The Perifocal Frame

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1 Where this note sits

The classical time-of-flight equations for the three conic types are already in hand. For an ellipse, Kepler's equation; for a parabola, Barker's equation; for a hyperbola, its hyperbolic cousin:

$$M = E - e\sin E, \qquad \sqrt{\frac{\mu}{p^3}}\,(t-t_p) = \tfrac{1}{2}\left(D + \tfrac{1}{3}D^3\right),\ D = \tan\tfrac{f}{2}, \qquad M_h = e\sinh H - H.$$

Here E, M, D, H and M_h are the eccentric, mean, parabolic and hyperbolic anomalies of the two-body note, and t_p is the time of periapsis passage. All three equations answer the same physical question: *given the elapsed time, where is the body along the conic?* But none of them produces a **state vector**. They give a *scalar* position along a curve, while flight software, tracking and manoeuvre design all need $(\mathbf{r}, \mathbf{v})$ in an inertial frame.

History. Kepler found the first two laws in 1609 (*Astronomia Nova*): ellipses with the Sun at a focus, and equal areas in equal times. The same book introduces the *eccentric* and *mean* anomalies, and **Kepler's equation** appears in its chapter 60. Newton's *Principia* (1687) proved the converse, that an inverse-square force makes the orbit a conic (Newton 1687); Jakob Hermann and Johann Bernoulli derived the orbits again with the calculus in 1710.

Two questions follow, and each has its own route.

1. Once the location on the conic is known, how do we express position and velocity as **vectors**?
2. Can we propagate the Cartesian state **directly**, without changing the formulation between elliptic, parabolic and hyperbolic motion?

The first question leads to the **perifocal frame**, the subject of this note. The second leads to the Lagrange coefficients and then to the universal-variable formulation (the universal-variables note); Section 6 shows how the two routes meet.

$$\begin{aligned} \text{classical, element-based:} &\quad \Delta t \to (E,\, D,\, H) \to f \to (\mathbf{r}, \mathbf{v})_{PQW} \to (\mathbf{r}, \mathbf{v})_I, \\ \text{universal, state-vector:} &\quad \Delta t \to \chi \to F,\, G,\, \dot{F},\, \dot{G} \to (\mathbf{r}, \mathbf{v}). \end{aligned}$$

> **Note**
>
> The two routes are not rivals. The classical one is transparent and is what orbital elements are *for*; the universal one is what you implement when the code must not branch on conic type.

The plan, with every step done in full:

1. **Where the frame comes from** (Section 2). Two vectors, $\mathbf{h}$ and $\mathbf{e}$, never change; they define an inertial frame tied to the orbit, in which the position is a two-line formula.
2. **The velocity** (Section 3). Both components of the velocity depend on the true anomaly only; the result, the *perifocal state vector*, holds for every conic.
3. **From the plane to space** (Section 4). One 3×3 rotation, built from Ω, i and ω, carries the perifocal state into the inertial one. We derive it twice, geometrically and by matrix multiplication.
4. **The chain, with numbers** (Section 5), and **a bridge to the Lagrange coefficients** (Section 6).

As in the other notes, a blue **Try it** box asks you to do a small calculation before reading on, and a green **Check yourself** box asks a short question about what you have just read. Each has a hidden answer. When you finish, you should be able to answer the following without hand-waving.

1. Why is the perifocal frame inertial, when the polar basis $\{\hat{\mathbf{r}}, \hat{\theta}\}$ is not?
2. Why does the same pair of formulas give the state on an ellipse, a parabola and a hyperbola?
3. Why is the velocity hodograph a circle, and what does the position of the origin inside or outside it tell you?
4. Which three rotations carry the orbital plane to its place in space, in which order, and how do you check the sign convention of a formula you have found in a book?

Table 1: Notation

Symbol	Meaning
μ	gravitational parameter of the central body ($398\,600.4418\ \mathrm{km}^3/\mathrm{s}^2$ for the Earth)
$\mathbf{h} = \mathbf{r} \times \mathbf{v}$, h	specific angular momentum and its magnitude
$\mathbf{e}$, e	eccentricity vector and eccentricity
$p = h^2/\mu$	semi-latus rectum (finite for **every** conic; a is not)
f	true anomaly: the angle from periapsis to the body, in the direction of motion
$\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}$	the perifocal triad (fixed in inertial space)
$\hat{\mathbf{r}}, \hat{\theta}$	the polar basis (carried by the body, turning at $\dot{f}$)
$\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$	the inertial triad
Ω, i, ω	longitude of the ascending node, inclination, argument of periapsis
$\mathbf{Q}$	the rotation matrix from perifocal to inertial components

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

2 Where the frame comes from

2.1 Two vectors that never change

The two-body equation is $\ddot{\mathbf{r}} = -\mu\,\mathbf{r}/r^3$. Two vectors built from the state are constant along the motion.

The angular momentum. Differentiate $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ with the product rule:

$$\frac{d\mathbf{h}}{dt} = \underbrace{\mathbf{v} \times \mathbf{v}}_{\mathbf{0}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \Big(-\frac{\mu}{r^3}\,\mathbf{r} \Big) = \mathbf{0},$$

because $\mathbf{r} \times \mathbf{r} = \mathbf{0}$: the force is central. So $\mathbf{h}$ is constant. Since $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is perpendicular to $\mathbf{r}$ at every instant, $\mathbf{r} \cdot \mathbf{h} = 0$ always: **the orbit lies in a fixed plane**, through the attracting centre and perpendicular to $\mathbf{h}$.

The eccentricity vector. Cross the equation of motion with $\mathbf{h}$ and expand the double cross product with $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$:

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\,\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = -\frac{\mu}{r^3}\Big[(\mathbf{r} \cdot \mathbf{v})\,\mathbf{r} - r^2\mathbf{v} \Big] = \mu\Big[\frac{\mathbf{v}}{r} - \frac{(\mathbf{r} \cdot \mathbf{v})\,\mathbf{r}}{r^3} \Big].$$

The bracket is the time derivative of the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$: with $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ we have $\dot{r} = (\mathbf{r} \cdot \mathbf{v})/r$, so $d(\mathbf{r}/r)/dt = \mathbf{v}/r - \mathbf{r}\,\dot{r}/r^2 = \mathbf{v}/r - (\mathbf{r} \cdot \mathbf{v})\,\mathbf{r}/r^3$. And because $\mathbf{h}$ is constant, $\ddot{\mathbf{r}} \times \mathbf{h} = d(\mathbf{v} \times \mathbf{h})/dt$. Hence

$$\frac{d}{dt}\Big(\mathbf{v} \times \mathbf{h} - \mu\,\hat{\mathbf{r}} \Big) = \mathbf{0}, \qquad \text{so} \qquad \boxed{\mathbf{e} \equiv \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \hat{\mathbf{r}} \ \text{ is constant.}} \tag{1}$$

Two more facts follow at once. First, $\mathbf{e}\cdot\mathbf{h} = (\mathbf{v}\times\mathbf{h})\cdot\mathbf{h}/\mu - \hat{\mathbf{r}}\cdot\mathbf{h} = 0 - 0$, so **e lies in the orbital plane**. Second, using $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$,

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) = (\mathbf{r}\times\mathbf{v})\cdot\mathbf{h} = h^2, \qquad \text{so} \qquad \mathbf{r}\cdot\mathbf{e} = \frac{h^2}{\mu} - r.$$

Let f be the angle between $\mathbf{e}$ and $\mathbf{r}$, so $\mathbf{r}\cdot\mathbf{e} = re\cos f$. Then $r\,(1 + e\cos f) = h^2/\mu$:

$$\boxed{r = \frac{p}{1 + e\cos f}, \qquad p \equiv \frac{h^2}{\mu}, \quad h = \sqrt{\mu p}.} \tag{2}$$

This is the polar equation of a conic with a focus at the attracting centre. It has its minimum $r = p/(1+e)$ at $f = 0$, where $\mathbf{r}$ is parallel to $\mathbf{e}$: **e points to periapsis**, and f is the true anomaly. The value of $e = |\mathbf{e}|$ gives the type: ellipse ($e < 1$), parabola ($e = 1$), hyperbola ($e > 1$); $e = 0$ is the circle. (The two-body note derives the same facts at more length, and shows what the conic looks like for each e.)

Take a circular orbit: $\mathbf{r} = r\hat{\mathbf{I}}$ and $\mathbf{v} = \sqrt{\mu/r}\,\hat{\mathbf{J}}$. Evaluate Equation 1. What is $\mathbf{e}$, and what does that mean for the direction $\hat{\mathbf{p}} = \mathbf{e}/e$ of the next section?

