Earth*. In the ECEF frame, $\mathbf{r}_{\text{ECEF}} = \mathbf{R}_3(-\theta_G)\mathbf{r}$ and $\dot{\theta}_G = \omega_\oplus$, so

$$\dot{\rho}_{\text{ECEF}} = \mathbf{R}_3(-\theta_G)\,\mathbf{v} - \omega_\oplus \times \mathbf{r}_{\text{ECEF}}, \qquad \omega_\oplus = \begin{bmatrix} 0 \\ 0 \\ \omega_\oplus \end{bmatrix}, \tag{7}$$

and in SEZ components $\dot{\rho}_{\text{SEZ}} = \mathbf{M}\,\dot{\rho}_{\text{ECEF}}$. The second term is the velocity of the Earth-fixed point *under* the satellite, $\omega_\oplus r \cos\varphi_{gc}$ (0.47 km/s at the equator), and it is not small next to the satellite's own 7.7 km/s: leaving it out changes $\dot{\rho}$ by up to 0.47 km/s, comparable with the whole difference between the left and right ends of the range-rate curve of Figure 3.

Why not just differentiate ρ using the inertial velocity $\mathbf{v}$?

Answer

Because the station is not at rest in the inertial frame: it moves at $\omega_\oplus \times \mathbf{r}_{\text{site}}$. The range rate is the rate of change of the distance between the satellite and the station, and the *station's* motion contributes. Equivalently, $\dot{\rho} = \hat{\rho}\cdot(\mathbf{v}-\mathbf{v}_{\text{site}})$ in the inertial frame, and $\mathbf{v}_{\text{site}} = \omega_\oplus \times \mathbf{r}_{\text{site}}$; Equation 7 is the same statement written in the frame that rotates with the station.

4.2 The three rates

The rate of the range is the component of the relative velocity along the line of sight. The azimuth and elevation rates follow from differentiating Equation 5. With $\rho_{\text{SEZ}} = (\rho_S, \rho_E, \rho_Z)$ and $\dot{\rho}_{\text{SEZ}} = (\dot{\rho}_S, \dot{\rho}_E, \dot{\rho}_Z)$,

$$\boxed{\dot{\rho} = \frac{\rho_{\text{SEZ}} \cdot \dot{\rho}_{\text{SEZ}}}{\rho}, \qquad \dot{A} = \frac{-\rho_S\,\dot{\rho}_E + \rho_E\,\dot{\rho}_S}{\rho_S^2 + \rho_E^2}, \qquad \dot{\varepsilon} = \frac{\dot{\rho}_Z - \dot{\rho}\,\sin\varepsilon}{\rho\cos\varepsilon}} \tag{8}$$

(for $\dot{A}$, write $A = \operatorname{atan2}(\rho_E, N)$ with $N = -\rho_S$ and use $\dot{(\arctan(E/N))} = (N\dot{E} - E\dot{N})/(N^2+E^2)$; for $\dot{\varepsilon}$, differentiate $\sin\varepsilon = \rho_Z/\rho$). Note the singularities: the azimuth is undefined, and its rate unbounded, when the satellite passes through the zenith ($\rho_S = \rho_E = 0$), and the elevation rate has $\cos\varepsilon$ in the denominator for the same reason. A pass very close to overhead swings through all azimuths in a few seconds, which tracking antennas find hard to follow.

4.3 Doppler

A transmitter on the satellite sends a carrier of frequency ν_t. The receiver sees it shifted by

$$\frac{\nu_r - \nu_t}{\nu_t} \approx -\frac{\dot\rho}{c} \tag{9}$$

to first order in $\dot\rho/c$: approaching ($\dot\rho < 0$) raises the frequency, receding lowers it. For a *two-way* link (the station transmits and a transponder returns the signal) the shift is twice as large, which is the $-\frac{1}{2}c\,\Delta\nu/\nu$ of the time note. A frequency is a clock measurement, so the clock error of the time note carries over: a fractional frequency error y in the reference is a range-rate error $c\,y$.

4.4 A pass

Figure 3 shows the whole pass of the example orbit over the station, with a 5° elevation mask. The satellite rises (**acquisition of signal**, AOS) in the south-west at azimuth 223.2°, reaches 63.8° in the north-west, and sets (**loss of signal**, LOS) in the north-east at 31.4°, a total of 507.9 s later. At AOS and LOS it is 1860 km away and $\dot\rho = \mp 6.83$ km/s; at closest approach it is 465 km away and $\dot\rho = 0$, and the one-way Doppler shift of a 2.2 GHz carrier sweeps from +50.1 kHz to −50.1 kHz.

