Plane Changes

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1 Plane changes

Suppose a spacecraft is on an orbit and we want to move it to an orbit in a different plane. Why would we need this? A geostationary satellite has to be in the equatorial plane in order to stay over one point on the Earth. But ISRO launches from Sriharikota, which is at a latitude of 13.7°, and the orbit a launcher reaches from there is inclined to the equator. In this note we will take that inclination to be 19.2°. So the plane has to be turned by this angle somewhere along the way.

We will first see why this angle is 19.2° and not simply the latitude, 13.7°. We will then answer three questions. Where can a plane change be made, and what does it cost? Can we reduce the cost by combining it with a burn that we have to make anyway? If two burns are available, how should the

turn be shared between them? We will then look at a different kind of plane change, in which the node moves and the inclination stays fixed.

This note follows the orbital-manoeuvres note, which derives the impulse formulas, and the Hohmann note, which derives the transfer used here. Throughout, $\mu = 398\,600.4418$ km^3/s^2 and $R_\oplus = 6378.137$ km. "LEO" means the circular orbit at 300 km altitude, $r_1 = 6678.137$ km, and "GEO" means the circular orbit of radius $r_2 = 42\,164$ km. On the Hohmann transfer between them, v_{c1} and v_{c2} are the circular speeds at r_1 and r_2, and v_{t1} and v_{t2} are the speeds on the transfer ellipse at the same two radii.

1.1 Why the inclination is not the latitude

Sriharikota is at a latitude of 13.7°, so why do we not simply take $\Delta i = 13.7°$? We will show that the latitude is only the lowest inclination that a launch can reach, and that it is reached only by a launch due east. ISRO's launches to transfer orbits do not head due east. We will find the inclination that their heading gives, and compare it with the inclination they actually reach.

1.1.1 The heading sets the inclination

At launch, the orbital plane contains the launch site and the centre of the Earth. A third piece of information then fixes the plane, which is the direction in which the rocket flies. Let φ be the latitude of the site. Let A be the launch **azimuth**, which is the heading of the rocket measured clockwise from north, so that due east is $A = 90°$.

We wish to find the inclination in terms of φ and A. At the launch site, take the unit vectors $\hat{\mathbf{r}}$ pointing up, $\hat{\mathbf{e}}$ pointing east and $\hat{\mathbf{n}}$ pointing north. These form a right-handed set, so $\hat{\mathbf{r}} \times \hat{\mathbf{e}} = \hat{\mathbf{n}}$ and $\hat{\mathbf{r}} \times \hat{\mathbf{n}} = -\hat{\mathbf{e}}$. The horizontal direction of flight is

$$\hat{\mathbf{u}} = \sin A\, \hat{\mathbf{e}} + \cos A\, \hat{\mathbf{n}}.$$

Any vertical part of the velocity stays in the same plane, so it does not matter here. The normal to the orbital plane is

$$\hat{\mathbf{h}} = \hat{\mathbf{r}} \times \hat{\mathbf{u}} = \sin A\, \hat{\mathbf{n}} - \cos A\, \hat{\mathbf{e}}.$$

The Earth's spin axis, in the same three vectors, is $\hat{\mathbf{K}} = \sin\varphi\, \hat{\mathbf{r}} + \cos\varphi\, \hat{\mathbf{n}}$. The inclination is the angle between $\hat{\mathbf{h}}$ and $\hat{\mathbf{K}}$, so $\cos i = \hat{\mathbf{h}} \cdot \hat{\mathbf{K}}$. Only the $\hat{\mathbf{n}}$ terms survive the dot product, which gives

$$\boxed{\cos i = \cos\varphi\, \sin A.} \tag{1}$$

This answers the question. Since $\sin A$ cannot exceed 1, we have $\cos i \leq \cos \varphi$, and therefore $i \geq \varphi$. The two are equal only when $\sin A = 1$, which is a launch due east. So the latitude is a lower limit on the inclination, and any heading away from due east gives an inclination above it.

1.1.2 Why the rocket does not head due east

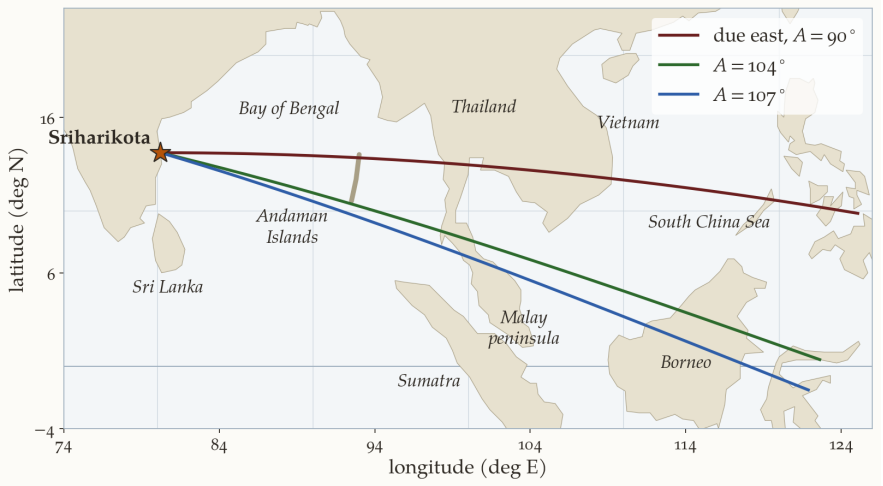


Figure 1: Launch tracks from Sriharikota on three headings, drawn as great circles with the Earth's rotation during the ascent ignored. *Script: fig-man-plane-launch-map.py.*

A launcher drops its spent stages along its path, and it can fail during the ascent. So its path is chosen to keep it away from populated land while it is low, and to let the stages fall into the open sea. For this reason, every launch site has a limited range of allowed headings.

