Time

Devendra Ghate

November 21, 2026

Table of contents

Where we are going **1**

1 Why time is a first-class variable **2**

2 The day is not one number **3**
2.1 Noon to noon and star to star 3
2.2 What the difference does to spaceflight 5

3 Time scales **6**
3.1 Julian date and sidereal time in practice 6

4 What a clock error costs: numerical experiments **7**
4.1 Two ways a clock enters navigation 7
4.2 Experiment 1: how far is Mars, in minutes? 7
4.3 Experiment 2: a clock that is slightly wrong 8

Summary and outlook **9**

Exercises **10**

Where we are going

Every formula we have written so far had a t in it: Kepler's equation $M = n(t-t_0)$, the position of a planet, the angle through which the Earth has turned. We treated t as a number that somebody else supplies. This note is about the somebody else: What is "one day"? Why is a geostationary satellite *not* in a 24-hour orbit? What are UT1, UTC, TAI, TT and TDB, and why do we need all of them? And, with numerical experiments, why a probe on its way to Mars needs clocks that are good to a few parts in 10^{12} or better. The sequel, Ground Tracks, uses the Earth's rotation angle of this note to turn orbital elements into a curve on a map.

> What you should be able to answer at the end
>
> (1) By how much, and why, does a sidereal day differ from a solar day? (2) What period must a geostationary satellite have, and what happens if you choose 24 h? (3) What is the difference between UTC, TAI, TT and TDB, and how large is each offset today? (4) How does an error of δt in a clock become an error in position, range and velocity? (5) How stable must a clock be for deep-space ranging and Doppler tracking?

The required background is the two-body problem, Kepler's equation and the orbital elements. The secular effect of J_2 on the node, quoted for launch windows, is derived in the note Beyond the Point Mass. The navigation experiments also

use a Lambert transfer, the hyperbolic excess speed v_∞ and $C_3 = v_\infty^2$ of the patched-conics note, and a state-transition matrix (the linear map from an initial error to a later one, met properly in the cw note); only their results are needed here.

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

1 Why time is a first-class variable

A spacecraft moves, so its position is a function of time, $\mathbf{r}(t)$, and a clock that is wrong by δt places it at the wrong point. To first order

$$\delta \mathbf{r} = \mathbf{v}\,\delta t, \qquad \delta r = v\,\delta t. \tag{1}$$

Figure 1 shows what this means for the objects we deal with. A second is a long time: the International Space Station covers 7.66 km in it, the Earth 29.8 km along its orbit, and a radio signal 299 792 km. A *millisecond* is worth 7.7 m for the ISS and, for a radio signal, 300 km. A nanosecond is 30 cm of light travel. Navigation to a metre needs nanoseconds.

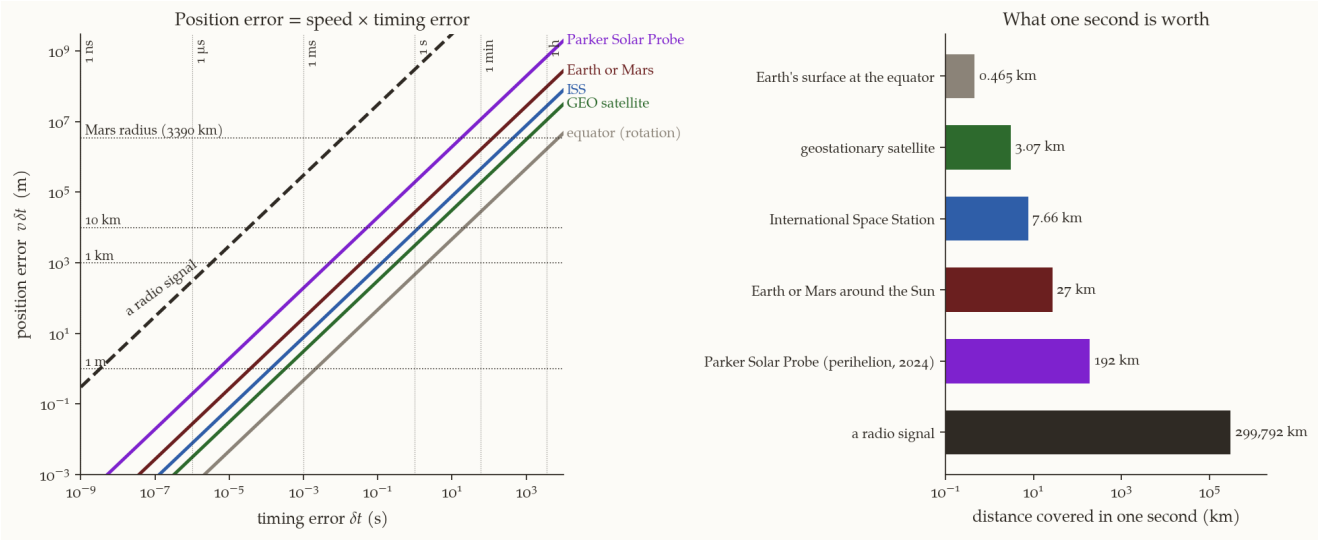


Figure 1: **Left:** the position error $v\,\delta t$ produced by a timing error δt for several objects. Read along a vertical line to see what one clock error costs each of them; read along a horizontal line to see how small the clock error must be to stay under a given position error. The dotted horizontal lines mark 1 m, 1 km, 10 km and the radius of Mars. **Right:** the same information for $\delta t = 1$ s. *Script: fig-time-days.py.*

Two questions to keep in mind while reading.

> **Two questions**
>
> 1. A geostationary satellite is "in a 24-hour orbit". Is that true?
> 2. The Sun is overhead at noon today. After how much time will *a star* that was overhead with it be overhead again?
>
> The answers are "no, it is 23 h 56 m 4 s" and "after 23 h 56 m 4.09 s, i.e. 3 m 55.91 s *before* the Sun". Section 2 explains why.

History: longitude is time. The Earth turns 15° per hour, so a difference of one hour between local noon and the time on a clock set at a reference meridian means 15° of longitude. In 1714 the British *Longitude Act* offered £20 000 for a method to find longitude at sea to half a degree, i.e. to two minutes of time, after a voyage of weeks. John Harrison's fourth chronometer (H4) lost only about five seconds in 81 days at sea on its 1761 trial to Jamaica. The problem of the clock is older than the problem of the orbit.