What you should find

$\mathbf{h} = r\sqrt{\mu/r}\,\hat{\mathbf{K}} = \sqrt{\mu r}\,\hat{\mathbf{K}}$, and $\mathbf{v}\times\mathbf{h} = \sqrt{\mu/r}\sqrt{\mu r}\,(\hat{\mathbf{J}}\times\hat{\mathbf{K}}) = \mu\hat{\mathbf{I}}$. So $\mathbf{e} = \hat{\mathbf{I}} - \hat{\mathbf{r}} = \mathbf{0}$. A circle has no periapsis, and $\hat{\mathbf{p}} = \mathbf{e}/e$ is $\mathbf{0}/0$: undefined. The warning box below deals with this case.

History: the vector with six names. The constant vector **e** first appears in letters exchanged by Jakob Hermann and Johann Bernoulli in 1710: Hermann found a constant of integration tied to the eccentricity of the orbit, and Bernoulli generalised it to an arbitrary orientation of the orbit in its plane. Laplace rediscovered it in his *Mécanique céleste* (1799), Hamilton again in 1845, and Gibbs wrote it down in his vector analysis in 1901. Runge used it in a German textbook on vectors, which Lenz cited in a paper on the hydrogen atom; after Pauli's quantum treatment (1926) it became the **Runge–Lenz** or **Laplace–Runge–Lenz** vector. Runge himself attributed it to Gibbs. The name is a standard example of Stigler's law of eponymy: nobody who carries it first found it.

2.2 The frame

Define the three unit vectors, centred at the attracting centre (the focus):

$$\hat{\mathbf{w}} = \frac{\mathbf{h}}{h}, \qquad \hat{\mathbf{p}} = \frac{\mathbf{e}}{e}, \qquad \hat{\mathbf{q}} = \hat{\mathbf{w}}\times\hat{\mathbf{p}}. \tag{3}$$

- $\hat{\mathbf{p}}$ points from the focus toward **periapsis**;
- $\hat{\mathbf{w}}$ is **normal** to the orbital plane;
- $\hat{\mathbf{q}}$ lies in the plane, 90° ahead of $\hat{\mathbf{p}}$ in the direction of motion.

They are orthonormal because $\mathbf{e}\perp\mathbf{h}$ (shown above). They are right-handed, $\hat{\mathbf{p}}\times\hat{\mathbf{q}} = \hat{\mathbf{w}}$, because $\hat{\mathbf{p}}\times(\hat{\mathbf{w}}\times\hat{\mathbf{p}}) = \hat{\mathbf{w}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}) - \hat{\mathbf{p}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{w}}) = \hat{\mathbf{w}}$.

Now the key point. Both **h** and **e** are **constant vectors** (Section 2.1). So $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{w}}$ are constant too: **the perifocal frame does not turn.** It is a non-rotating frame centred on the attracting body, so the relative two-body equation keeps its form in it and no fictitious forces appear. (Strictly, its origin accelerates, because the attracting body itself moves about the barycentre; but the relative equation of motion already accounts for that, so for the two-body problem the frame behaves as an inertial one.) Contrast the frames carried by the body: the polar basis $\{\hat{\mathbf{r}}, \hat{\theta}\}$ turns at the rate $\dot{f}$, and the LVLH frame of the relative-motion note turns at the orbital rate. Everything difficult about those frames, the extra terms from differentiating a moving basis, is absent here (Figure 1).

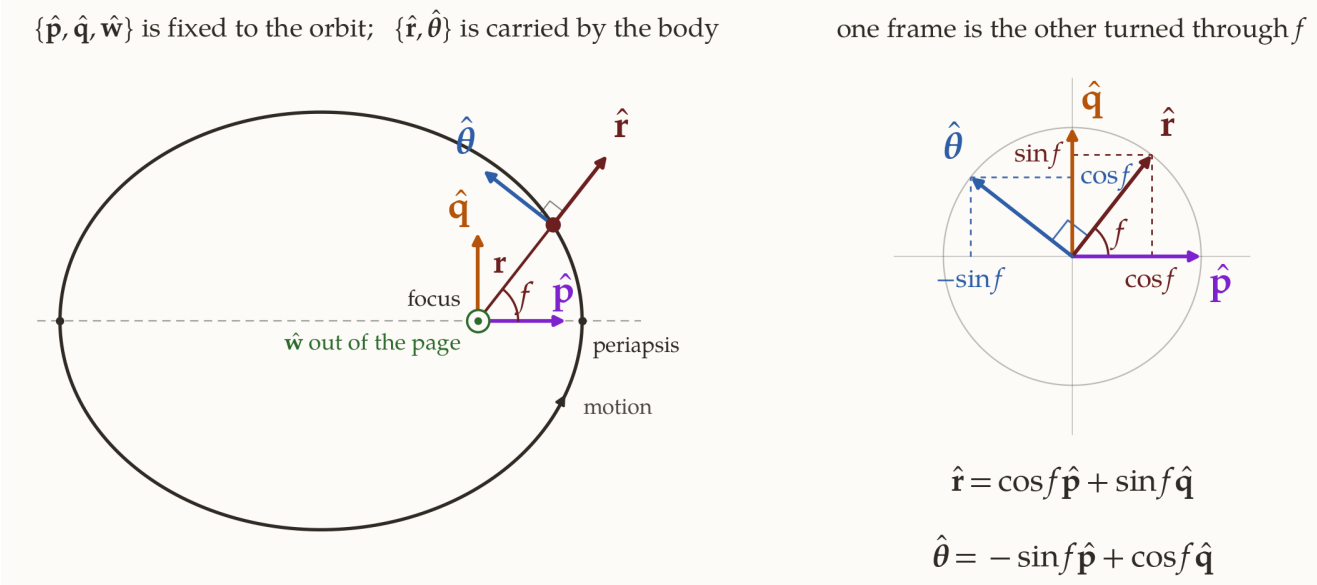

Figure 1: **Left:** the conic in its own plane. The perifocal basis $\{\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}\}$ is pinned to the *orbit* — $\hat{\mathbf{p}}$ from the focus to periapsis, $\hat{\mathbf{q}}$ a right angle ahead of it in the direction of motion, $\hat{\mathbf{w}} = \mathbf{h}/h$ out of the plane — so it never turns. The polar basis $\{\hat{\mathbf{r}}, \hat{\theta}\}$ is carried by the *body*, so it turns at the rate $\dot{f}$. **Right:** the only relation between them. Both are orthonormal and both lie in the orbital plane, so one is the other rotated through the true anomaly f. *Script: fig-perifocal-frame.py.*

Naming. The frame is called **perifocal** (the frame at the focus, aligned with periapsis) or **PQW**, after the names of its three axes, in much of the astrodynamics literature. Some texts use $\hat{\mathbf{e}}$, $\hat{\mathbf{p}}$, $\hat{\mathbf{h}}$ for the same three vectors, or $\hat{\mathbf{P}}$, $\hat{\mathbf{Q}}$, $\hat{\mathbf{W}}$; check the symbols, not the names.

> **A degenerate case**
>
> For a **circular** orbit $e = 0$, so $\mathbf{e} = \mathbf{0}$ and there is no periapsis: $\hat{\mathbf{p}}$ is undefined. All the formulas below still hold if $\hat{\mathbf{p}}$ is *chosen* (any direction in the plane will do, and the usual choice is the ascending node), and the true anomaly is then measured from that choice. The same degeneracy returns in Section 4, where it is handled with the argument of latitude.

2.3 Position in the perifocal frame

The position vector $\mathbf{r}$ lies in the orbital plane ($\mathbf{r} \cdot \mathbf{h} = 0$), so it is a combination of $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ alone. The orbit equation fixes the angle between $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ only up to sign ($\cos f$ is what it determines), and we choose the sign so that f **grows in the direction of motion**, the sense of $\hat{\mathbf{q}}$ (Section 3.1 confirms this is consistent with $\dot{f} > 0$). Then

$$\hat{\mathbf{r}} = \cos f\,\hat{\mathbf{p}} + \sin f\,\hat{\mathbf{q}}, \tag{4}$$

which is the right-hand panel of Figure 1. With $\mathbf{r} = r\hat{\mathbf{r}}$ and Equation 2:

$$\boxed{\mathbf{r}_{PQW} = r \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix} = \frac{p}{1 + e\cos f} \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix}.} \tag{5}$$

The third component is identically zero. **That is the whole point of the frame**: a three-dimensional problem has been reduced to a two-dimensional one, and the missing dimension is restored later by a single rotation.

Four remarks on the range of f and on special points.