Figure 1 shows the choice at Sriharikota. A track due east, in maroon, passes over the Andaman Islands and then runs along the length of Thailand, Cambodia and Vietnam. The tracks about 15° south of east, in green and blue, stay over the sea for much longer. They cross the narrow Malay peninsula and do not reach land again until Borneo, far downrange.

We should be careful about how much we read into this map. It suggests why these headings are preferred, but the mission documents that we use below give the heading and not the reasons for it. What we can test is the effect of the heading on the inclination.

So we now test Equation 1 against real flights, using the heading and inclination that ISRO publishes for each mission.

Table 1: The launch heading and the inclination of two flights from Sriharikota to transfer orbits.

Mission	Heading A	i from Equation 1	i published by ISRO
GSLV-F12	104°	19.49°	19.36°
LVM3-M5	107°	21.71°	21.4°

The formula agrees with both flights to within a few tenths of a degree. It leaves out the Earth's rotation and the steering of the rocket during the ascent, so we should not expect exact agreement. The heading alone therefore explains why the inclination is well above 13.7°. Flights from Sriharikota have reached inclinations from about 19° up to 21.7°, depending on the mission. In this note we will use 19.2°, which is a representative value near the low end of this range.

For a launch from a latitude of $\varphi = 13.7°$, use $\cos i = \cos\varphi \sin A$ to find the heading A that gives $i = 19.2°$. Choose the solution that heads south of east.

What you should find

We have $\sin A = \cos 19.2° / \cos 13.7° = 0.9720$. This has two solutions, $A = 76.4°$ and $A = 103.6°$, which give the same inclination. The one south of east is $A = 103.6°$, close to the heading of GSLV-F12.

You may have read that rockets from Sriharikota make a turn called a *dog-leg* to avoid Sri Lanka. That turn belongs to a different kind of launch. Sri Lanka lies south of Sriharikota, so it is in the way of southbound launches to polar orbits. Launches to transfer orbits head east-south-east, away from it. The agreement in Table 1 shows that these launches do not need a large dog-leg to explain their inclination.

What does this heading cost a mission? It gives up a little of the Earth's rotation, which helps a launch by the speed $\omega_\oplus R_\oplus \cos\varphi \sin A$ along the track. This is about 13 m/s less at $A = 104°$ than due east. It also leaves a larger turn to make later. The exercise at the end of this note finds the cost of that larger turn, after we have the tools to do so.

1.2 The cost of turning the plane

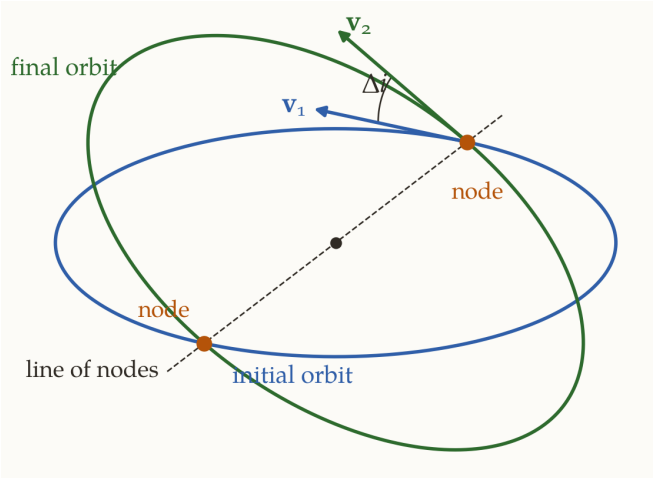


Figure 2: The initial orbit (blue) and the final orbit (green), whose planes differ by the angle Δi. The dashed line is the line of nodes, where the two planes meet. The arrows show the velocity on each orbit at one of the nodes. *Script: fig-man-plane-sketches.py*.

Figure 2 shows the initial orbit in blue and the desired final orbit in green. Each orbital plane passes through the centre of the Earth, so the two planes meet along a straight line through the centre. This is called the **line of nodes**.

Where can the burn be made? An impulse changes the velocity of the spacecraft but not its position. So the burn has to be made at a point that lies on both orbits. Such a point has to lie in both planes, and therefore on the line of nodes. In Figure 2 these points are the two nodes.

What does the burn have to do? At a node, it has to turn the velocity from $\mathbf{v}_1$ to $\mathbf{v}_2$. We want to find the cost of this turn. We will show that it is proportional to the speed, and this will tell us where to make the plane change.

1.2.1 The velocity triangle

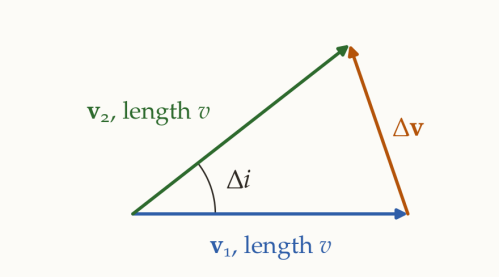


Figure 3: A turn of the velocity through the angle Δi at constant speed. *Script: fig-man-plane-sketches.py*.