2 The day is not one number

2.1 Noon to noon and star to star

Two natural clocks are available in the sky: the Sun and the stars. The interval between two successive transits of the Sun across your meridian is the **solar day**. The interval between two transits of a distant star, which is the time the Earth needs for one complete turn relative to inertial space, is the **sidereal day**.[1] They differ because the Earth moves along its orbit while it turns (Figure 2).

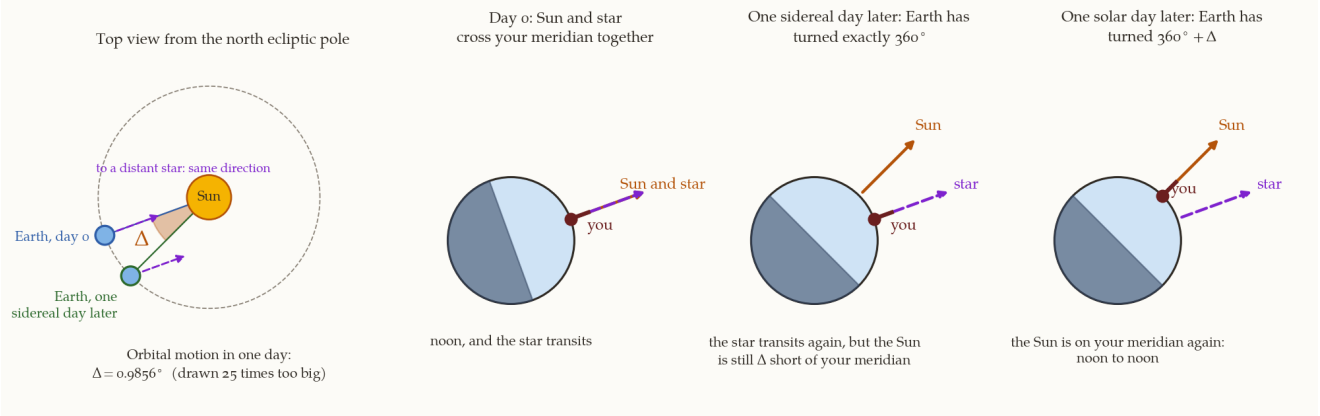


Figure 2: **Left:** the Earth moves through $\Delta = 360°/365.2422 = 0.9856°$ in one day (drawn 25 times too big). The direction to a distant star does not change, the direction to the Sun does. **Right three:** you at noon, when the Sun and a star are on your meridian; one *sidereal* day later, when the Earth has turned exactly 360° and the star is back but the Sun is still short by Δ; one *solar* day later, when the Earth has turned $360° + \Delta$ and the Sun is back. *Script: fig-time-days.py.*

Let $\omega_\oplus$ be the rotation rate of the Earth relative to the stars (strictly, 2π over the stellar day of Table 1; against the moving equinox the period is the sidereal day $T_{\rm sid}$, 8.4 ms shorter, a difference that matters only at the millisecond level) and $n_\odot = 2\pi/Y$ the rate at which the Sun's direction moves as the Earth goes round (the mean motion of the Earth, with Y the length of the year). The Sun returns to your meridian when the Earth has turned through 2π plus the angle the Sun's direction has moved:

$$\omega_\oplus T_{\rm sol} = 2\pi + n_\odot T_{\rm sol}, \qquad \omega_\oplus = \frac{2\pi}{T_{\rm sid}}, \quad n_\odot = \frac{2\pi}{Y} \quad\Longrightarrow\quad \frac{1}{T_{\rm sid}} = \frac{1}{T_{\rm sol}} + \frac{1}{Y}. \tag{2}$$

With $T_{\rm sol} = 1$ mean solar day and $Y = 365.242189$ days (the tropical year, the one that defines the seasons), Equation 2 gives

$$T_{\rm sid} = \frac{1}{1 + 1/365.242189}\ \text{day} = 0.99726957\ \text{day} = 86\,164.0905\ \text{s} = 23\,\text{h}\,56\,\text{m}\,4.0905\,\text{s}.$$

The Earth therefore has to turn an extra $\Delta = 360°/365.2422 = 0.9856°$ to bring the Sun back, which takes $86\,400 - 86\,164.0905 = 235.909$ s $= 3$ m 55.909 s. Equivalently: **one tropical year contains** 365.2422 **solar days but** 366.2422 **sidereal days.** The animation below shows this on a planet whose year is only 24 days long, so that the extra turn is 15° per day and cannot be missed.

[1] Strictly there are two "sidereal" days. The *mean sidereal day* is measured against the mean equinox of date, which drifts westward against the stars because of precession. The *stellar day* (the rotation period relative to the International Celestial Reference Frame) is 8.4 ms longer. We return to this difference below; for the geometry of this section the two are the same.