- **Ellipse and circle**: f runs over a full turn.
- **Parabola** ($e = 1$): $r = p/(1 + \cos f) \to \infty$ as $f \to \pm\pi$, so $|f| < \pi$.

- **Hyperbola** ($e > 1$): $1 + e\cos f > 0$ requires $\cos f > -1/e$, so $|f| < f_\infty = \arccos(-1/e)$. For $e = 1.5$ this is 131.81°: the body never gets "behind" the focus.
- **Periapsis and apoapsis**: $r_p = p/(1+e)$ at $f = 0$, and $r_a = p/(1-e)$ at $f = \pi$ (ellipse only). Putting $e = 0$ gives $\mathbf{r}_{PQW} = p\,(\cos f, \sin f, 0)$, a circle of radius p.

A hyperbola has $e = 2$ and $p = 10\,000$ km. Find the range of f, the periapsis radius, and $\mathbf{r}_{PQW}$ at $f = 90°$.

What you should find

$f_\infty = \arccos(-1/2) = 120°$, so $-120° < f < 120°$. $r_p = p/(1+e) = 3333.33$ km. At $f = 90°$, $\cos f = 0$, so $r = p$ and $\mathbf{r}_{PQW} = (0,\ 10\,000,\ 0)$ km: on the $\hat{\mathbf{q}}$ axis, at the end of the semi-latus rectum. That is true for every conic, and it is where the name p comes from.

3 The velocity

3.1 The polar basis moves; the perifocal one does not

Alongside $\hat{\mathbf{r}}$ define $\hat{\theta} = \hat{\mathbf{w}} \times \hat{\mathbf{r}}$, the unit vector 90° ahead of $\hat{\mathbf{r}}$. Using $\hat{\mathbf{w}} \times \hat{\mathbf{p}} = \hat{\mathbf{q}}$ and $\hat{\mathbf{w}} \times \hat{\mathbf{q}} = -\hat{\mathbf{p}}$ on Equation 4,

$$\begin{aligned} \hat{\mathbf{r}} &= \cos f\,\hat{\mathbf{p}} + \sin f\,\hat{\mathbf{q}}, \\ \hat{\theta} &= -\sin f\,\hat{\mathbf{p}} + \cos f\,\hat{\mathbf{q}}. \end{aligned} \tag{6}$$

Now differentiate in time. $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are **constant**, so only the trigonometric coefficients change, and no transport theorem is needed:

$$\begin{aligned} \dot{\hat{\mathbf{r}}} &= \dot{f}\,(-\sin f\,\hat{\mathbf{p}} + \cos f\,\hat{\mathbf{q}}) = \dot{f}\,\hat{\theta}, \\ \dot{\hat{\theta}} &= \dot{f}\,(-\cos f\,\hat{\mathbf{p}} - \sin f\,\hat{\mathbf{q}}) = -\dot{f}\,\hat{\mathbf{r}}. \end{aligned} \tag{7}$$

(The unit vector $\hat{\mathbf{r}}$ cannot change length, only direction, and it turns at the rate $\dot{f}$.) Then

$$\mathbf{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{f}\,\hat{\theta}, \tag{8}$$

and the angular momentum is $\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{f}\hat{\theta}) = r^2\dot{f}\,\hat{\mathbf{w}}$, using $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\mathbf{w}}$. Two consequences:

$$h = r^2\dot{f}, \qquad \text{and} \qquad \dot{f} > 0. \tag{9}$$

$\dot{f} > 0$ confirms that f grows in the direction of motion (the sense of $\hat{\mathbf{q}}$), as we assumed in Section 2.3. And $\frac{1}{2}r^2\dot{f} = h/2$ is the areal velocity: **equal areas in equal times**.

History: the area law. Kepler's second law (1609) is exactly $h = r^2\dot{f}$ = const. Newton showed in the first proposition of the *Principia* that the area law holds for *any* central force, not just gravity, which is why h is conserved here without using the inverse-square form of the force.

The two terms in Equation 8 have different meanings:

term	name	vanishes when
$\dot{r}$	radial speed v_r	at the apsides, $f = 0, \pi$
$r\dot{f}$	transverse speed v_θ	never (for $h \neq 0$)

We now evaluate each in terms of (p, e, f) alone.

3.2 The transverse term

From Equation 9 and $h = \sqrt{\mu p}$, and with $1/r = (1 + e\cos f)/p$ from Equation 2,

$$r\dot{f} = \frac{h}{r} = \sqrt{\mu p}\,\frac{1 + e\cos f}{p} = \sqrt{\frac{\mu}{p}}\,(1 + e\cos f). \tag{10}$$

3.3 The radial term

Differentiate the orbit equation in the form $r\,(1 + e\cos f) = p$ **with respect to time**, with p constant:

$$\dot{r}\,(1 + e\cos f) - r\,e\sin f\,\dot{f} = 0 \qquad \Longrightarrow \qquad \dot{r} = \frac{r\,e\sin f}{1 + e\cos f}\,\dot{f}.$$

Since $1 + e\cos f = p/r$, we have $r/(1 + e\cos f) = r^2/p$. Then $\dot{r} = \dfrac{r^2\dot{f}}{p}\,e\sin f = \dfrac{h}{p}\,e\sin f$, and $h/p = \sqrt{\mu p}/p = \sqrt{\mu/p}$:

$$\dot{r} = \sqrt{\frac{\mu}{p}}\,e\sin f. \tag{11}$$

(The same result follows from the chain rule, $\dot{r} = (dr/df)\dot{f}$ with $dr/df = pe\sin f/(1 + e\cos f)^2 = r^2 e\sin f/p$.) Notice that r has cancelled completely: **both velocity components depend on f only**, never on where the body is. This is the algebraic root of the hodograph being a circle (Section 3.7).

3.4 The velocity in the polar basis

Collecting Equation 10 and Equation 11 in Equation 8:

$$\boxed{\mathbf{v} = \sqrt{\frac{\mu}{p}}\Big[\,e\sin f\;\hat{\mathbf{r}} + (1 + e\cos f)\;\hat{\theta}\,\Big].} \tag{12}$$

Three things can be read off at once.

- **Flight-path angle.** The angle γ between $\mathbf{v}$ and the local horizontal ($\hat{\theta}$) obeys $\tan\gamma = v_r/v_\theta = e\sin f/(1 + e\cos f)$. It is zero at the apsides and, for $e = 0$, everywhere.
- **Speed.** $v^2 = \dfrac{\mu}{p}[e^2\sin^2 f + (1 + e\cos f)^2] = \dfrac{\mu}{p}(1 + 2e\cos f + e^2)$, because $\sin^2 f + \cos^2 f = 1$.
- **A check against energy.** The vis-viva equation, with $a = p/(1 - e^2)$ and $r = p/(1 + e\cos f)$, gives $\mu(2/r - 1/a) = \dfrac{\mu}{p}[2(1 + e\cos f) - (1 - e^2)] = \dfrac{\mu}{p}(1 + 2e\cos f + e^2)$: **the same**. We did not use energy anywhere in deriving Equation 12, so this agreement is a genuine check. Equivalently the specific energy is

  $$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2p}[1 + 2e\cos f + e^2 - 2 - 2e\cos f] = -\frac{\mu\,(1 - e^2)}{2p},$$

  independent of f: energy is conserved.

On an ellipse the flight-path angle is zero at both apsides, so it has a largest value somewhere in between. Differentiate $\tan\gamma = e\sin f/(1+e\cos f)$ with respect to f to find where, and find $\gamma_{\max}$ for $e = 0.4$.

What you should find

The quotient rule gives a numerator $e\cos f\,(1+e\cos f) + e^2\sin^2 f = e\,(\cos f + e)$, which vanishes at $\cos f = -e$. There $\sin f = \sqrt{1-e^2}$ and $1 + e\cos f = 1 - e^2$, so $\tan\gamma_{\max} = e/\sqrt{1-e^2}$, that is $\sin\gamma_{\max} = e$. For $e = 0.4$: $\gamma_{\max} = \arcsin 0.4 = 23.578°$, at $f = \arccos(-0.4) = 113.578°$. The point $\cos f = -e$ is the end of the minor axis, where $r = a$.