Consider an orbit where the velocity at the node is perpendicular to the line of nodes. This is true on a circular orbit, and at a node that is also an apsis. Suppose the speed v is to stay the same. Then $\mathbf{v}_1$, $\mathbf{v}_2$ and $\Delta\mathbf{v}$ form the isosceles triangle of Figure 3. Its two equal sides have length v and the angle between them is Δi. The law of cosines gives

$$\Delta v^2 = v^2 + v^2 - 2v^2 \cos \Delta i = 2v^2(1 - \cos \Delta i).$$

Using $1 - \cos \Delta i = 2 \sin^2(\Delta i/2)$, this becomes $\Delta v^2 = 4v^2 \sin^2(\Delta i/2)$. Taking the square root gives the cost of the plane change.

$$\boxed{\Delta v = 2v \sin \frac{\Delta i}{2}.} \tag{2}$$

Notice the direction of this impulse. The base angles of the triangle are each $90° - \Delta i/2$, so $\Delta\mathbf{v}$ makes an angle of $90° + \Delta i/2$ with $\mathbf{v}_1$. The engine does not fire exactly sideways. It also points a little backward. A burn exactly at right angles to $\mathbf{v}_1$ would make the spacecraft go faster, and we want its speed to stay the same.

What if the velocity at the node is not perpendicular to the line of nodes? This happens on an eccentric orbit when the node is not at an apsis. The line of nodes is a radial line, so the velocity then has a component v_r along it, and turning the plane about this line leaves v_r unchanged. Only the transverse component v_θ turns. The same triangle then has equal sides of length v_θ, and the cost is $2v_\theta \sin(\Delta i/2)$. Every example below has its node at an apsis or on a circle, so Equation 2 applies directly.

What does a 60° plane change cost, according to Equation 2? What does a 180° plane change cost, and what does that turn do to the orbit?

Answer

For 60° the cost is $2v \sin 30° = v$, which is the whole orbital speed. For 180° the cost is $2v \sin 90° = 2v$. This turn reverses the direction of motion, which means stopping the spacecraft and then accelerating it to the same speed the other way.

1.2.2 Where to make the turn

Equation 2 says that the cost is proportional to v. So the turn is cheapest where the spacecraft is slowest. This is the opposite of what we found for energy, where a burn is most effective at high speed. A burn that adds energy belongs where the speed is high, and a burn that turns the plane belongs where the speed is low.

Example. A launch from Sriharikota leaves a satellite in LEO with an inclination of 19.2°. The satellite is wanted in GEO, in the equatorial plane. Find the cost of the 19.2° plane change if it is made in LEO, in GEO, and at the apoapsis of the Hohmann transfer ellipse between them.

Solution. The speed in each place is different, so we find the three speeds first. The two circular speeds are

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} = 7.726 \text{ km/s}, \qquad v_{c2} = \sqrt{\frac{\mu}{r_2}} = 3.075 \text{ km/s}.$$

The transfer ellipse has $a_t = (r_1 + r_2)/2 = 24\,421$ km, and vis-viva gives its speed at apoapsis.

$$v_{t2} = \sqrt{\mu \left(\frac{2}{r_2} - \frac{1}{a_t} \right)} = 1.608 \text{ km/s}.$$

All three places have the velocity perpendicular to the line of nodes, provided that the apse line of the transfer ellipse lies along the line of nodes. So we can use Equation 2 with $\Delta i = 19.2°$.

Where	v (km/s)	$\Delta v = 2v \sin 9.6°$ (km/s)
LEO	7.726	2.577
GEO	3.075	1.026
Apoapsis of the transfer ellipse	1.608	0.536

The same turn is 4.8 times cheaper at the best of these places than at the worst. Apoapsis of the transfer ellipse is the best, because the speed there is lower than the speed on either circle.

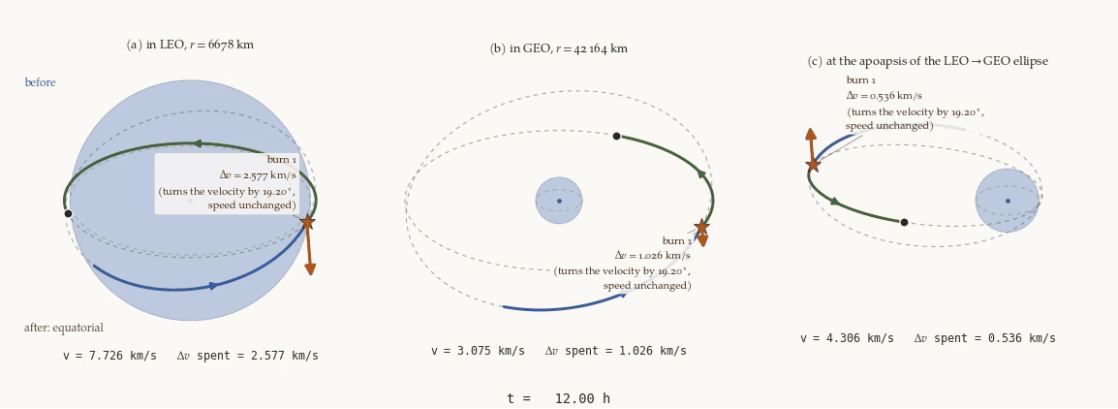


Figure 4: The same 19.2° plane change in the three places of this example, on one clock. The orbit before the burn is blue and the equatorial orbit after it is green. Each burn is made at a node and turns the velocity at constant speed. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-man.py, with the engine _manim.py.*