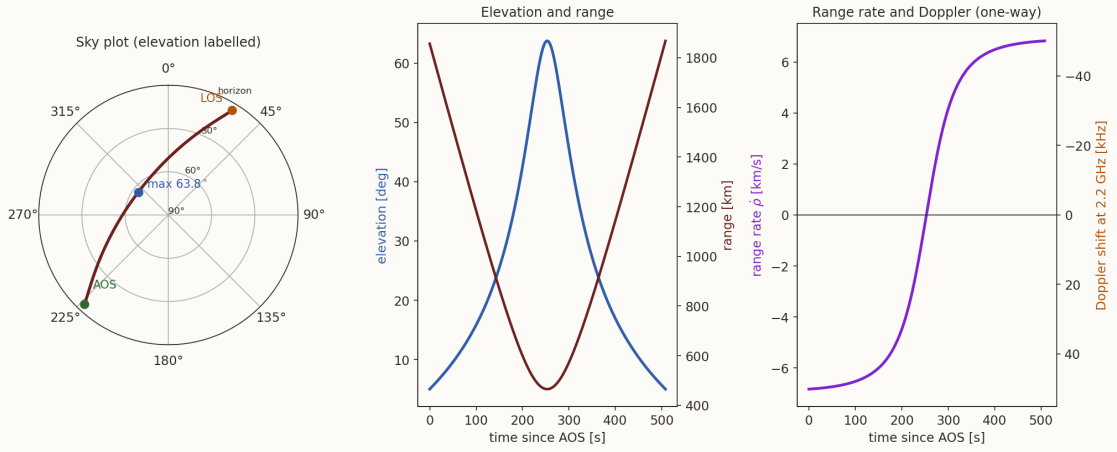


Figure 3: A pass of an ISS-like orbit (420 km, $i = 51.64°$) over Bengaluru, above a 5° mask. **Left:** the sky plot: azimuth round the edge, zenith distance outwards (the centre is the zenith). **Middle:** elevation and range against time. **Right:** range rate and the corresponding one-way Doppler shift at 2.2 GHz. The range rate passes through zero at closest approach. *Script: fig-topo-pass.py.*

At the instant of the worked observation above, find the rates $\dot A$, $\dot\varepsilon$, $\dot\rho$ and the one-way Doppler shift at 2.2 GHz.

What you should find

$\dot A = 0.1417°/\text{s}$, $\dot\varepsilon = 0.2494°/\text{s}$ and $\dot\rho = -6.008$ km/s (approaching), so the

one-way shift is +44.09 kHz. (The sign check: this is between AOS, where the shift is +50 kHz, and closest approach, where it is zero.)

5 From a radar observation to an orbit

A radar gives (ρ, A, ε) and, with Doppler, their rates. That is six numbers at one instant, exactly what a state vector needs. The conversion is the forward chain run backwards.

1. The line of sight in SEZ components, $\rho_{\text{SEZ}} = \rho\hat{\mathbf{u}}$ from Equation 6, and its derivative $\dot{\rho}_{\text{SEZ}} = \dot{\rho}\,\hat{\mathbf{u}} + \rho\,\dot{\hat{\mathbf{u}}}$, with

$$\dot{\hat{\mathbf{u}}} = \begin{bmatrix} \cos\varepsilon \sin A\,\dot{A} + \sin\varepsilon \cos A\,\dot{\varepsilon} \\ \cos\varepsilon \cos A\,\dot{A} - \sin\varepsilon \sin A\,\dot{\varepsilon} \\ \cos\varepsilon\,\dot{\varepsilon} \end{bmatrix}.$$

2. Rotate to ECEF with $\mathbf{M}^{\mathsf{T}}$ (the transpose is the inverse): $\mathbf{r}_{\text{ECEF}} = \mathbf{r}_{\text{site}} + \mathbf{M}^{\mathsf{T}}\rho_{\text{SEZ}}$ and $\dot{\rho}_{\text{ECEF}} = \mathbf{M}^{\mathsf{T}}\dot{\rho}_{\text{SEZ}}$.
3. Rotate to the inertial frame, adding back the Earth's rotation: $\mathbf{r} = \mathbf{R}_3(\theta_G)\,\mathbf{r}_{\text{ECEF}}$ and, inverting Equation 7, $\mathbf{v} = \mathbf{R}_3(\theta_G)(\dot{\rho}_{\text{ECEF}} + \omega_\oplus \times \mathbf{r}_{\text{ECEF}})$.
4. Convert $(\mathbf{r}, \mathbf{v})$ to orbital elements with the algorithm of the elements note.