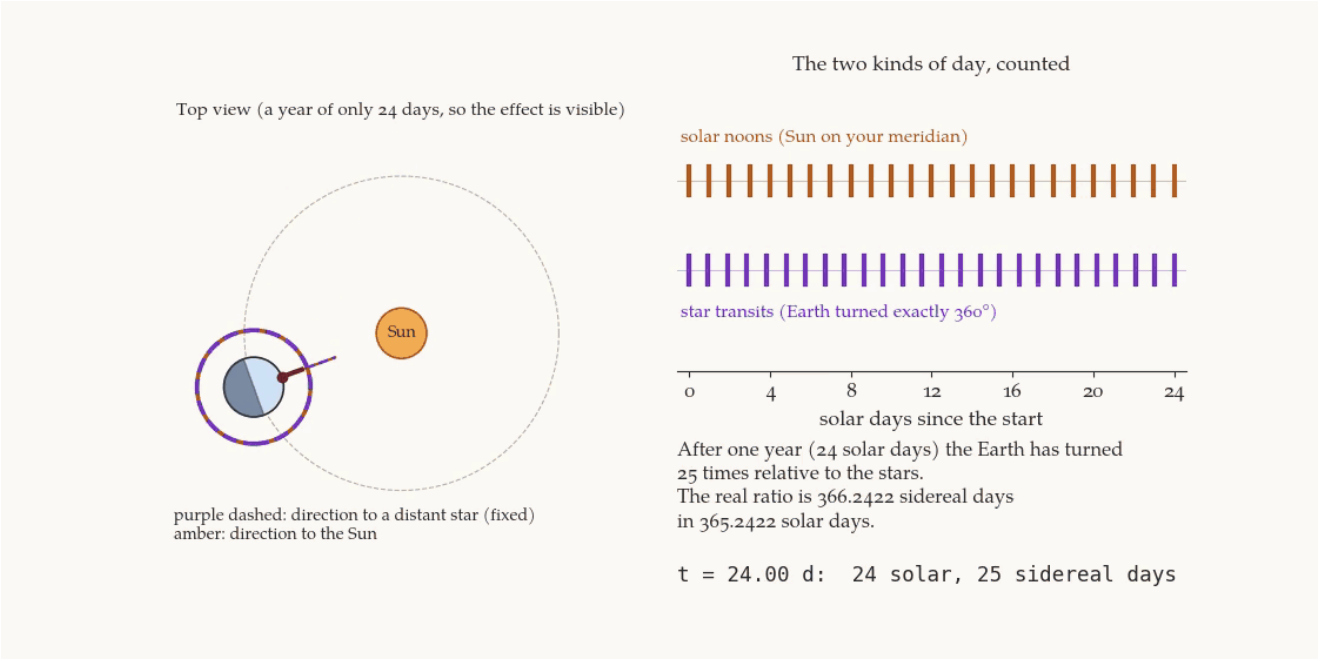


Figure 3: A planet with a year of 24 days. The observer's flag turns once per sidereal day; every time it points at the distant star (dashed purple) a purple tick is drawn, and every time it points at the Sun (amber) an amber tick. After one year there are 24 amber and 25 purple ticks. *Script: anim-time.py*.

The rotation rate follows directly:

$$\omega_\oplus = \frac{2\pi}{86\,164.0989\ \mathrm{s}} = 7.2921151 \times 10^{-5}\ \mathrm{rad/s} = 15.04107^\circ/\mathrm{h} = 360.98561^\circ/\text{solar day}, \tag{3}$$

where 86 164.0989 s is the stellar day (Table 1). At the equator the surface moves at $\omega_\oplus R_\oplus = 0.4651$ km/s.

Table 1: The days. Computed in fig-time-days.py and _timeutil.py.

Quantity	Symbol	Value
Mean solar day (definition, SI seconds)	T_{sol}	86 400 s
Mean sidereal day (against the mean equinox)	T_{sid}	86 164.0905 s = 23 h 56 m 4.0905 s
Stellar day (rotation against the ICRF)	$2\pi/\omega_\oplus$	86 164.0989 s
Solar day minus sidereal day		235.909 s = 3 m 55.909 s
Rotation rate against the stars	$\omega_\oplus$	7.2921151×10^{-5} rad/s
Extra turn per day to catch the Sun	Δ	0.9856°

Sidereal day or stellar day? An 8 ms difference

The equinox, the reference for sidereal time, is not fixed among the stars: the Earth's axis precesses with a period of about 25 800 years, so the equinox slides westward along the ecliptic by 50.29″ per year. Projected on the equator, which is what matters for an hour angle, the drift in right ascension

is 46.1″ = 3.075 s of time per year. A clock following the *equinox* therefore loses 3.075/365.25 = 8.4 ms per day against one following the *stars*. That is exactly the difference between 86 164.0905 s and 86 164.0989 s. In practice: **sidereal time** (GMST) is the angle of the Greenwich meridian from the *equinox*; the **Earth rotation angle** (ERA) is the angle from a non-rotating origin fixed to the stars. The two differ by the accumulated precession in right ascension, which is 1.28° per century. The modern transformations use ERA; textbooks, and our Julian-date formulas in Section 3.1, use GMST.

2.2 What the difference does to spaceflight

Geostationary orbit. A satellite stays above one point of the equator only if it goes round in the time the Earth takes to turn once *relative to inertial space*, which is the sidereal (stellar) day and not 24 h. Kepler's third law $a^3 = \mu\,(T/2\pi)^2$ gives

$$T = 86\,164.0989\,\text{s} \;\Rightarrow\; a = 42\,164.17\,\text{km},\ h = 35\,786.03\,\text{km}; \qquad T = 86\,400\,\text{s} \;\Rightarrow\; a = 42\,241.10\,\text{km},\ h = 35\,862.96\,\text{km}.$$

The two orbits differ by only 76.9 km in radius. A satellite placed in the 24-hour orbit turns through 360° per 86 400 s while the Earth turns through 360.9856°: it falls behind by 0.9856° per day, drifting **west** at the rate at which the Sun moves along the ecliptic. After 60 days it is 59° from where it started, and after a year it has been all the way round (Figure 4). One degree of longitude on the geostationary ring is 736 km.