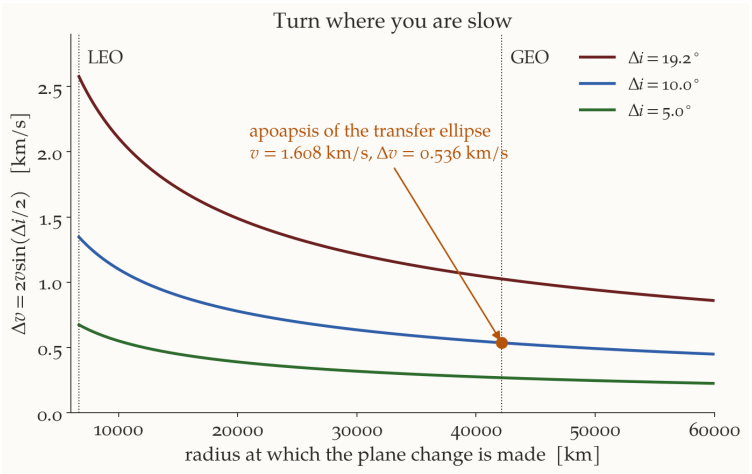


Figure 5: The cost of a plane change against the radius at which it is made, for three values of Δi. The dot is the apoapsis of the transfer ellipse from the example, with $\Delta i = 19.2°$. It lies below the 19.2° curve because the speed on the ellipse there is lower than the circular speed at that radius. *Script: fig-man-plane-change.py.*

What does this mean for a mission? Figure 5 shows that the cost falls quickly as the radius grows, so a turn in low orbit is very expensive. Take the 60° turn of the check above. In LEO it costs the whole orbital speed, 7.726 km/s. With the rocket equation and a typical chemical engine with $I_{sp} = 320$ s, only 8.5% of the spacecraft's mass would remain after the burn. This is why missions avoid large plane changes in low orbit, and launch close to the plane they want.

The speeds in LEO and GEO are 7.726 km/s and 3.075 km/s. What does a plane change of only 5° cost in each orbit, using $\Delta v = 2v \sin(\Delta i/2)$? Is the difference between the two orbits still large?

What you should find

The costs are $2(7.726) \sin 2.5° = 0.674$ km/s in LEO and $2(3.075) \sin 2.5° = 0.268$ km/s in GEO. Even a small turn is expensive in LEO. The ratio is the ratio of the speeds, 2.5, for any angle.

1.3 Combining the plane change with a transfer burn

So far we have treated the plane change as a separate burn. But a transfer from LEO to GEO has two burns that we must make anyway. Can we turn the plane during one of them? If we do, there is one impulse instead of two, and we pay for the size of a sum of two vectors instead of the sum of their sizes. The size of a sum is never greater than the sum of the sizes. So at any one point, combining the two burns can only help. We will now find out how much it helps, and at which point we should do it.

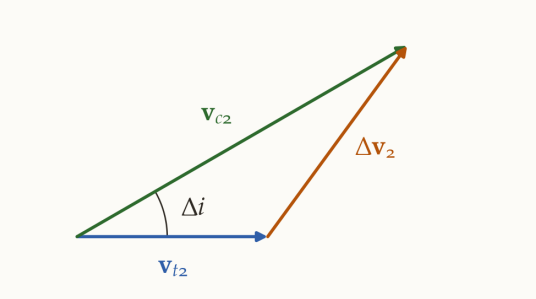


Figure 6: The burn at GEO when it also turns the plane. The spacecraft arrives with velocity $\mathbf{v}_{t2}$ and has to leave with velocity $\mathbf{v}_{c2}$ in the new plane. *Script: fig-man-plane-sketches.py.*

Figure 6 shows the burn at GEO. The spacecraft arrives with the speed v_{t2} and leaves with the speed v_{c2}, and the angle between the two velocities is Δi. The two sides of the triangle are now of different lengths, but the law of cosines still applies.

$$\boxed{\Delta v_2 = \sqrt{v_{t2}^2 + v_{c2}^2 - 2v_{t2}v_{c2}\cos\Delta i}\,.} \tag{3}$$

We can check this formula in two limits. If $\Delta i = 0$, the square root becomes $v_{c2} - v_{t2}$, which is the ordinary tangential burn. If $v_{t2} = v_{c2} = v$, it becomes $2v \sin(\Delta i/2)$, which is Equation 2.

Example. A spacecraft launched from Sriharikota goes from LEO to GEO with $\Delta i = 19.2^\circ$. Compare the total Δv of three strategies. In strategy (a) it flies a Hohmann transfer and then makes a separate plane change in GEO. In strategy (b) it makes the whole plane change in the LEO burn. In strategy (c) it makes the whole plane change in the GEO burn. Which is the cheapest?

Solution. We need the speed v_{t1} at the periapsis of the transfer ellipse as well as the three speeds of the previous example. Vis-viva gives

$$v_{t1} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_t}\right)} = 10.151 \text{ km/s}.$$

The tangential burns without any turn are $v_{t1} - v_{c1} = 2.42573$ km/s at LEO and $v_{c2} - v_{t2} = 1.46682$ km/s at GEO. We keep five decimals in these sums, so that the rounded values add up correctly.

In strategy (a), we add the separate plane change of the previous example to these two burns.

$$\Delta v = 2.42573 + 1.46682 + 1.02552 = 4.91807 \text{ km/s}.$$

In strategy (b), the LEO burn turns the velocity as well. Equation 3, with the speeds at LEO, gives $\sqrt{7.726^2 + 10.151^2 - 2(7.726)(10.151)\cos 19.2^\circ} = 3.82218$ km/s, so

$$\Delta v = 3.82218 + 1.46682 = 5.28900 \text{ km/s}.$$

In strategy (c), the GEO burn turns the velocity. Equation 3 gives $\sqrt{1.608^2 + 3.075^2 - 2(1.608)(3.075)\cos 19.2^\circ} = 1.64363$ km/s, so

$$\Delta v = 2.42573 + 1.64363 = 4.06936 \text{ km/s}.$$

Strategy (c) is the cheapest. It saves 0.849 km/s, or about 17%, against strategy (a). Strategy (b) is the worst of the three, even though it combines two burns. It makes the turn where the speeds are highest, and this costs more than a separate turn at the low speed of GEO. So where we make the turn matters more than whether we combine it.