Take the observation of the worked example, $(\rho, A, \varepsilon) = (868.822\text{ km}, 232.2122^\circ, 25.5051^\circ)$ with rates $(-6.0078\text{ km/s}, 0.141695^\circ/\text{s}, 0.249444^\circ/\text{s})$ and $\theta_G = 100.727^\circ$. Rebuild the state vector and the elements.

What you should find

The chain returns $\mathbf{r} = (-6668.678, 815.911, 1038.120)$ km and $\mathbf{v} = (0.3193, -4.8834, 5.8893)$ km/s (to the digits of the input). The elements are $a = 6798.137$ km, $e = 0$ (to 10^{-16}: the orbit was built circular), $i = 51.64^\circ$ and $\Omega = 166.0^\circ$, with argument of latitude 11.23° (since $n\,t = 11.23^\circ$ after 174.009 s on the orbit's own clock, and the argument of latitude was 0 at the epoch). With noise on the six measurements, of course, the elements would not be so tidy; Section 8 says how large the errors are.

6 Angles only: right ascension and declination

An optical telescope, or a radar that cannot range, gives only the direction of the line of sight, two numbers. It is usually reported in the *inertial* sky coordinates: the **topocentric right ascension** α and **declination** δ, which are the right ascension and declination of the direction from the site to the satellite. The direction cosines of the line of sight in the inertial frame are

$$\hat{\rho} = (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)^{\mathsf{T}} = \mathbf{R}_3(\theta_G)\,\mathbf{M}^{\mathsf{T}}\hat{\mathbf{u}}, \tag{10}$$

where $\hat{\mathbf{u}}$ is the unit line of sight in SEZ components of Equation 6. So, given A and ε, we get (α, δ) by the rotation, and vice versa. The same relation is the classical spherical-trigonometry triangle of positional astronomy (zenith, celestial pole, satellite). With the **local sidereal time** $\Theta = \theta_G + \lambda$ (the right ascension of the meridian over the site) and the **hour angle** $H = \Theta - \alpha$ (positive west of the meridian), it reads

$$\sin\delta = \sin\varphi\sin\varepsilon + \cos\varphi\cos\varepsilon\cos A, \qquad \cos\delta\,\sin H = -\cos\varepsilon\sin A, \qquad \cos\delta\,\cos H = \cos\varphi\sin\varepsilon - \tag{11}$$

(here φ is the geodetic latitude, which is the *astronomical* latitude to the accuracy of a few arcseconds that we need). Given (δ, H) the same three equations, solved for ε and A, give the horizon coordinates.

For the worked observation ($A = 232.2122°$, $\varepsilon = 25.5051°$, $\varphi = 12.97°$, $\lambda = 77.59°$, $\theta_G = 100.727°$), find δ from Equation 11, the local sidereal time, and the topocentric α.

What you should find

$\sin\delta = \sin 12.97°\sin 25.505° + \cos 12.97°\cos 25.505°\cos 232.21° = -0.4423$, so $\delta = -26.25°$. The local sidereal time is $\Theta = 100.727° + 77.59° = 178.317°$ (11 h 53.3 min). From the other two equations, $H = 52.68°$, so $\alpha = \Theta - H = 125.64°$. Rotating Equation 10 directly gives the same $(\alpha, \delta) = (125.64°, -26.25°)$.

> **Topocentric is not geocentric.** The *geocentric* direction of the same satellite, from the position vector $\mathbf{r}$ alone, has $\alpha = 173.02°$ and $\delta = +8.78°$. The topocentric direction has $\alpha = 125.64°$ and $\delta = -26.25°$: the two directions are 57.8° apart. This is **parallax**: for a satellite 870 km from the observer and 6 800 km from the centre, the observer and the centre see it in quite different directions. (The angle between them is the angle at the satellite between the lines to the observer and to the centre, which is the nadir angle η of the visibility section, $\sin\eta \approx (r_{\text{site}}/r)\cos\varepsilon$.) Parallax matters only for near objects. Mixing the two is a classic error in angles-only orbit determination; for a planet or a star it would be a non-issue.