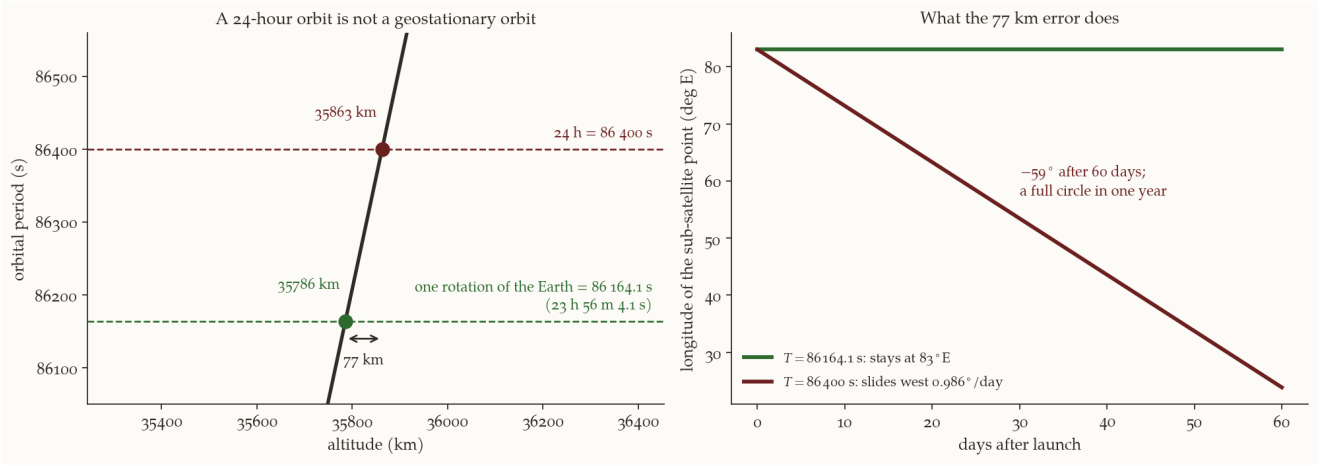


Figure 4: **Left:** orbital period near the geostationary altitude. The 24-hour orbit is 76.9 km higher than the geostationary orbit. **Right:** longitude of the sub-satellite point of a satellite launched to 83°E in each orbit. *Script: fig-time-days.py.*

Launch windows. To launch into a given orbital plane, the launch site must pass through that plane, and the plane is fixed in inertial space (to zeroth order). The site returns to the same inertial direction once per *sidereal* day, so an instantaneous launch window opens 3 m 56 s earlier each day. If the plane itself turns, as the orbit of the International Space Station does at $\dot\Omega = -4.95°/\text{day}$ because of J_2, the site meets the plane again after turning exactly 360° *relative to the plane*. Because the plane has meanwhile turned 4.95° westwards, that is less than a full turn against the stars: the Earth turns relative to the plane at $\omega_\oplus - \dot\Omega = 365.94°/\text{day}$, so the windows recur every $2\pi/(\omega_\oplus - \dot\Omega) = 23.61$ h, about 23 minutes earlier each day. Figure 5 draws it.

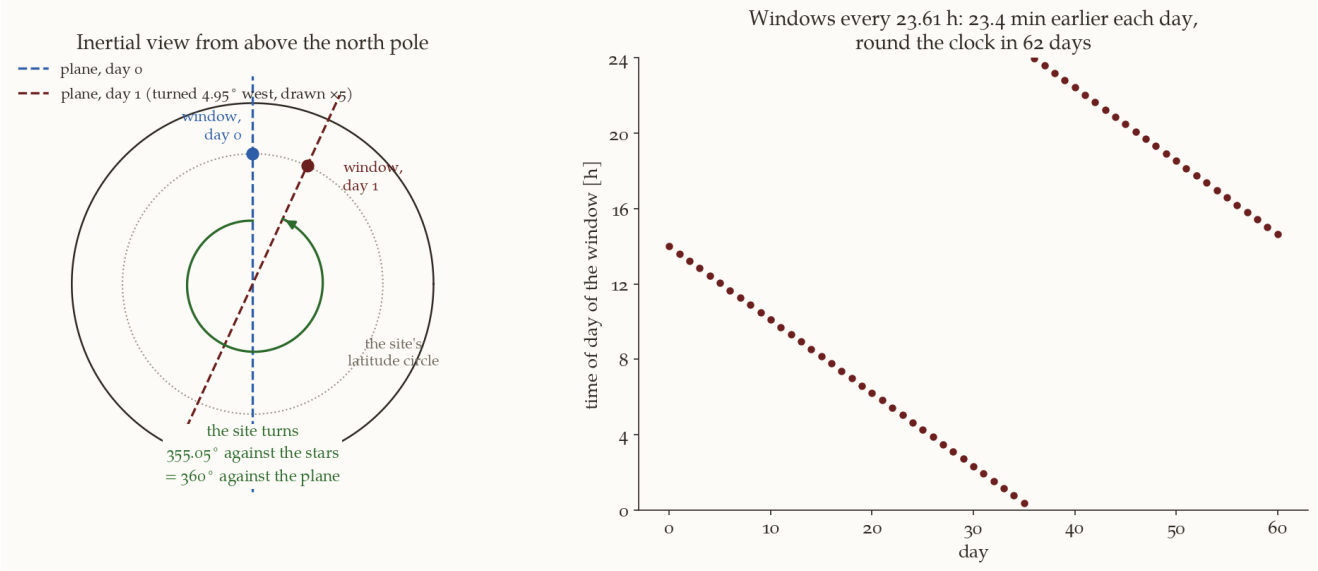


Figure 5: **Left:** the ISS plane seen from above the north pole, in inertial axes; the Earth and the launch site turn anticlockwise (eastwards). The window is where the plane meets the site's latitude circle. In a day the plane regresses 4.95° westwards (drawn five times larger), so the site meets it again after turning 355.05° against the stars, which is exactly 360° against the plane. **Right:** the time of day of the window, 23.4 min earlier each day. *Script: fig-time-launchwindow.py.*

Sun-synchronous orbits. The condition for an orbit to keep a fixed angle to the Sun is that its plane turns at 0.9856°/day, the rate at which the Sun's direction moves in Figure 2 (see orbital-elements note). For such an orbit the Earth turns through *exactly* 360° relative to the plane in one mean *solar* day, which is why repeat cycles of Earth-observation satellites are quoted in solar days.

Frames. The rotation between the inertial (ECI) and Earth-fixed (ECEF) frames is a rotation about the polar axis through the Greenwich sidereal angle. This is the only place where the Earth's rotation, and thus this section, enters a ground-track or ground-station calculation (ground-tracks note).

3 Time scales

3.1 Julian date and sidereal time in practice

The course text (Curtis, chapter 5) gives working formulas, valid for 1901 to 2099. The Julian date at 0 h UT of calendar date (y, m, d) is

$$J_0 = 367\,y - \mathrm{INT}\left\{\frac{7\left[y + \mathrm{INT}\left(\frac{m+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275\,m}{9}\right) + d + 1\,721\,013.5, \qquad \mathrm{JD} = J_0 + \frac{\mathrm{UT}}{24}, \tag{4}$$

with UT in hours. The Greenwich sidereal time at 0 h UT follows from $T_0 = (J_0 - 2\,451\,545)/36\,525$,

$$\theta_{G0} = 100.4606184^\circ + 36\,000.77004^\circ\,T_0 + 0.000387933^\circ\,T_0^2 - 2.583\times10^{-8}\,T_0^3, \tag{5}$$

and at a later universal time UT (hours),

$$\theta_G = \theta_{G0} + 360.98564724^\circ\,\frac{\mathrm{UT}}{24}. \tag{6}$$

The coefficient 360.98564724 is 360° times the number of sidereal days in a solar day, 1.00273790935, and is the rate of Equation 3 measured against the equinox. The local sidereal time at east longitude λ_E is $\theta_G + \lambda_E$; it equals the right ascension of the stars on your meridian.