What is this saving worth? Table 3 uses the rocket equation with $I_{sp} = 320$ s, for the idealised case of one engine making all the burns. Strategy (c) delivers 1.31 times as much mass to GEO as strategy (a). The way we arrange the burns has made a large difference, with no new hardware at all.

Table 3: The three strategies for LEO to GEO with $\Delta i = 19.2°$.

Strategy	Total Δv (km/s)	Mass left in GEO
(a) Separate plane change in GEO	4.918	20.9%
(b) Whole turn in the LEO burn	5.289	18.5%
(c) Whole turn in the GEO burn	4.069	27.3%

1.3.1 Why launch sites near the equator are valuable

We can now see why the latitude of a launch site matters so much for geostationary missions. A launch from a site close to the equator leaves a smaller turn to be made at GEO. How does Sriharikota compare with other sites? Table 4 uses strategy (c) with the inclinations at which transfer orbits typically start from three launch sites.

Table 4: The whole turn in the GEO burn, for three launch sites.

Launch site	Δi	GEO burn (km/s)	Total Δv (km/s)	Mass left in GEO
Kourou	7°	1.492	3.917	28.7%
Sriharikota	19.2°	1.644	4.069	27.3%
Cape Canaveral	28.5°	1.830	4.256	25.8%

The same spacecraft, launched from Sriharikota, arrives in GEO with 1.06 times the mass it would have from Cape Canaveral. Launched from Kourou, it would arrive with 1.05 times the mass it has from Sriharikota. That extra mass can be payload, or propellant for a longer life on orbit. The site of a launch is therefore a real part of the design of the mission.

1.4 Sharing the turn between two burns

Strategy (c) put the whole turn into the GEO burn. Is that the best we can do? So the natural question is, should a little of the turn go into the LEO burn as well? We will show that it should, and find how much.

Let the LEO burn turn the velocity by an angle δ and let the GEO burn turn it by the rest, $\Delta i - \delta$. Each burn is then a triangle like Figure 6, so the total cost is

$$\Delta v(\delta) = \underbrace{\sqrt{v_{c1}^2 + v_{t1}^2 - 2v_{c1}v_{t1}\cos\delta}}_{\Delta v_1,\ \text{at } r_1} + \underbrace{\sqrt{v_{t2}^2 + v_{c2}^2 - 2v_{t2}v_{c2}\cos(\Delta i - \delta)}}_{\Delta v_2,\ \text{at } r_2}.$$

To find the best δ, we differentiate each square root. The derivative of $\sqrt{X}$ is $X'/(2\sqrt{X})$. For the first term, $X' = 2v_{c1}v_{t1}\sin\delta$. For the second term, the chain rule brings in a minus sign, because δ appears as $\Delta i - \delta$, so $X' = -2v_{t2}v_{c2}\sin(\Delta i - \delta)$. Therefore

$$\frac{d\Delta v}{d\delta} = \frac{v_{c1}v_{t1}\sin\delta}{\Delta v_1} - \frac{v_{t2}v_{c2}\sin(\Delta i - \delta)}{\Delta v_2}.$$

The best share makes this derivative zero, which gives the condition

$$\boxed{\frac{v_{c1}v_{t1}\sin\delta}{\Delta v_1} = \frac{v_{t2}v_{c2}\sin(\Delta i - \delta)}{\Delta v_2}.} \tag{4}$$

Should δ be zero, as in strategy (c)? We can settle this without solving anything.

The derivative of the total cost is $d\Delta v/d\delta = v_{c1}v_{t1}\sin\delta/\Delta v_1 - v_{t2}v_{c2}\sin(\Delta i - \delta)/\Delta v_2$. Evaluate it at $\delta = 0$. What does its sign tell you?

Answer

At $\delta = 0$ the first term vanishes and the second is negative, so $d\Delta v/d\delta < 0$. Moving a little of the turn into the first burn therefore always reduces the cost.

Why does this happen? The LEO burn is mainly a long tangential vector, of length $v_{t1} - v_{c1}$. A small tilt adds a short vector at right angles to it. By Pythagoras, this changes its length only at second order in δ. The saving at GEO is first order in δ. For a small enough δ the saving must win.

Example. A spacecraft launched from Sriharikota goes from LEO to GEO with $\Delta i = 19.2°$. Find the share δ of the turn that should be made in the LEO burn, and the total cost. How much does this save against strategy (c)?

Solution. Equation 4 has no closed-form solution, so we solve it numerically. We find $\delta^* = 1.64°$, which leaves $17.56°$ for the GEO burn. To check this, we evaluate the two burns.

$$\Delta v_1 = 2.43901 \text{ km/s}, \qquad \Delta v_2 = 1.61619 \text{ km/s}.$$

With these values both sides of Equation 4 equal 0.92263 km/s, as they should. The total is 4.05520 km/s, against 4.06936 km/s for strategy (c). The saving is 14.2 m/s.

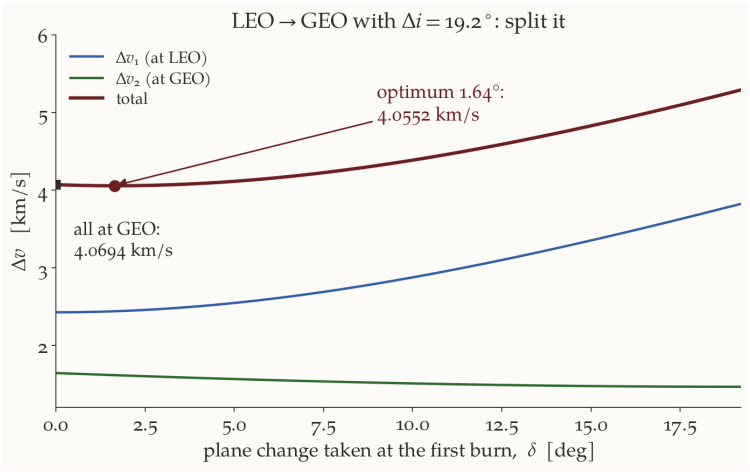


Figure 7: The total cost and the two burns against the share δ of the turn made in the first burn. The minimum of the total is at $\delta = 1.64°$, and it is shallow. *Script: fig-man-plane-change.py*.