Three such directions at three times, with the observer's position known, are enough in principle to determine an orbit. That is *Gauss's method* of angles-only orbit determination, the most famous example of preliminary

orbit determination, which the orbit-determination note will take up; it takes the direction cosines of Equation 10 and the site positions of Equation 2 as its data.

7 Is the satellite visible?

A station sees a satellite while its elevation is above the **mask** $\varepsilon_{\min}$, a limit set by terrain, buildings, and atmospheric noise (typically 5° to 10°). For a circular orbit of radius r and a spherical Earth, the ground-tracks note gives the largest Earth central angle ψ_0 between the station and the satellite's sub-point at which the satellite is above the mask:

$$\sin\eta = \frac{R_\oplus}{r}\cos\varepsilon_{\min}, \qquad \psi_0 = 90^\circ - \varepsilon_{\min} - \eta,$$

where η is the nadir angle at which the station is seen from the satellite. If the satellite passes directly overhead and the Earth were not rotating, it crosses $2\psi_0$ of its orbit above the mask, taking

$$T_{\text{pass}} \approx \frac{2\psi_0}{n}, \qquad n = \sqrt{\mu/r^3}. \tag{12}$$

For the ISS orbit above a 5° mask, $\eta = 69.17^\circ$, $\psi_0 = 15.83^\circ$ and $T_{\text{pass}} = 490.5$ s, 8.2 min. The pass of Figure 3, 63.8° at its highest and not overhead, lasts 507.9 s: the Earth's rotation, which carries the station along with the satellite, makes it a few percent longer. A satellite that reaches only 20° at its highest is above the mask for a much shorter part of its orbit, which is why pass durations in the tables of the ground-tracks note are averages over all passes.

Why does a *higher* orbit give a longer pass, though the satellite moves more slowly?

Answer

Both effects work in the same direction. A larger orbit radius r makes η smaller (the horizon is farther away, $\sin\eta = R_\oplus\cos\varepsilon_{\min}/r$), so ψ_0 is larger: more of the orbit is in view. And it makes n smaller, so each radian takes longer. In Equation 12 both increase T_{pass}. At GEO the satellite never sets for the stations that see it at all.

8 What limits the accuracy

Each measurement has an error, and the error in position at the satellite is the error of the observation times a lever arm.

- **Angles.** An angular error $\delta\theta$ moves the apparent position of the satellite *across the line of sight* by $\rho\,\delta\theta$. A pointing error of 0.01° at a range of 1000 km is 175 m. Refraction in the atmosphere bends the line of sight toward the zenith by an angle (Bennett's empirical formula (Meeus 1998))

$$R = \cot\!\left(\varepsilon + \frac{7.31}{\varepsilon + 4.4}\right) \text{ arcminutes} \qquad (\varepsilon \text{ in degrees}),$$

  which is 34.5′ on the horizon, 9.9′ at $\varepsilon = 5°$ and 1.0′ at $\varepsilon = 45°$. A satellite at 5° is seen 0.16° too high, 2.9 km at 1000 km: this is why angle measurements below about 10° are avoided or corrected.
- **The local vertical.** Using the *geocentric* latitude in $\mathbf{M}$ tilts the frame by up to 0.192°, which is 3.4 km at 1000 km (Section 1.1).
- **Range.** A radar range is a time measurement, $\rho = \frac{1}{2}c\,\tau$, so its error is a clock error (the time note): 10 ns in τ is 1.5 m. Range errors are *along* the line of sight and, for a radar, typically far smaller than the angle errors translated to metres, which is why a radar's range is its best number.
- **Range rate.** A frequency error of 0.1 Hz on a 2.2 GHz carrier is 13.6 mm/s in $\dot\rho$.
- **Light time.** The signal received at time t left the satellite at $t - \rho/c$. At 1860 km that is 6.2 ms, during which the satellite moves 48 m. Precise orbit determination corrects for it, and the *antenna's pointing* has to lead the line of sight by the same amount.
- **Station position.** The site vector of Equation 2 is known to centimetres with GPS and to a few metres from a map; a 10 m error in the station is a 10 m error in the satellite position.