Both formulas are implemented in _timeutil.py (the functions for the Julian date and for Greenwich mean sidereal time).

For example, J_0 for 2026 September 29 is 2 461 312.5, so noon UT that day is JD 2 461 313.0.

Worked example: the sidereal time at Bengaluru. Find the local sidereal time at Bengaluru (77.59°E) at 12:00 IST on 2026 September 29.

Solution

IST is UT +5.5 h, so UT = 6.5 h. From Equation 4, $J_0 = 2\,461\,312.5$, and $T_0 = (J_0 - 2\,451\,545)/36\,525 = 0.26741958$. Equation 5 gives $\theta_{G0} = 9727.7713°$, which is 27 whole turns plus 7.7713°. Equation 6 adds $360.98564724° \times 6.5/24 = 97.7669°$, so $\theta_G = 105.538°$. Adding the east longitude, the local sidereal time is $105.538° + 77.59° = 183.13°$, or 12 h 12.5 min. A star with right ascension 12 h 12 min is crossing the meridian at Bengaluru at local noon.

The sidereal-time formulas agree with astropy's IAU 1982 model to about 10^{-4} degrees once UT1 (not UTC) is used for the argument, and our leap-second table agrees with the ERFA table at more than a hundred dates between 1972 and 2026. The code is in _timeutil.py.

4 What a clock error costs: numerical experiments

We now put numbers on the question of how good a clock must be. All experiments use the planetary positions of a real ephemeris (the built-in one of astropy, good to arcseconds, ample for this purpose).

4.1 Two ways a clock enters navigation

Time tags. Every state, every ephemeris lookup, every command is stamped with an epoch. If the stamp is wrong by δt, Equation 1 applies directly.

Measurements. Deep-space navigation is done with radio signals, and each measurement is a comparison with a clock. A **two-way range** measures the round-trip time τ of a signal transmitted at t_t and received at t_r by the same ground clock:

$$\rho = \tfrac{1}{2}\, c\,(t_r - t_t) = \tfrac{1}{2} c\tau. \tag{7}$$

A **Doppler** measurement compares the frequency of the received carrier with that of the transmitted one, and gives the range rate $\dot\rho = -\tfrac{1}{2}\, c\,\Delta\nu/\nu$ for a two-way link ($\dot\rho = -c\,\Delta\nu/\nu$ one-way), with ν the carrier frequency. Therefore *range is a time measurement* and *range rate is a frequency measurement*: an error of one nanosecond in τ is 15 cm of range; a fractional error y in the frequency reference is a velocity error $c\,y$. (We write y for a fractional frequency error, as the clock literature does.)

4.2 Experiment 1: how far is Mars, in minutes?

The first thing a clock has to do at planetary distances is to bridge the light time. Figure 6 shows the one-way light time between Earth and Mars for the next five years, and the animation shows the geometry behind it. The light time varies between 5.6 and 5.4 min at the oppositions of 2027 and 2029 (0.68 and 0.65 AU) and 21.2 min at the conjunction of June 2030 (2.55 AU). Over several decades the extremes are 3.1 and 22.3 min (0.37 and 2.68 AU).

Consequences: there is no possibility of "flying" a spacecraft from Earth. A command is executed at a time *on the spacecraft clock* that was chosen at least one light time earlier, so the ground must know the relation between the spacecraft clock and UTC to far better than the accuracy demanded of the event. When

India's Mars Orbiter Mission was inserted into orbit on 24 September 2014 the distance was 1.499 AU (224 million km), the one-way light time 12.5 min and the round trip 24.9 min; the whole braking sequence had to be uploaded in advance.

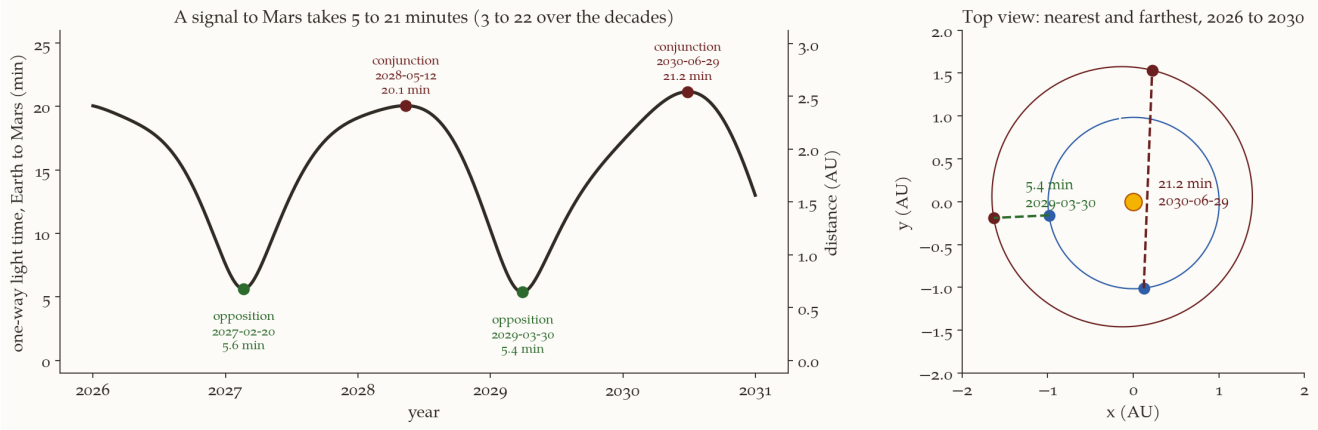


Figure 6: **Left:** one-way light time (and distance, right axis) between Earth and Mars, 2026 to 2030, from the ephemeris. **Right:** the geometry at the closest approach (5.4 min, 30 March 2029) and at the greatest distance (21.2 min, 29 June 2030) in the plotting window. *Script: fig-time-nav.py.*