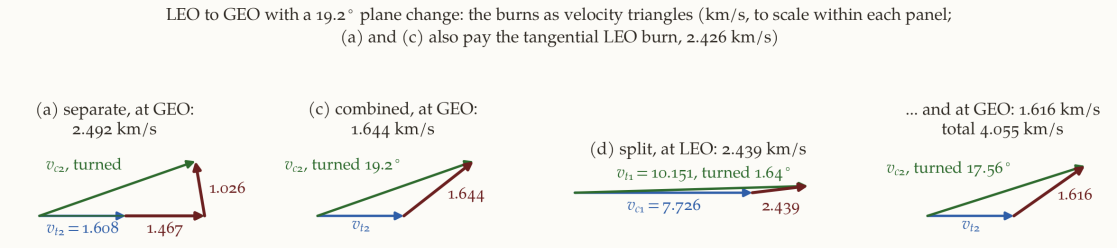


Figure 8: The burns as velocity triangles, for strategy (a) in the first panel, strategy (c) in the second, and the shared turn, which we call strategy (d), in the last two. In strategy (d) the LEO burn grows by only 13 m/s, while the GEO burn falls by 27 m/s. *Script: fig-man-plane-triangles.py*.

Figure 7 shows that the minimum is shallow, so the exact value of δ is not critical. Figure 8 shows the reason for the saving. In the LEO burn, the small turn lengthens the burn by only 13 m/s. In the GEO burn, it shortens the burn by 27 m/s.

Is a saving of 14.2 m/s worth having? It increases the final mass by only about 0.5%. But it costs nothing, since it needs no extra burn and no extra hardware. It only changes the direction of a burn that is being made anyway. Sharing the turn in this way is standard practice on missions to geostationary orbit.

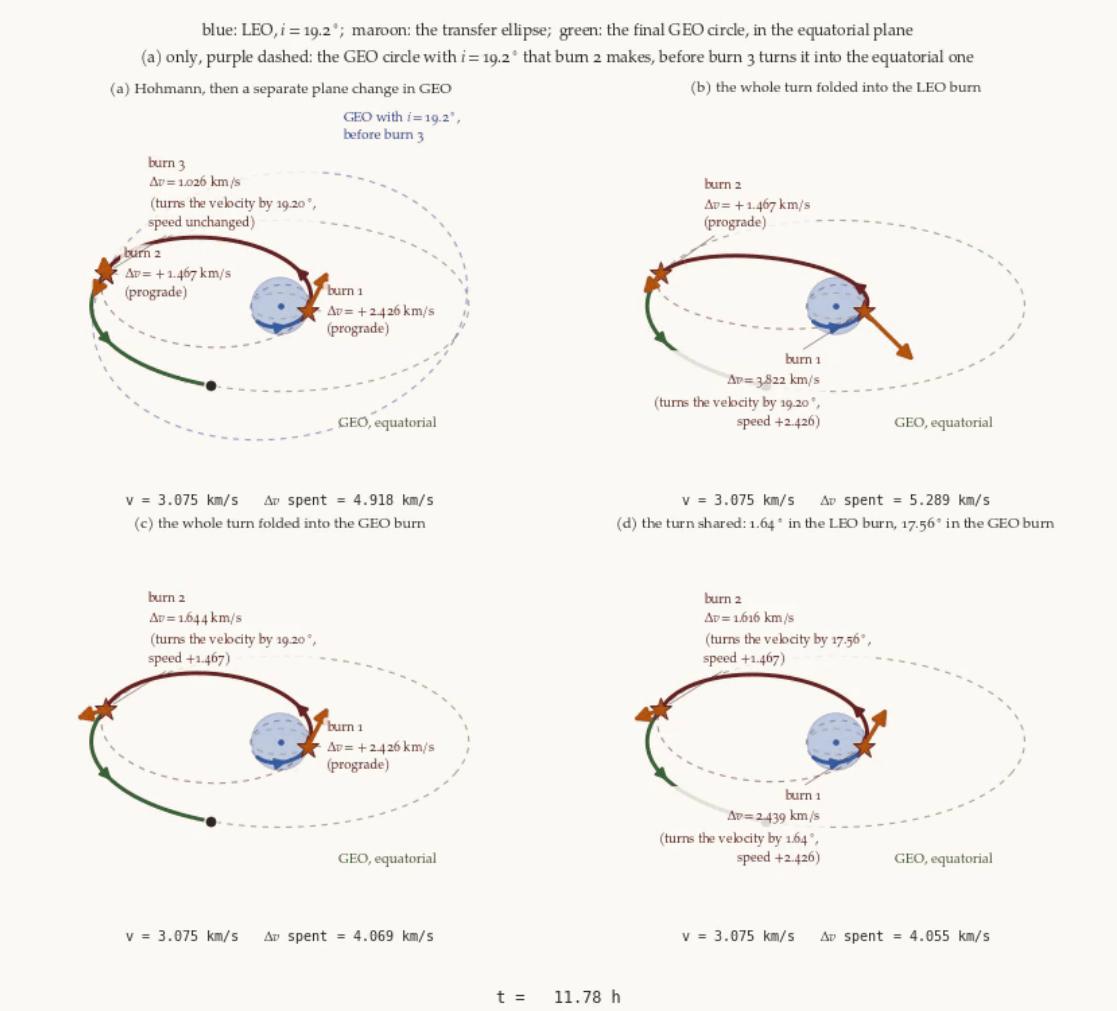


Figure 9: The four strategies on one clock, ending on the equatorial GEO circle. (**a**) A Hohmann transfer, then a separate plane change in GEO. (**b**) The whole turn in the LEO burn. (**c**) The whole turn in the GEO burn. (**d**) The turn shared between the two burns. The dashed purple circle in (a) is the GEO radius at the old inclination. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-man.py, with the engine _manim.py.*