In one paragraph. A station sits at geodetic latitude φ on the ellipsoid, at $\mathbf{r}_{\text{site}}$ of Equation 2. Its SEZ axes are the rows of $\mathbf{M}$ (Equation 3). The satellite's position is rotated into ECEF, subtracted from $\mathbf{r}_{\text{site}}$, and rotated by $\mathbf{M}$ to give ρ_{SEZ} and, with Equation 5, the range, azimuth (from north, clockwise) and elevation. The rates use the velocity relative to the rotating Earth (Equation 7), and $\dot\rho$ is what a Doppler measures. Run backwards, six radar numbers give a state vector and then the elements. An optical observation is the direction only, in right ascension and declination, which differ from the geocentric ones by parallax. The accuracy is set by the angle error times the range, the refraction, the local vertical, and the clock.

Summary

The numbers of this note are checked by topocentric-verify.py, using the helper module _topo.py; the figure scripts fig-topo-frames.py and fig-topo-pass.py draw the figures.

What we left out. (i) *Precession, nutation and polar motion*, which make the ECEF frame of this note an approximation at the metre level (the time note says so). (ii) *Deflection of the vertical*: the plumb line differs from the ellipsoid normal by up to tens of arcseconds because of local mass anomalies, so the "geodetic latitude" and the "astronomical latitude" are not quite equal. (iii) *Aberration*, the apparent shift of direction by v/c (about 5″ for a LEO satellite), and *light-time*, treated here only as a size. (iv) *Atmospheric delay of ranging signals* (ionosphere, troposphere), which adds metres to a range at low elevation.

Further reading. H. D. Curtis, *Orbital Mechanics for Engineering Students*, chapter 5 (topocentric coordinate systems, orbit determination from angle and range measurements); D. A. Vallado, *Fundamentals of Astrodynamics and Applications* (coordinate transformations, observation types, sensor models); the WGS 84 definition (National Imagery and Mapping Agency 2000); J. Meeus, *Astronomical Algorithms* (Meeus 1998), for refraction and the astronomical triangle.

Exercises

1. **The site vector.** Compute $\mathbf{r}_{\text{site}}$ at the equator at the prime meridian, at sea level, and at the north pole. Check $|\mathbf{r}_{\text{site}}| = a$ and b. (Answer: $(6378.137, 0, 0)$ km; $(0, 0, 6356.752)$ km.)
2. **Latitudes.** Show that $\tan\varphi_{gc} = (1-e^2)\tan\varphi$ at $h = 0$, and find $\varphi - \varphi_{gc}$ at $\varphi = 45°$. (Answer: 0.192°.)
3. **Matrix.** Show that $\mathbf{M}$ of Equation 3, with the rows reordered to $(\hat{\mathbf{E}}, -\hat{\mathbf{S}}, \hat{\mathbf{Z}})$, is the ENU matrix, and that its first row at $\lambda = 0$ is $(0, 1, 0)$.
4. **Overhead.** A satellite is directly above the site at height h. Find ρ_{SEZ}, the range and the elevation. What is the azimuth? (Answer: $(0, 0, h)$, h, 90°, and it is undefined.)
5. **Doppler rate.** A 2.2 GHz transmitter on the ISS orbit passes at closest approach 465 km from a station. Treating the relative motion as a straight line at speed $v \approx 7$ km/s, show that $\ddot{\rho} = v^2/\rho_{\min}$ at closest approach, and hence estimate the Doppler rate $\dot{\nu} = -\nu\ddot{\rho}/c$ in Hz/s. (Answer: $\ddot{\rho} \approx 0.105$ km/s^2 and $\dot{\nu} \approx -770$ Hz/s; the exact value for the pass of Figure 3 is -800 Hz/s.)
6. **Radar to state.** Using the helper module, build the state from the observation of the worked example, and check that the elements come out as $a = 6798.137$ km, $i = 51.64°$, $\Omega = 166.0°$.
7. **Refraction.** From Bennett's formula, find the elevation at which refraction falls to one arcminute. (Answer: about 45°.)
8. **Parallax.** A satellite at $r = 7000$ km is seen at $\varepsilon = 10°$ from a site on a spherical Earth of radius 6378 km. Find the angle between the topocentric and the geocentric directions of the satellite. (Hint: it is the

angle at the satellite in the triangle formed with the site and the centre, and the sine rule gives it. Answer: $\eta = \arcsin(\frac{6378}{7000}\cos 10^\circ) = 63.8^\circ$.)

Meeus, Jean. 1998. *Astronomical Algorithms*. 2nd ed. Richmond, Virginia: Willmann-Bell.

National Imagery and Mapping Agency. 2000. "Department of Defense World Geodetic System 1984: Its Definition and Relationships with Local Geodetic Systems." Technical Report TR8350.2. Third. National Imagery; Mapping Agency.