3.5 Projecting onto the perifocal axes

Equation 12 is complete, but in the *turning* basis $\{\hat{\mathbf{r}}, \hat{\theta}\}$, which is useless for adding vectors at two different instants. Convert to the fixed basis with Equation 6:

$$\mathbf{v} = \sqrt{\frac{\mu}{p}}\Big[e\sin f\,(\cos f\,\hat{\mathbf{p}} + \sin f\,\hat{\mathbf{q}}) + (1+e\cos f)(-\sin f\,\hat{\mathbf{p}} + \cos f\,\hat{\mathbf{q}})\Big].$$

Collect the coefficients of each unit vector.

$$\begin{aligned}\hat{\mathbf{p}}:&\quad e\sin f\cos f - \sin f - e\sin f\cos f = -\sin f,\\ \hat{\mathbf{q}}:&\quad e\sin^2 f + \cos f + e\cos^2 f = e + \cos f.\end{aligned}$$

Both simplifications are pure cancellation: the $e\sin f\cos f$ terms in the first, the Pythagorean identity in the second.

The same result by differentiating the components. Since $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are constant, $\mathbf{v}$ is just the time derivative of Equation 5 component by component, $(v_P, v_Q) = (\dot r\cos f - r\dot f\sin f,\ \dot r\sin f + r\dot f\cos f)$. This is the rotation of (v_r, v_θ) through the angle f:

$$\begin{bmatrix} v_P \\ v_Q \end{bmatrix} = \begin{bmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \sqrt{\frac{\mu}{p}}\begin{bmatrix} e\sin f\cos f - (1+e\cos f)\sin f \\ e\sin^2 f + (1+e\cos f)\cos f \end{bmatrix},$$

which simplifies to $\sqrt{\mu/p}\,(-\sin f,\ e+\cos f)^{\mathsf T}$ by the same two cancellations.

3.6 The perifocal state vector

$$\boxed{\mathbf{r}_{PQW} = \frac{p}{1+e\cos f}\begin{bmatrix}\cos f\\ \sin f\\ 0\end{bmatrix}, \qquad \mathbf{v}_{PQW} = \sqrt{\frac{\mu}{p}}\begin{bmatrix}-\sin f\\ e+\cos f\\ 0\end{bmatrix}.} \tag{13}$$

Since $h = \sqrt{\mu p}$, the velocity is often written $\mathbf{v}_{PQW} = \dfrac{\mu}{h}\,(-\sin f,\ e+\cos f,\ 0)^{\mathsf T}$, the form used when h, rather than a, is carried as the size element.

> **Important**
>
> Nothing in this derivation assumed $e < 1$. The pair Equation 13 holds for **every conic** — ellipse, circle, parabola ($e = 1$, where a is useless but p is finite) and hyperbola — because it was built from p, e and f, never from a.

Three consistency checks.

Angular momentum. The $\hat{\mathbf{w}}$ component of $\mathbf{r}\times\mathbf{v}$ is

$$r_P v_Q - r_Q v_P = r\sqrt{\frac{\mu}{p}}\left[\cos f\,(e+\cos f) + \sin^2 f\right]$$
$$= r\sqrt{\frac{\mu}{p}}\,(1+e\cos f) = p\sqrt{\frac{\mu}{p}} = \sqrt{\mu p}.$$

Right size, and *positive*: the motion is counter-clockwise about $\hat{\mathbf{w}}$.

Eccentricity vector. From Equation 1 and $\mathbf{v}\times(\mathbf{r}\times\mathbf{v}) = \mathbf{r}\,v^2 - \mathbf{v}\,(\mathbf{r}\cdot\mathbf{v})$ one has the equivalent form $\mathbf{e} = [(v^2-\mu/r)\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v}]/\mu$. Evaluated on Equation 13 it returns $(e,0,0)^{\mathsf{T}}$: it points along $\hat{\mathbf{p}}$, as it must. (Checked symbolically in perifocal-verify.py.)

Periapsis. At $f=0$: $\mathbf{r} = (p/(1+e),0,0)$ and $\mathbf{v} = \sqrt{\mu/p}\,(0,\,1+e,\,0)$. The velocity is purely transverse, perpendicular to $\mathbf{r}$.

3.7 Two facts you can read off

The velocity hodograph is a circle. The tip of $\mathbf{v}$ has components $v_P = -\sqrt{\mu/p}\sin f$ and $v_Q - e\sqrt{\mu/p} = \sqrt{\mu/p}\cos f$. Square and add:

$$v_P^2 + \left(v_Q - e\sqrt{\frac{\mu}{p}}\right)^2 = \frac{\mu}{p}. \tag{14}$$

As f runs over the orbit, the tip of $\mathbf{v}$ traces a circle of radius $\sqrt{\mu/p} = \mu/h$ centred at $(0,\,e\mu/h)$: **offset, but a circle for every eccentricity** (Figure 2). The origin of the velocity plane is inside the circle for a bound orbit ($e<1$), so the velocity vector turns through a full 360°; it lies *on* the circle for the parabola, where $\mathbf{v}\to 0$ as $f\to\pm\pi$; and it is *outside* the circle for the hyperbola, where only the arc with $|f|<f_\infty$ is traversed and the velocity direction is confined to a wedge. The chord ends are the asymptotic velocities, of magnitude $v_\infty = \sqrt{\mu/p}\,\sqrt{e^2-1}$, which agrees with the energy $\varepsilon = v_\infty^2/2 = \mu(e^2-1)/2p$ found above.

History: the hodograph. William Rowan Hamilton introduced the word and the curve in a paper communicated to the Royal Irish Academy in December 1846 (published in 1847) (Hamilton 1847): the *hodograph* is the curve traced by the tip of the velocity vector. He showed that under Newton's law of attraction it is a circle, and that this property singles out the inverse-square law among central forces. He later learned that Möbius had already described the curve in his *Mechanik des Himmels* (1843).

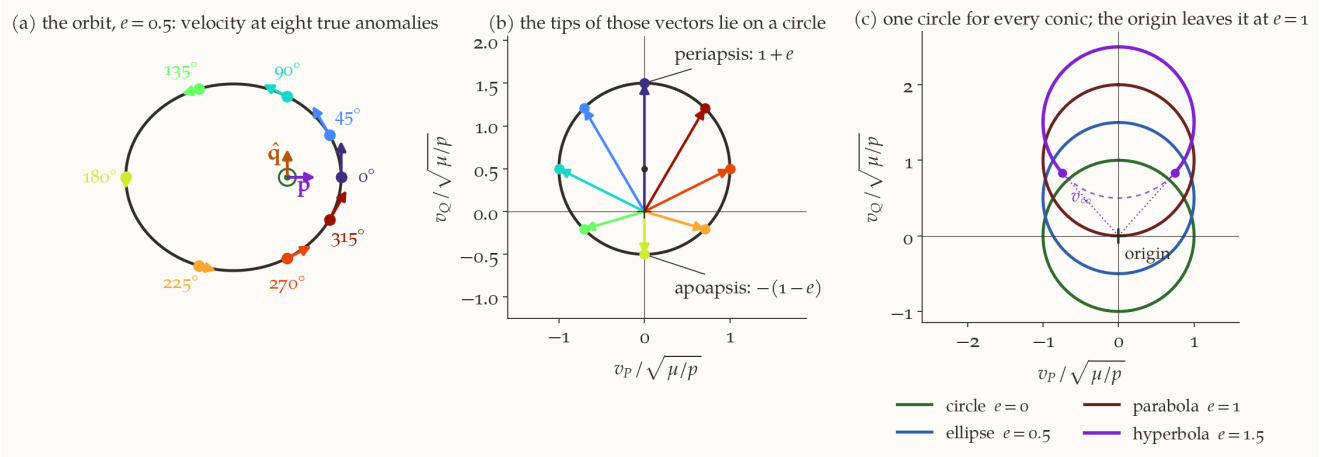

Figure 2: The velocity hodograph. (**a**) An ellipse with $e=0.5$, with the velocity drawn at eight true anomalies. (**b**) The same eight velocities with their tails at the origin of the velocity plane: the tips lie on a circle of unit radius, in units of $\sqrt{\mu/p}$, centred at $(0,e)$. (**c**) The same circle for $e=0,\,0.5,\,1,\,1.5$: only the centre moves. For the hyperbola the solid arc is the part actually traversed, and the dotted lines run to the asymptotic velocities v_∞. *Script: fig-perifocal-hodograph.py.*

Periapsis and apoapsis speeds follow by setting $f=0$ and $f=\pi$ in v_Q:

$$v_p = \sqrt{\frac{\mu}{p}}\,(1+e), \qquad v_a = \sqrt{\frac{\mu}{p}}\,(1-e), \qquad \frac{v_p}{v_a} = \frac{1+e}{1-e},$$

the ratio that sets the cost of every apsidal manoeuvre. (With $p = a(1-e^2)$: $v_p = \sqrt{\mu/a}\,\sqrt{(1+e)/(1-e)}$, the more familiar form.)

Read v_p and v_a off the hodograph instead. In units of $\sqrt{\mu/p}$ the circle has radius 1 and its centre is at distance e from the origin. Which points of the circle are nearest to and farthest from the origin, and which true anomalies do they belong to?