1.4.1 How the best share depends on the size of the turn

We solved for one value of Δi. Can we find a simple rule for any small turn? We will expand each square root for small angles. Rewrite the expression under the first root as

$$v_{c1}^2 + v_{t1}^2 - 2v_{c1}v_{t1}\cos\delta = (v_{t1} - v_{c1})^2 + 2v_{c1}v_{t1}(1 - \cos\delta).$$

For a small δ in radians, $1 - \cos\delta \approx \delta^2/2$. For a small ϵ, we also have

$\sqrt{X^2+\epsilon} \approx X + \epsilon/(2X)$. Together these give

$$\Delta v_1 \approx (v_{t1} - v_{c1}) + \frac{1}{2}\,\frac{v_{c1}v_{t1}}{v_{t1}-v_{c1}}\,\delta^2.$$

The GEO burn works in the same way, with $\Delta i - \delta$ in place of δ. So the total cost is

$$\Delta v \approx \text{const} + \frac{A}{2}\,\delta^2 + \frac{B}{2}\,(\Delta i - \delta)^2, \qquad A = \frac{v_{c1}v_{t1}}{v_{t1}-v_{c1}}, \quad B = \frac{v_{t2}v_{c2}}{v_{c2}-v_{t2}}.$$

The constants A and B measure how quickly each burn grows when it is tilted. Setting the derivative with respect to δ to zero gives $A\delta = B(\Delta i - \delta)$, and therefore

$$\delta^* \approx \frac{B}{A+B}\,\Delta i \qquad \text{for small } \Delta i.$$

For the transfer from LEO to GEO, the speeds of our examples give $A = 32.33$ km/s and $B = 3.370$ km/s, so $\delta^* \approx 0.0944\,\Delta i$. Only about a tenth of a small turn goes into the first burn. The reason is that A is about 9.6 times B. Both speeds are high at LEO, so the product $v_{c1}v_{t1} = 78.4$ km^2/s^2 is about sixteen times the product $v_{t2}v_{c2} = 4.94$ km^2/s^2 at GEO. The same tilt therefore costs much more at LEO.

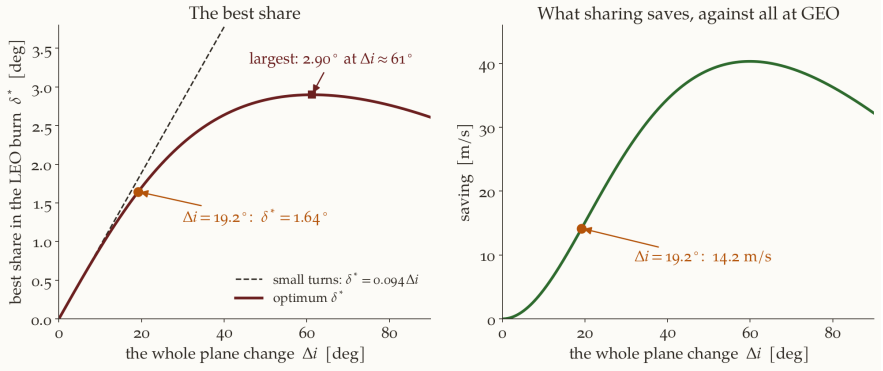


Figure 10: **Left:** the best share δ^* against the whole turn Δi, for the transfer from LEO to GEO. The dashed line is the small-turn result. **Right:** what the best share saves against putting the whole turn in the GEO burn. *Script: fig-man-plane-change.py.*

The left panel of Figure 10 shows the exact optimum, found numerically, for turns from 0 to 90°. It follows the dashed small-turn line at first and then

falls below it. At $\Delta i = 19.2°$ the small-turn rule predicts 1.81° and the exact value is 1.64°, which is 8.6% of the turn. The largest value of δ^* is 2.90°, at about $\Delta i = 61°$, and after that it declines slowly.

The right panel shows the saving. It grows from 14.2 m/s at 19.2° to about 40 m/s near 60°. So for any plane change on this transfer, sharing the turn is a small adjustment to the aim of the burns. Most of the benefit comes from the step we took earlier, which is making the turn in the GEO burn.

1.5 Changing the node instead

So far we have changed the inclination. A mission may instead need to change the right ascension of the ascending node Ω and keep the inclination i fixed. Two satellites in a constellation, for example, can have the same inclination and different nodes. This also turns the orbital plane. We wish to find the angle through which the plane turns, because Equation 2 then gives the cost.