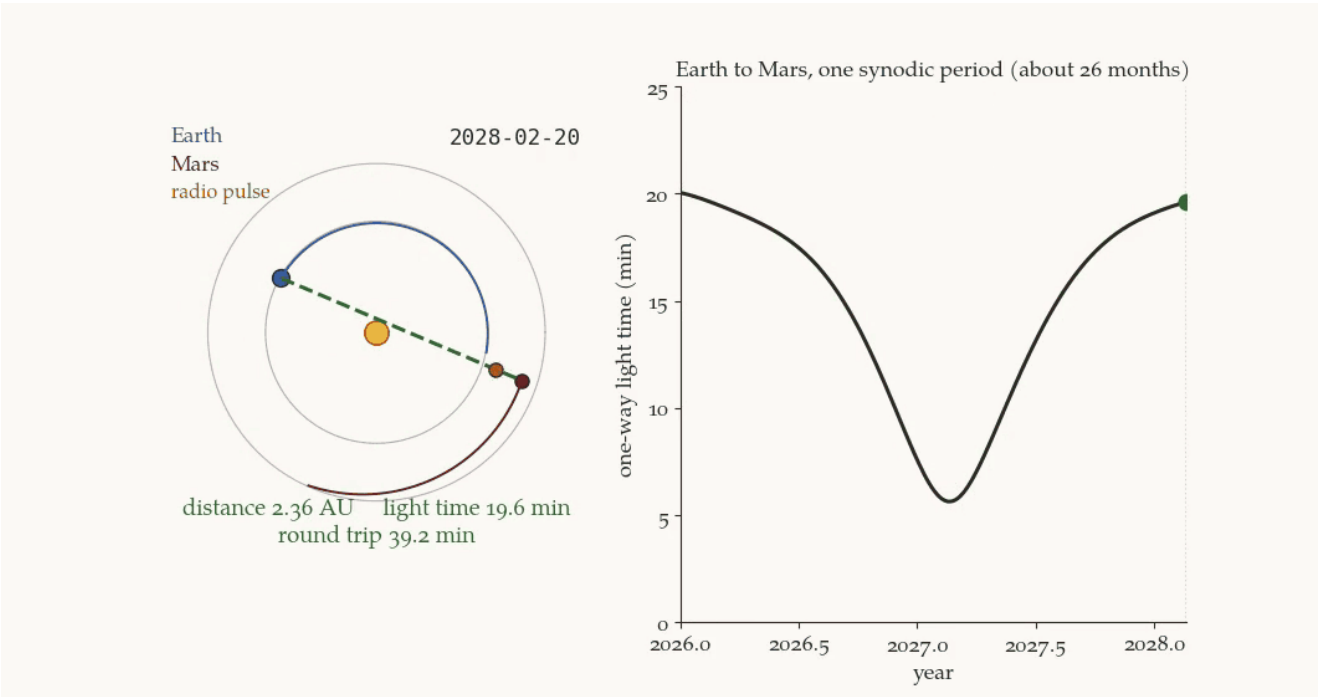


Figure 7: One synodic period (780 days) of the Earth–Mars geometry, with the light time. The dot moving along the line is a radio pulse at a schematic speed. *Script: anim-time.py.*

4.3 Experiment 2: a clock that is slightly wrong

Suppose a ground clock runs fast or slow by the fractional amount y. After a time t it is wrong by yt (Figure 8, left), and a ranging measurement that spans a round trip $\tau = 2\rho/c$ is wrong by $y\tau$, i.e. the range error is

$$\delta\rho = y\,\rho. \tag{8}$$

It is the *fractional* error of the clock that matters, and it multiplies the whole distance (Figure 8, right). Table 2 lists representative values for oscillator classes

(orders of magnitude of an uncalibrated frequency offset, not datasheet values).

Table 2: Oscillators, and what a fractional error y does to a two-way range at 1.5 AU. The last column is $y\rho$ with $\rho = 2.24 \times 10^{11}$ m.

Oscillator	y	Time error after 1 day	$\delta\rho$ at 1.5 AU
quartz crystal	10^{-6}	86 ms	224 km
oven-controlled quartz	10^{-9}	86 µs	224 m
rubidium	10^{-11}	0.86 µs	2.2 m
caesium beam	10^{-13}	8.6 ns	2.2 cm
hydrogen maser	10^{-15}	0.09 ns	0.22 mm
optical lattice (laboratory)	10^{-17}	0.9 ps	2 µm

To range to 1 m the round-trip time must be timed to $2\rho/c$ within 6.7 ns, and over the light time of Mars the clock must be better than $y = 4.5 \times 10^{-12}$ at 1.5 AU and 2.7×10^{-12} at 2.5 AU. A rubidium clock is marginal; a caesium clock or a hydrogen maser is comfortable. A quartz oscillator is hopeless: at 1.5 AU it would place Mars 224 km away from where it is.