Answer

The farthest point is on the line through the origin and the centre, beyond the centre, at distance $1+e$: the top of the circle, $(v_P, v_Q) = (0,\ 1+e)$, which is $f = 0$. The nearest is the bottom, at distance $1-e$, $(0,\ -(1-e))$, which is $f = \pi$. So the speed runs from $(1-e)\sqrt{\mu/p}$ to $(1+e)\sqrt{\mu/p}$ and is extreme at the apsides, with no calculus. At apoapsis $v_Q < 0$: the body is moving in the $-\hat{\mathbf{q}}$ direction, as it must on the far side of the orbit.

4 From the plane to space

4.1 Two independent pieces of information

The perifocal frame separates two kinds of information. The conic equation describes the geometry **within** the orbital plane, $r = p/(1+e\cos f)$. The angles Ω, i, ω describe how that plane, and the apse line in it, is **oriented** in inertial space. Thus

conic geometry + orbital orientation = inertial state.

Everything hard about the *dynamics* lives on the left, and everything hard about the *bookkeeping* lives on the right. The perifocal frame is the natural bridge between scalar orbital geometry and three-dimensional vectors.

4.2 The three angles

Let the inertial frame be $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$, with $\hat{\mathbf{K}}$ along the reference pole. The orbital plane and the reference plane intersect in the **line of nodes**. Let $\hat{\mathbf{N}}$ be the unit vector along it, pointing to the **ascending node**, the point where the body crosses the reference plane going north. Then (Figure 3):

- i, the **inclination**, is the angle between $\hat{\mathbf{K}}$ and $\hat{\mathbf{w}}$;
- Ω, the **longitude of the ascending node**, is the angle from $\hat{\mathbf{I}}$ to $\hat{\mathbf{N}}$, measured about $\hat{\mathbf{K}}$;
- ω, the **argument of periapsis**, is the angle from $\hat{\mathbf{N}}$ to $\hat{\mathbf{p}}$, measured in the orbital plane in the direction of motion.

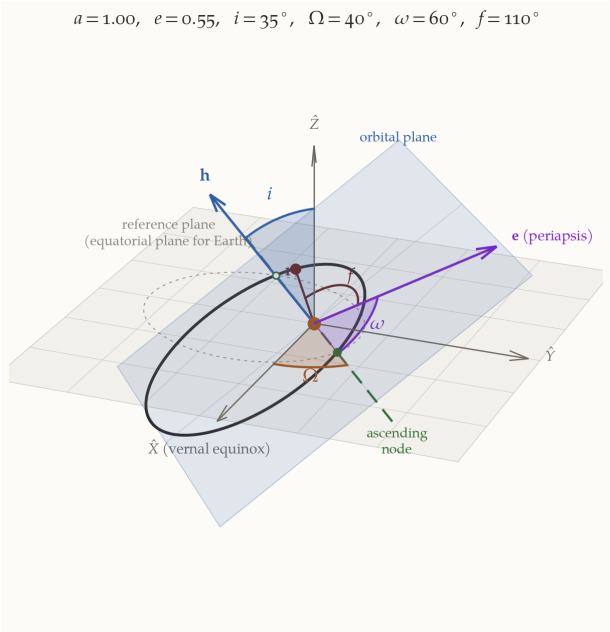

Figure 3: The orbit is fixed in inertial space by six numbers: two for the conic (p or a, and e), three for the orientation of its plane (i, Ω, ω), and one for where the body is on it (f). The dashed curve is the orbit projected onto the reference plane; the green dashed line is the line of nodes. The inertial axes are drawn as $\hat{X}, \hat{Y}, \hat{Z}$; they are the $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$ of this note. *Script: orbital-elements-3d.py.*

History: Euler angles for an orbit. Euler proved in 1775 that any displacement of a rigid body about a fixed point is equivalent to a single rotation about some axis. The orientation of a rotated frame is conventionally given by three successive angles that bear his name. The three angles Ω, i, ω are exactly Euler angles for the orientation of the orbital plane, in the 3–1–3 sequence below. Computing a body's place in the sky from its orbital elements is what an *ephemeris* is. Gauss's orbit computation of 1801 for the lost minor planet Ceres, from a short arc of observations, allowed astronomers to find it again; his *Theoria motus corporum coelestium* (1809) set out the method.

4.3 Building the frame from the angles: the geometric route

We need $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{w}}$ in inertial components. Start from the node vector, which lies in the reference plane at angle Ω from $\hat{\mathbf{I}}$:

$$\hat{\mathbf{N}} = \begin{bmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{bmatrix}.$$

Step 1: the normal $\hat{\mathbf{w}}$. The orbital plane is the reference plane tilted about the line of nodes through the angle i, so $\hat{\mathbf{w}}$ is $\hat{\mathbf{K}}$ rotated about $\hat{\mathbf{N}}$ by i. For a vector perpendicular to the axis (here $\hat{\mathbf{K}} \perp \hat{\mathbf{N}}$), a right-handed rotation through i gives $\cos i\,\hat{\mathbf{K}} + \sin i\,(\hat{\mathbf{N}} \times \hat{\mathbf{K}})$. With $\hat{\mathbf{N}} \times \hat{\mathbf{K}} = (\sin\Omega, -\cos\Omega, 0)^{\mathsf{T}}$:

$$\hat{\mathbf{w}} = \begin{bmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{bmatrix}.$$

(Check: $\hat{\mathbf{K}} \times \hat{\mathbf{w}} = (\cos\Omega\sin i,\ \sin\Omega\sin i,\ 0)^{\mathsf{T}} = \sin i\,\hat{\mathbf{N}}$, so $\hat{\mathbf{N}}$ is indeed the direction of the **node vector** $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h}$. The orbital-elements note, which comes next, uses $\mathbf{n}$ unnormalised, with $|\mathbf{n}| = h\sin i$.)

Step 2: the in-plane partner of $\hat{\mathbf{N}}$. The vector $\hat{\mathbf{M}} \equiv \hat{\mathbf{w}} \times \hat{\mathbf{N}}$ lies in the orbital plane, 90° ahead of $\hat{\mathbf{N}}$ in the direction of motion. Its components are

$$\hat{\mathbf{M}} = \hat{\mathbf{w}} \times \hat{\mathbf{N}} = \begin{bmatrix} -\cos i\sin\Omega \\ \cos i\cos\Omega \\ \sin i \end{bmatrix},$$

(the third component is $\sin\Omega\sin i \cdot \sin\Omega + \cos\Omega\sin i \cdot \cos\Omega = \sin i$).

Step 3: $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$. Inside the plane, $\hat{\mathbf{p}}$ is $\hat{\mathbf{N}}$ turned through ω in the direction of motion, and $\hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}}$. Since $\hat{\mathbf{w}} \times \hat{\mathbf{M}} = -\hat{\mathbf{N}}$,

$$\begin{aligned} \hat{\mathbf{p}} &= \cos\omega\, \hat{\mathbf{N}} + \sin\omega\, \hat{\mathbf{M}}, \\ \hat{\mathbf{q}} &= -\sin\omega\, \hat{\mathbf{N}} + \cos\omega\, \hat{\mathbf{M}}. \end{aligned} \tag{15}$$

Written out, $\hat{\mathbf{p}} = (\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i,\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i,\ \sin\omega\sin i)^{\mathsf{T}}$, and likewise $\hat{\mathbf{q}}$.