The angle α between two planes is the angle between their normals. The unit normal of an orbit with inclination i and node Ω is $\hat{\mathbf{h}} = (\sin i \sin\Omega,\ -\sin i \cos\Omega,\ \cos i)$, as we saw in the orbital-elements note. Both orbits have the same i. So their dot product is

$$\cos\alpha = \hat{\mathbf{h}}_1 \cdot \hat{\mathbf{h}}_2 = \sin^2 i\,(\sin\Omega_1 \sin\Omega_2 + \cos\Omega_1 \cos\Omega_2) + \cos^2 i.$$

The bracket is $\cos(\Omega_1 - \Omega_2)$, which we write as $\cos\Delta\Omega$. This gives the angle, and Equation 2 gives the cost.

$$\boxed{\cos\alpha = \cos^2 i + \sin^2 i \cos\Delta\Omega, \qquad \Delta v = 2v \sin\frac{\alpha}{2}.} \tag{5}$$

The burn is again made where the two planes intersect. On a circular orbit, the velocity there is perpendicular to the line of intersection, so it turns through the full angle α. On an eccentric orbit, we would replace v by v_θ, as we found in Section 1.2.1, so Equation 5 then gives too high a cost.

Using $\cos\alpha = \cos^2 i + \sin^2 i \cos\Delta\Omega$, find α for a polar orbit ($i = 90°$) and for an orbit that is nearly equatorial ($i \to 0$). Do the results make sense?

Answer

For $i = 90°$, $\cos\alpha = \cos\Delta\Omega$, so $\alpha = \Delta\Omega$. The planes of all polar orbits meet at the poles, and changing the node is then the whole turn. For $i \to 0$, $\cos\alpha \to 1$, so $\alpha \to 0$. An equatorial orbit has no node to move.

Example. A space station orbits at $r = 6778$ km with $i = 51.6°$. Find the cost of changing its node by $10°$ with a burn. Then find the time the same change would take if we simply waited for the Earth's oblateness to move the node. That drift is $\dot{\Omega} = -\frac{3}{2} J_2 \, (R_\oplus / p)^2 \, n \cos i$, with $J_2 = 1.08263 \times 10^{-3}$ and n the mean motion (derived in the gravity note).

Solution. Equation 5 gives

$$\cos\alpha = \cos^2 51.6° + \sin^2 51.6° \cos 10° = 0.99067, \qquad \text{so} \quad \alpha = 7.833°.$$

The orbit is circular, with speed $v = \sqrt{\mu/r} = 7.669$ km/s. The cost of the burn is therefore

$$\Delta v = 2(7.669) \sin 3.917° = 1.048 \text{ km/s}.$$

For the drift we use $p = r$, since the orbit is circular, and $n = \sqrt{\mu/r^3}$. This gives $\dot{\Omega} = -5.00°$ per day. At this rate the node moves by $10°$ in 2.0 days, without using any propellant.

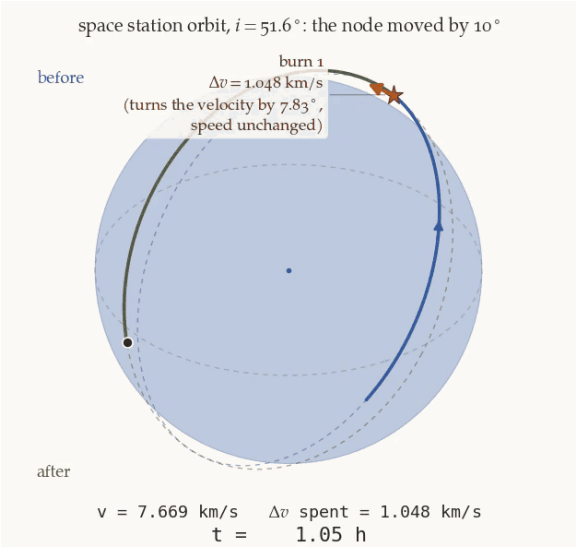


Figure 11: The node change of this example. The two orbits are drawn close together because their planes differ by only $\alpha = 7.83°$. The burn is made on the line where the planes meet. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-man.py, with the engine _manim.py.*

1.5.1 Burn or wait?

The burn takes a moment and uses 28% of the spacecraft's mass, with $I_{sp} = 320$ s. Waiting takes two days and uses none. So when time allows, a mission lets the Earth do the work. There are two cautions.

The first is the direction. For $i < 90°$ the node moves only westward, so $\dot{\Omega}$ is negative. A change of $10°$ to the east would mean waiting for the node to travel the other $350°$, which takes 70 days.

The second is that the drift moves the planes of nearby spacecraft as well. Suppose we want to change our plane relative to another satellite in the same orbit, such as a space station. Both planes drift at the same rate, so waiting achieves nothing. To gain on it, we can park for a while at a different altitude or inclination, where the drift rate is different. This uses less propellant than the direct burn, but it takes more time.

The same thinking applies to a launch. A launcher that misses its moment does not fire harder to reach the plane. The Earth's rotation brings the launch site back under the plane within about a day, so the launch simply waits.

2 Summary and exercise

A plane change costs $2v \sin(\Delta i/2)$ when it is a separate burn, so it should be made where the speed is low. It costs less again when it is combined with a transfer burn, and a little less when it is shared between two burns. A change of the node can be left to J_2 when time allows, at no cost in propellant.

> **A common mistake**
> Do not make a large plane change in LEO. At 7.726 km/s, a $19.2°$ turn alone costs 2.577 km/s. Fold the turn into the slowest burn of the mission.

Exercise. Repeat the optimal split for an idealised launch from Sriharikota due east, so that $\Delta i = 13.7°$, the latitude of the launch site. How much does sharing the turn save? How much does heading south of east cost, in total Δv from LEO to GEO? (The answers are $\delta^* = 1.23°$ and a total of 3.977 km/s, against 3.985 km/s with the whole turn at GEO, which is a saving of 8.0 m/s. Against the total of 4.055 km/s with $\Delta i = 19.2°$, the extra turn costs about 78 m/s.)

All the numbers in this note are checked by plane-changes-verify.py.