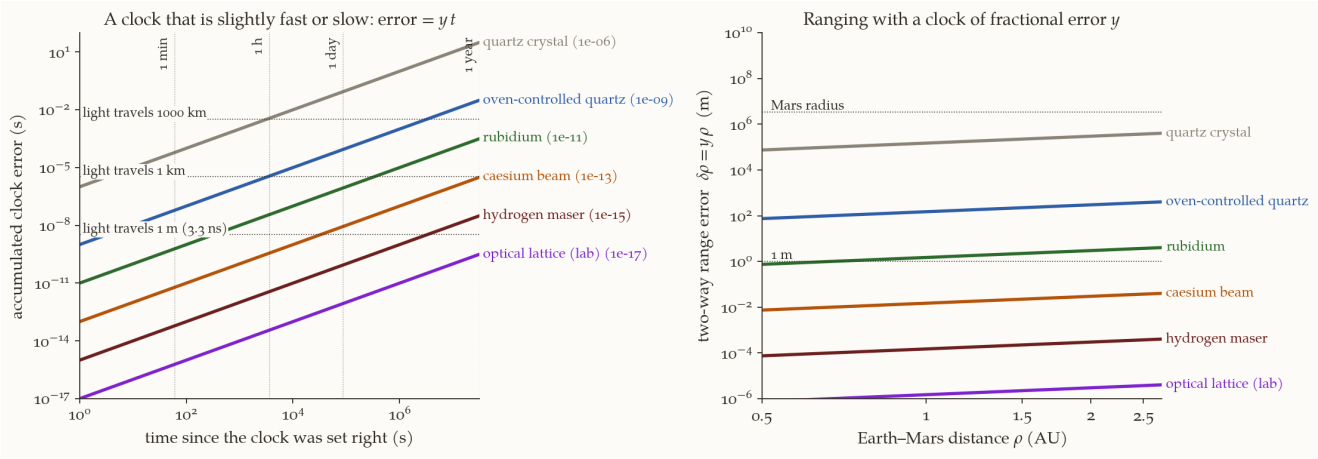


Figure 8: **Left:** accumulated error of clocks with the fractional errors of Table 2; the horizontal dotted lines show the time it takes light to travel 1 m, 1 km and 1000 km. **Right:** the range error $\delta\rho = y\rho$ over the Earth–Mars distance. *Script: fig-time-nav.py.*

In one paragraph. Position error is speed times time error (Equation 1); range is a time measurement and Doppler a frequency measurement. A day is 86 400 s of solar time but 86 164.09 s for one turn of the Earth, and the difference (3 m 56 s) explains the geostationary altitude, launch windows and sun-synchronous orbits. Time comes in three families: UT1 (rotation), atomic (TAI, UTC, GPS) and dynamical (TT, TDB); today TAI − UTC = 37 s, TT − UTC = 69.184 s and TDB differs from TT by ±1.66 ms, which follows from **?@eq-rate**. For a Mars mission range needs nanoseconds and velocity needs fractional frequency stability of order 10^{-13} or better, which is hydrogen-maser territory.

Summary and outlook

The numbers that follow from formulas in this note are checked by time-verify.py; the ephemeris experiments are computed in their figure scripts.

What we left out. (i) *Precession, nutation and polar motion.* The Earth's orientation is here a single rotation angle, which is all that the experiments need; the full IAU 2006/2000A transformation adds precession, nutation and polar motion, which move points on the ground by up to a few hundred metres. (ii) *The complete time scales*, in particular the SPICE system used by space agencies, which handles the conversion between spacecraft clock, UTC, TDB and ephemeris time with kernels.

Constants used in this note.

Table 3: The full list for all the notes is on the notation page.

Quantity	Symbol	Value
speed of light	c	299 792.458 km/s (exact)
astronomical unit	AU	149 597 870.700 km (exact)
Sun's gravitational parameter	$\mu_\odot$	$1.32712440018 \times 10^{11}$ km^3/s^2
Earth's gravitational parameter	$\mu_\oplus$	398 600.4418 km^3/s^2
Earth's equatorial radius	$R_\oplus$	6378.137 km
Earth's oblateness coefficient	J_2	1.08263×10^{-3}
Earth's rotation rate (against the stars)	$\omega_\oplus$	7.2921151×10^{-5} rad/s
mean sidereal day / stellar day	T_{sid}	86 164.0905 s / 86 164.0989 s
tropical / sidereal year		365.242189 d / 365.256363 d
obliquity of the ecliptic	ϵ	23.44°
Earth's orbital eccentricity	e	0.0167
TT − TAI		32.184 s

Further reading. H. D. Curtis, *Orbital Mechanics for Engineering Students* (Julian date, sidereal time); D. A. Vallado, *Fundamentals of Astrodynamics and Applications* (time scales, the IAU transformations); the IERS Conventions (2010) and the *Explanatory Supplement to the Astronomical Almanac* for time scales and the equation of time; N. Ashby, *Relativity in the Global Positioning System*, Living Reviews in Relativity 6 (2003), for the GPS clock corrections; J. Taylor and colleagues at JPL on deep-space Doppler tracking and Allan deviation.

Exercises

1. **Sidereal day.** Use Equation 2 with the tropical year 365.242189 d to show that $T_{\text{sid}} = 86\,164.0905$ s, and with the *sidereal* year 365.256363 d to obtain the stellar day to within about a millisecond. Why do the two answers differ, and how is the difference related to the 8.4 ms between the sidereal and the stellar day?

Solution. With the tropical year. Equation 2 gives, in days,

$$T = \frac{1}{1 + 1/365.242189} = 0.99726957 \text{ d} = 86\,164.0905 \text{ s}.$$

With the sidereal year.

$$T = \frac{1}{1 + 1/365.256363} = 0.99726967 \text{ d} = 86\,164.0997 \text{ s},$$

which is the stellar day of Table 1, 86 164.0989 s, to within 0.8 ms.

Why they differ. The year that goes into Equation 2 decides what the Earth's turn is measured against. The tropical year is the return of the Sun to the equinox. Because of precession the equinox slides westward to meet the Sun, so the tropical year is 20.4 min shorter than the sidereal year, which is the return to the same stars. The first answer is therefore the turn against the moving equinox, the sidereal day, and the second is the turn against the stars, the stellar day.

The two answers differ by 9.1 ms, a little more than the 8.4 ms between the sidereal and the stellar day. The 8.4 ms comes from the precession measured along the equator, 46.1″ a year, which is what an hour angle sees. The difference of the two years comes from the precession along the ecliptic, 50.29″ a year. Their ratio, $50.29/46.1 = 1.09$, is the ratio $9.1/8.4$. So Equation 2 with the sidereal year gives the stellar day only to within this small projection effect, the 0.8 ms we found.

Reading the result. These milliseconds do not matter for an orbit, but they show that "one turn of the Earth" must always say against what. For frame rotations the Earth rotation angle, against the stars, is the modern standard.

2. **Revolutions per day.** A satellite is in a circular orbit at 400 km altitude. How many revolutions does it make per solar day and per sidereal day? Take $\mu = 398\,600.4418 \text{ km}^3/\text{s}^2$ and $R_\oplus = 6378.137$ km.