Step 4: the rotation matrix. A vector with perifocal components (x, y, z) is $x\hat{\mathbf{p}} + y\hat{\mathbf{q}} + z\hat{\mathbf{w}}$, so its inertial components are the combination of the three columns. The matrix whose columns are $\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}$ **is** the rotation from perifocal to inertial components:

$$\mathbf{Q}_{PQW\to I} = [\ \hat{\mathbf{p}}\ \ \hat{\mathbf{q}}\ \ \hat{\mathbf{w}}\ ]_I. \tag{16}$$

4.4 The same matrix as a product of three rotations

The standard orbital-element sequence is the 3–1–3 Euler sequence. For an **active** rotation (of the *vector*, with the frame held still), the elementary rotations about the 3 and 1 axes are

$$\mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

(Check on $\mathbf{R}_3$: it takes $\hat{\mathbf{I}} = (1,0,0)^{\mathsf{T}}$ to $(\cos\theta, \sin\theta, 0)^{\mathsf{T}}$, a counter-clockwise turn.) The claim is

$$\mathbf{Q}_{PQW\to I} = \mathbf{R}_3(\Omega)\, \mathbf{R}_1(i)\, \mathbf{R}_3(\omega). \tag{17}$$

We verify it by multiplying, right to left. Write $c_\theta = \cos\theta$, $s_\theta = \sin\theta$. First,

$$\mathbf{R}_1(i)\, \mathbf{R}_3(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_i & -s_i \\ 0 & s_i & c_i \end{bmatrix} \begin{bmatrix} c_\omega & -s_\omega & 0 \\ s_\omega & c_\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\omega & -s_\omega & 0 \\ c_i s_\omega & c_i c_\omega & -s_i \\ s_i s_\omega & s_i c_\omega & c_i \end{bmatrix}.$$

Then left-multiplying by $\mathbf{R}_3(\Omega)$ leaves the third row alone and mixes the first two: the new first row is c_Ω (row 1) $- s_\Omega$ (row 2), the new second row is s_Ω (row 1) $+ c_\Omega$ (row 2):

$$\mathbf{Q} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega s_\omega c_i & -c_\Omega s_\omega - s_\Omega c_\omega c_i & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega s_\omega c_i & -s_\Omega s_\omega + c_\Omega c_\omega c_i & -c_\Omega s_i \\ s_\omega s_i & c_\omega s_i & c_i \end{bmatrix}. \tag{18}$$

The first column is $\hat{\mathbf{p}}$ of Equation 15, the third is $\hat{\mathbf{w}}$ of Step 1, and the second is $\hat{\mathbf{q}}$: **the two derivations agree entry by entry**. The product also has two geometric readings. Read it **right to left, on fixed axes**, as three rotations applied in turn to a triad that starts on $(\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$: $\mathbf{R}_3(\omega)$ turns $\hat{\mathbf{p}}$ forward by ω about $\hat{\mathbf{K}}$, inside the reference plane; $\mathbf{R}_1(i)$ then tilts the plane up by i about $\hat{\mathbf{I}}$, which is where the line of nodes lies at that moment; and $\mathbf{R}_3(\Omega)$ swings the node line from $\hat{\mathbf{I}}$ round to its true place. Read it **left to right, about the triad's own axes**, it is the same motion described from the moving triad: first Ω about $\hat{\mathbf{K}}$, then i about the new node line, then ω about the new normal. The orbital-elements note uses the same sequence to go from the elements to $(\mathbf{r}, \mathbf{v})$. Figure 4 draws the first reading.

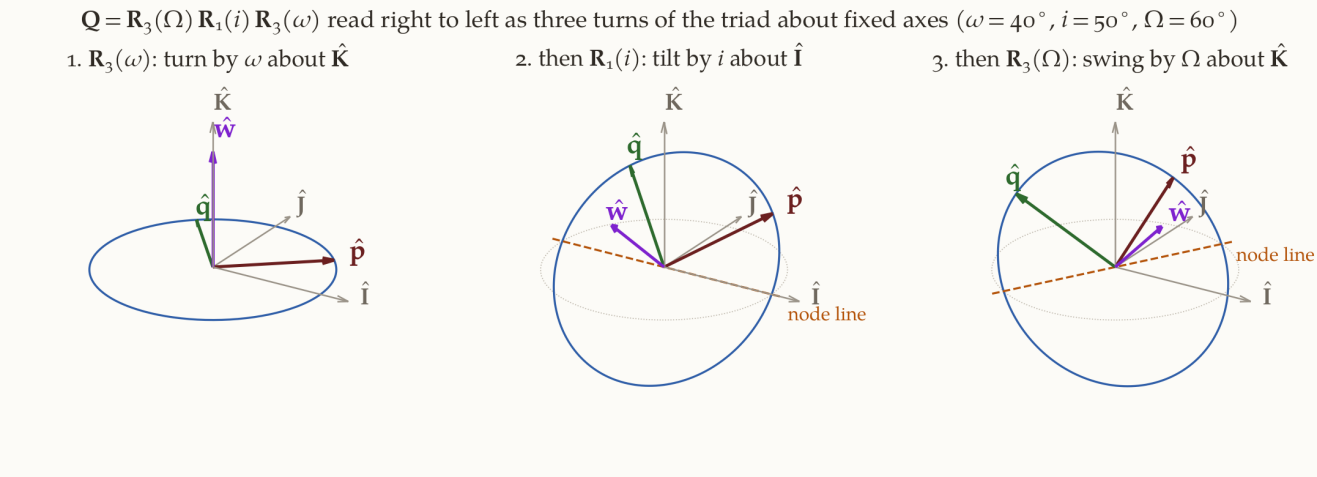

Figure 4: The right-to-left reading of Equation 17, on fixed axes $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$ (grey). **1.** $\mathbf{R}_3(\omega)$ turns the triad by ω in the equatorial plane. **2.** $\mathbf{R}_1(i)$ tilts the plane by i about the fixed $\hat{\mathbf{I}}$ axis, which becomes the line of nodes. **3.** $\mathbf{R}_3(\Omega)$ swings the tilted plane about $\hat{\mathbf{K}}$, carrying the node from $\hat{\mathbf{I}}$ to longitude Ω. Drawn for $\omega = 40°$, $i = 50°$, $\Omega = 60°$. *Script: fig-perifocal-313.py.*

> **Convention trap**
>
> Many texts, Curtis among them, define $\mathbf{R}_3$ as rotating the *frame* rather than the vector, and so tabulate the **transpose** of everything above. For one elementary rotation the difference is just $\theta \to -\theta$. For the product it is more: the transpose of $\mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega)$ is $\mathbf{R}_3(-\omega)\mathbf{R}_1(-i)\mathbf{R}_3(-\Omega)$, with the **order reversed** as well as the signs. A formula copied with the signs flipped but the order kept has the roles of Ω and ω swapped. Check before trusting a formula: $\Omega = i = \omega = 0$ must give $\mathbf{Q} = \mathbf{1}$; and with $i = 90°$, the case $\Omega = 90°$, $\omega = 0$ must give $\hat{\mathbf{p}} = \hat{\mathbf{J}}$ and $\hat{\mathbf{w}} = \hat{\mathbf{I}}$, while $\Omega = 0$, $\omega = 90°$ must give $\hat{\mathbf{p}} = \hat{\mathbf{K}}$. The second pair catches a reversed order; a test with $\omega = 0$ alone cannot.

Use Equation 18 on a polar orbit with $i = 90°$ and $\Omega = \omega = 0$. Write down $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{w}}$. In which direction is the body moving as it passes periapsis, and does the triad pass the right-hand test $\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \hat{\mathbf{w}}$?

What you should find

With $c_\Omega = c_\omega = 1$, $s_\Omega = s_\omega = 0$, $c_i = 0$, $s_i = 1$, the three columns are $\hat{\mathbf{p}} = \hat{\mathbf{I}}$, $\hat{\mathbf{q}} = \hat{\mathbf{K}}$ and $\hat{\mathbf{w}} = -\hat{\mathbf{J}}$. Periapsis is at the ascending node ($\omega = 0$), and the velocity there is along $\hat{\mathbf{q}}$: straight **north**, as an ascending node requires. And $\hat{\mathbf{I}} \times \hat{\mathbf{K}} = -\hat{\mathbf{J}}$, so the triad is right-handed.

4.5 What the matrix does, and checks on it

- **Both vectors transform with the same matrix**, $\mathbf{r}_I = \mathbf{Q}\,\mathbf{r}_{PQW}$ and $\mathbf{v}_I = \mathbf{Q}\,\mathbf{v}_{PQW}$, because $\mathbf{Q}$ is **constant** (the perifocal frame does not turn), so $d(\mathbf{Q}\mathbf{r})/dt = \mathbf{Q}\,\dot{\mathbf{r}}$. Between a frame that *does* turn (LVLH, Earth-fixed) and an inertial one, velocities pick up an extra $\omega \times \mathbf{r}$ term; here they do not.
- $\mathbf{Q}$ **is orthogonal with** $\det \mathbf{Q} = +1$. Its columns are an orthonormal right-handed triad, so $\mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{1}$, and the inverse map is free: $\mathbf{r}_{PQW} = \mathbf{Q}^{\mathsf{T}}\mathbf{r}_I$.
- **The third column carries no** ω: turning the apse line inside the plane cannot move the plane's normal.
- **The third row,** $(s_\omega s_i,\ c_\omega s_i,\ c_i)$**, holds the** $\hat{\mathbf{K}}$ **components** of $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{w}}$. Its first two entries carry $\sin i$ and vanish at $i = 0$, and the third is then 1: an equatorial orbit's $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ lie in the reference plane and its normal is $\hat{\mathbf{K}}$.

To go from a frame that turns with the spacecraft (such as the local-vertical, local-horizontal frame used later for rendezvous, in the cw note) to inertial axes, a relative velocity needs an extra term $\omega \times \rho$, with ω the frame's angular velocity and ρ the relative position. Why does $\mathbf{v}_I = \mathbf{Q}\,\mathbf{v}_{PQW}$ need no such term?

Answer

Because $\mathbf{Q}$ does not depend on time. Ω, i and ω are constants of the two-body motion, so $d(\mathbf{Q}\mathbf{r}_{PQW})/dt = \mathbf{Q}\,\dot{\mathbf{r}}_{PQW}$. The LVLH frame turns with the target, so its rotation matrix has a time derivative, and that derivative is the $\omega \times \rho$ term. (Under perturbations such as J_2, Ω and ω drift slowly. Then $\mathbf{Q}$ changes too, and the perifocal frame is only *instantaneously* the right frame.)

There is a pleasant closed form for the inertial position. Substituting Equation 15 in Equation 4 and using the angle-addition formulas,

$$\begin{aligned}\hat{\mathbf{r}} &= \cos f\,(\cos\omega\,\hat{\mathbf{N}} + \sin\omega\,\hat{\mathbf{M}}) + \sin f\,(-\sin\omega\,\hat{\mathbf{N}} + \cos\omega\,\hat{\mathbf{M}}) \\ &= \cos u\,\hat{\mathbf{N}} + \sin u\,\hat{\mathbf{M}}, \qquad u \equiv \omega + f,\end{aligned}$$

the **argument of latitude** u: the angle from the ascending node to the body, measured in the orbital plane. Component by component this is the classical set

$$\begin{aligned} x &= r\,(\cos\Omega\cos u - \sin\Omega\cos i\sin u), \\ y &= r\,(\sin\Omega\cos u + \cos\Omega\cos i\sin u), \\ z &= r\sin i\,\sin u. \end{aligned} \tag{19}$$

The third line says that the body crosses the reference plane at $u = 0$ and $u = 180°$ (the nodes) and reaches its greatest height $r\sin i$ at $u = 90°$. It also explains the degeneracy: **only the sum $\omega + f$ ever matters for the position**.

Take the orbit of the worked example in Section 5.2 ($a = 10\,000$ km, $e = 0.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$) at $f = 120°$. Use Equation 19, not the matrix, to find $\mathbf{r}_I$. Where on the orbit is the body?

What you should find

$r = 8400/(1 + 0.4\cos 120°) = 8400/0.8 = 10\,500$ km, and $u = 60° + 120° = 180°$, so $\cos u = -1$ and $\sin u = 0$. Then $x = -r\cos\Omega = -8043.47$ km, $y = -r\sin\Omega = -6749.27$ km and $z = 0$. The body is crossing the reference plane going south: it is at the **descending node**, $\mathbf{r}_I = -r\hat{\mathbf{N}}$. Three products, no matrix.

When the angles break down

- **Circular orbit, $e = 0$:** there is no periapsis, so ω and f are separately undefined, but their sum $u = \omega + f$ is not, and Equation 19 needs only u.
- **Equatorial orbit, $i = 0°$ or $180°$:** the node line does not exist, so Ω and ω are separately undefined. Only one combination fixes the apse line: $\Omega + \omega$ for $i = 0°$, and $\Omega - \omega$ for $i = 180°$, where Equation 19 becomes $x = r\cos(\Omega - u)$, $y = r\sin(\Omega - u)$.

The orbital-elements note ("When the elements break down") gives the replacements: argument of latitude, longitude of periapsis, true longitude, and the non-singular equinoctial elements.

5 The chain, with numbers

5.1 The classical propagation chain

$$\boxed{\Delta t \;\to\; \text{anomaly } (E,\, D,\, H) \;\to\; f \;\to\; \mathbf{r}_{PQW}, \mathbf{v}_{PQW} \;\to\; \mathbf{r}_I, \mathbf{v}_I.}$$

Each arrow is one idea:

1. $\Delta t \to$ anomaly is **Kepler's equation** (or Barker's, or the hyperbolic one): the only transcendental step.
2. anomaly $\to f$ is a half-angle identity.
3. $f \to \mathbf{r}_{PQW}, \mathbf{v}_{PQW}$ is Equation 13: the conic, in its own plane.
4. $\mathbf{r}_{PQW}, \mathbf{v}_{PQW} \to \mathbf{r}_I, \mathbf{v}_I$ is one 3×3 rotation, Equation 18, built from Ω, i, ω.

Step 3 is where the geometry becomes a vector; step 4 is where the vector acquires an address in the sky. Only f changes with time, so **five elements are the orbit and the sixth is the clock**.

5.2 A worked example

Given. $a = 10\,000$ km, $e = 0.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$, $f = 30°$ and $\mu = 398\,600.4418$ km^3/s^2.

Step 1: the conic. $p = a(1-e^2) = 8400$ km. $r = p/(1+e\cos f) = 8400/1.34641 = 6238.81$ km, and $\sqrt{\mu/p} = 6.88857$ km/s.

Step 2: the perifocal state, Equation 13:

$$\mathbf{r}_{PQW} = 6238.81 \begin{bmatrix} \cos 30° \\ \sin 30° \\ 0 \end{bmatrix} = \begin{bmatrix} 5402.97 \\ 3119.41 \\ 0 \end{bmatrix} \text{km},$$

$$\mathbf{v}_{PQW} = 6.88857 \begin{bmatrix} -\sin 30° \\ 0.4 + \cos 30° \\ 0 \end{bmatrix} = 6.88857 \begin{bmatrix} -0.5 \\ 1.26603 \\ 0 \end{bmatrix} = \begin{bmatrix} -3.44429 \\ 8.72111 \\ 0 \end{bmatrix} \text{km/s}.$$

Step 3: the rotation, Equation 18 with $\Omega = 40°$, $i = 30°$, $\omega = 60°$:

$$\mathbf{Q} = \begin{bmatrix} -0.09907 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25000 & 0.86603 \end{bmatrix}.$$

Step 4: the inertial state.

$$\mathbf{r}_I = \mathbf{Q}\,\mathbf{r}_{PQW} = \begin{bmatrix} -3472.96 \\ 4138.92 \\ 3119.41 \end{bmatrix} \text{km}, \qquad \mathbf{v}_I = \mathbf{Q}\,\mathbf{v}_{PQW} = \begin{bmatrix} -7.87188 \\ -5.04776 \\ 0.68886 \end{bmatrix} \text{km/s}.$$

Checks.

- $|\mathbf{r}_I| = 6238.81$ km $= r$, and $|\mathbf{v}_I| = 9.37661$ km/s $= \sqrt{(\mu/p)(1 + 2e\cos f + e^2)}$: $\mathbf{Q}$ preserves lengths.
- $z = r \sin i \, \sin(\omega + f) = 6238.81 \times 0.5 \times \sin 90° = 3119.41$ km, matching: $u = \omega + f = 90°$, so the body is at its **northernmost point**.
- $|\mathbf{h}| = |\mathbf{r}_I \times \mathbf{v}_I| = 57\,864.01$ km^2/s $= \sqrt{\mu p}$, and $\mathbf{h}/h = (0.32139, -0.38302, 0.86603) = \hat{\mathbf{w}}$, the third column of $\mathbf{Q}$.
- Energy: $v^2/2 - \mu/r = -19.9300$ km^2/s^2 $= -\mu/2a$.

- **Round trip.** Computing $\mathbf{h}$, the node vector $\hat{\mathbf{K}} \times \mathbf{h}$ and the eccentricity vector from $(\mathbf{r}_I, \mathbf{v}_I)$, with the two-argument arctangent for the angles, recovers $(a, e, i, \Omega, \omega, f) = (10\,000,\ 0.4,\ 30^\circ,\ 40^\circ,\ 60^\circ,\ 30^\circ)$ to 10^{-9}. The forward map has no quadrant problem at all, every step being a sine or cosine of a known angle; all the difficulty is in *inverting* it (the orbital-elements note, "From $(\mathbf{r}, \mathbf{v})$ to the elements").

The same two formulas, every conic. Take $f = 90^\circ$ on conics with the same $p = 10\,000$ km. There $\cos f = 0$ for every e, so $r = p$ and $\mathbf{v}_{PQW} = \sqrt{\mu/p}\,(-1,\ e,\ 0)$:

e	conic	speed [km/s]	$\varepsilon = v^2/2 - \mu/r$ [km^2/s^2]
0	circle	6.3135	−19.930
0.5	ellipse	7.0587	−14.948
1	parabola	8.9286	0
1.5	hyperbola	11.3818	+24.913

The speed is $\sqrt{\mu/p}\,\sqrt{1+e^2}$ in every row. The sign of the energy flips exactly at $e = 1$, and at the parabola the speed is $\sqrt{2\mu/r}$, the escape speed at $r = p$. The same pair of formulas has handled all four conics without a branch. (perifocal-verify.py reproduces every number in this section.)