Solution. The radius is $a = 6778.137$ km, so the period is

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 5553.6 \text{ s} = 92.56 \text{ min}.$$

In a solar day the satellite makes $86\,400/5553.6 = 15.56$ revolutions. In a sidereal day it makes $86\,164.09/5553.6 = 15.51$.

Reading the result. The difference, 0.04 revolution per day, is the 236 s by which the two days differ. It matters when we count revolutions against the ground. The Earth turns once relative to the stars in a sidereal day, so a ground track without J_2 repeats when a whole number of revolutions fits into a whole number of sidereal days, as in the ground-tracks note.

3. **The 24-hour orbit.** Derive the drift rate 0.9856°/day of a satellite in a circular equatorial orbit of period 86 400 s from the ratio of the periods, and its semi-major axis.

Solution. The drift. In one solar day of 86 400 s the satellite goes once round, through 360°. The Earth turns through $360° \times 86\,400/86\,164.0989 = 360.9856°$ in the same time. So each day the satellite falls behind the ground by

$$360°\Big(\frac{86\,400}{86\,164.0989} - 1\Big) = 0.9856°,$$

and it drifts **west**. This is the extra turn that the Earth makes to bring the Sun back to the meridian, Table 1.

The semi-major axis. Kepler's third law with $T = 86\,400$ s gives $a = (\mu T^2/4\pi^2)^{1/3} = 42\,241.10$ km, which is 76.9 km above the geostationary radius.

Reading the result. A satellite in a "24-hour" orbit goes round the Earth once a year relative to the ground. One degree on the geostationary ring is 736 km, so after two months it is thousands of kilometres from its slot. The difference of 77 km in radius between the two orbits is the whole mistake.

4. **Launch window.** The ISS has $\dot{\Omega} = -4.95^\circ$/day. By how many minutes does an instantaneous launch window move from one day to the next?

Solution. The window opens when the launch site passes through the orbit plane, so what counts is how fast the Earth turns **relative to the plane**. The Earth turns at 360.9856° per solar day against the stars (Exercise 3). The plane turns west at 4.95° per day, towards the site. Relative to the plane the Earth therefore turns at

$$\omega_\oplus - \dot{\Omega} = 360.9856^\circ + 4.95^\circ = 365.936^\circ \text{ per day.}$$

One full turn relative to the plane takes 360/365.936 of a day, which is 23.61 h. The window therefore comes $24 - 23.61 = 0.389$ h $= 23.4$ min earlier each day.

Reading the result. For a plane fixed in space the window would move only 3 min 56 s a day. For the station it moves six times faster, so a launch that slips by one day must be ready nearly half an hour earlier, and a slip of a week moves the window by almost three hours. Launch teams plan around this, and the ground-tracks note shows the same rotation of the plane in the drift of the ISS's local time.

5. **TDB.** Verify the amplitude 1.657 ms of TDB − TT from **?@eq-tdb**. What would the amplitude of the corresponding periodic term be for a clock on Mars ($a = 1.5237$ AU, $e = 0.0934$)?

Solution. The Earth. The amplitude in **?@eq-tdb** is $2\sqrt{\mu_\odot a}\, e/c^2$. With $\mu_\odot = 1.32712 \times 10^{11}$ km^3/s^2, $a = 1.495979 \times 10^8$ km, $e = 0.016709$ and $c = 299\,792.458$ km/s,

$$\sqrt{\mu_\odot a} = 4.4557 \times 10^9 \text{ km}^2/\text{s}, \qquad \frac{2 \times 4.4557 \times 10^9 \times 0.016709}{(299\,792.458)^2} = 1.6568 \times 10^{-3} \text{ s},$$

the 1.657 ms of the note.

Mars. The amplitude scales as $\sqrt{a}\, e$. So

$$1.6568 \text{ ms} \times \sqrt{1.5237} \times \frac{0.0934}{0.016709} = 1.6568 \times 1.2344 \times 5.590 = 11.4 \text{ ms}.$$

Reading the result. A clock on Mars would wander by about ±11 ms against the barycentre over a Martian year, seven times as much as a clock on the Earth. Most of the increase comes from the larger eccentricity of Mars, which makes its speed and its distance from the Sun change more over its orbit. A future Martian time scale would need this correction, just as TDB corrects TT.

6. **Ranging.** To what accuracy in ns must the round-trip time be measured for a 1 m range? Which clock class of Table 2 gives $\delta\rho < 1$ m at 2.5 AU?

Solution. The timing. By Equation 7, $\rho = \frac{1}{2}c\tau$, so $\delta\rho = \frac{1}{2}c\,\delta\tau$. For $\delta\rho = 1$ m,

$$\delta\tau = \frac{2\,\delta\rho}{c} = \frac{2}{2.998 \times 10^8} \text{ s} = 6.7 \text{ ns}.$$

The clock. By Equation 8 the range error from a fractional frequency error y is $y\rho$. At 2.5 AU $= 3.74 \times 10^{11}$ m we need $y < 1/(3.74 \times 10^{11}) = 2.7 \times 10^{-12}$.

From Table 2, a rubidium clock (10^{-11}) gives 3.7 m, which is marginal, while a caesium beam (10^{-13}) gives 3.7 cm and a hydrogen maser 0.4 mm. A caesium clock is sufficient.

Reading the result. Note the two separate demands. The timing electronics must resolve a few nanoseconds, and the clock that keeps the time must not drift by more than a few parts in 10^{12} over the whole round trip, which at 2.5 AU takes 42 minutes. The farther the spacecraft, the better the clock must be, because the error grows with the distance.

7. **Ledger.** By how many kilometres is a 500 km satellite displaced along its orbit if GPS time is mistaken for UTC?

Solution. GPS time is ahead of UTC by 18 s (**?@tbl-scales**). The circular speed at 500 km is $\sqrt{\mu/(R_\oplus + 500)} = 7.613$ km/s. By Equation 1 the satellite is placed along its track by

$$\delta r = v\,\delta t = 7.613 \times 18 = 137 \text{ km}.$$

Reading the result. The two clocks agree to the nanosecond in their seconds and differ only in the count, yet the mistake puts the satellite 137 km off. A ground station that pointed its antenna using the wrong time would miss a pass entirely, since 137 km at a range of a thousand kilometres is an angle of several degrees. Time-scale bookkeeping is one of the most common sources of large errors.