6 A bridge to the Lagrange coefficients

The second route, propagating the Cartesian state directly, uses the **Lagrange coefficients**. They too fall out of the perifocal state vector. We write them F and G, as the universal-variables note and the Lambert note do, to avoid a clash with the true anomaly f.

Let $(\mathbf{r}_0, \mathbf{v}_0)$ be the state at true anomaly f_0, and $(\mathbf{r}, \mathbf{v})$ the state at $f = f_0 + \Delta f$. All four vectors lie in the orbital plane, and $\mathbf{r}_0$, $\mathbf{v}_0$ are not parallel (since $h \neq 0$), so they form a basis of the plane:

$$\mathbf{r} = F\,\mathbf{r}_0 + G\,\mathbf{v}_0, \qquad \mathbf{v} = \dot{F}\,\mathbf{r}_0 + \dot{G}\,\mathbf{v}_0.$$

To find the coefficients, take cross products with $\mathbf{v}_0$ and $\mathbf{r}_0$; every cross product is along $\hat{\mathbf{w}}$, so only its $\hat{\mathbf{w}}$-component matters, and $\mathbf{r}_0 \times \mathbf{v}_0 = h\,\hat{\mathbf{w}}$:

$$F = \frac{\mathbf{r} \times \mathbf{v}_0}{h}, \qquad G = \frac{\mathbf{r}_0 \times \mathbf{r}}{h}, \qquad \dot{F} = \frac{\mathbf{v} \times \mathbf{v}_0}{h}, \qquad \dot{G} = \frac{\mathbf{r}_0 \times \mathbf{v}}{h}.$$

Now insert Equation 13. For G, $\mathbf{r}_0 \times \mathbf{r} = r_0\, r \sin\Delta f\ \hat{\mathbf{w}}$ at once. For F:

$$(\mathbf{r} \times \mathbf{v}_0)_w = r\sqrt{\frac{\mu}{p}}\Big[\cos f\,(e + \cos f_0) + \sin f \sin f_0\Big] = r\sqrt{\frac{\mu}{p}}\Big[e\cos f + \cos\Delta f\Big],$$

and $e\cos f = p/r - 1$ by Equation 2, so the bracket is $p/r - 1 + \cos\Delta f$. Dividing by $h = \sqrt{\mu p}$ gives F. For $\dot{G}$ the bracket is $e\cos f_0 + \cos\Delta f$ with $e\cos f_0 = p/r_0 - 1$. For $\dot{F}$,

$$\begin{aligned}(\mathbf{v} \times \mathbf{v}_0)_w &= \frac{\mu}{p}\Big[-\sin f\,(e + \cos f_0) + (e + \cos f)\sin f_0\Big] \\ &= \frac{\mu}{p}\Big[e\,(\sin f_0 - \sin f) - \sin\Delta f\Big],\end{aligned}$$

since $\sin f_0 \cos f - \cos f_0 \sin f = -\sin \Delta f$. The results:

$$\boxed{\begin{aligned} F &= 1 - \frac{r}{p}\,(1-\cos\Delta f), \quad G = \frac{r\,r_0}{\sqrt{\mu p}}\,\sin\Delta f, \\ \dot G &= 1 - \frac{r_0}{p}\,(1-\cos\Delta f), \quad \dot F = \frac{\sqrt{\mu}}{p^{3/2}}\Big[e\,(\sin f_0 - \sin f) - \sin\Delta f\Big]. \end{aligned}} \tag{20}$$

Taking the $\hat{\mathbf{w}}$ component of $\mathbf{r}\times\mathbf{v} = (F\dot G - \dot F G)\,\mathbf{r}_0\times\mathbf{v}_0$ and using the constancy of h gives the identity

$$F\dot G - \dot F\,G = 1,$$

a Wronskian-type identity. It is conservation of angular momentum in disguise: with $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$ and $\mathbf{v} = \dot F\mathbf{r}_0 + \dot G\mathbf{v}_0$,

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = (F\dot G - \dot F\,G)\,\mathbf{r}_0\times\mathbf{v}_0 = (F\dot G - \dot F\,G)\,\mathbf{h},$$

and $\mathbf{h} \neq \mathbf{0}$. As a numerical check on Equation 20, take the example above ($e = 0.4$, $p = 8400$ km, $f_0 = 30^\circ \to f = 110^\circ$): $F = 0.042679$, $G = 1033.277$ s, $\dot F = -9.518\times10^{-4}$ s^{-1}, $\dot G = 0.386256$, and $F\dot G - \dot F G = 1.000000$.

What must F, G, $\dot F$ and $\dot G$ be for a step $\Delta f = 0$? Check that Equation 20 gives them.

Answer

With no step, $\mathbf{r} = \mathbf{r}_0$ and $\mathbf{v} = \mathbf{v}_0$, so $F = \dot G = 1$ and $G = \dot F = 0$. In Equation 20, $1 - \cos 0 = 0$ gives $F = \dot G = 1$, $\sin 0 = 0$ gives $G = 0$, and in $\dot F$ both $\sin f_0 - \sin f$ and $\sin\Delta f$ vanish. Then $F\dot G - \dot F G = 1$ holds trivially.

So the two routes meet: Equation 20 is the classical, element-based expression of the Lagrange coefficients, in terms of the **geometry** (p, e, f_0, f). The universal formulation re-expresses the same coefficients in a variable (χ) that does not care which conic the body is on, and so replaces the Δf that is awkward to find (it needs Kepler's equation and its parabolic and hyperbolic siblings).

History. Lagrange gave the first direct and theoretically complete methods for determining the orbit of a comet from observations in the Berlin memoirs of 1778 and 1783. The coefficients F and G that express a later state through an earlier one are called **Lagrange coefficients**; they are the working tool of the orbit-determination methods of Gauss and Gibbs, which find an orbit from several observed positions.

7 Summary

1. $\mathbf{h}$ and the eccentricity vector $\mathbf{e} = (\mathbf{v}\times\mathbf{h})/\mu - \hat{\mathbf{r}}$ are **constants** of the two-body motion, with $\mathbf{e}\perp\mathbf{h}$. They define the **perifocal frame** $\hat{\mathbf{p}} = \mathbf{e}/e$, $\hat{\mathbf{w}} = \mathbf{h}/h$, $\hat{\mathbf{q}} = \hat{\mathbf{w}}\times\hat{\mathbf{p}}$, which is **fixed in inertial space** (no fictitious forces; no transport theorem).
2. In it the position is $\mathbf{r}_{PQW} = \dfrac{p}{1+e\cos f}(\cos f,\ \sin f,\ 0)^{\mathsf T}$ and the velocity is $\mathbf{v}_{PQW} = \sqrt{\mu/p}\,(-\sin f,\ e+\cos f,\ 0)^{\mathsf T}$: both depend on (p, e, f) only, and **hold for every conic**, because a never appears.
3. The velocity components depend on f alone, never on r, so the **hodograph is a circle** of radius μ/h centred at $(0, e\mu/h)$. Its origin is inside, on, or outside the circle for $e < 1, = 1, > 1$.
4. The state is checked three ways: $h = \sqrt{\mu p}$ along $+\hat{\mathbf{w}}$, the energy $-\mu(1-e^2)/2p$ independent of f, and the eccentricity vector returned as $(e, 0, 0)$.
5. The inertial state is $\mathbf{r}_I = \mathbf{Q}\mathbf{r}_{PQW}$, $\mathbf{v}_I = \mathbf{Q}\mathbf{v}_{PQW}$ with $\mathbf{Q} = [\hat{\mathbf{p}}\ \hat{\mathbf{q}}\ \hat{\mathbf{w}}] = \mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega)$: orthogonal, $\det = +1$, and derived here both geometrically and by multiplication. Only $\omega + f$ enters the position, which is why $e = 0$ needs the argument of latitude.

6. The classical chain is $\Delta t \to$ anomaly $\to f \to (\mathbf{r}, \mathbf{v})_{PQW} \to (\mathbf{r}, \mathbf{v})_I$: one transcendental step, one rotation. The same perifocal state gives the Lagrange coefficients $F, G, \dot{F}, \dot{G}$ in terms of Δf, the bridge to the universal formulation.

Everything in this note can be reproduced with perifocal-verify.py: the symbolic checks of the constants of motion, the velocity, the hodograph, the rotation (both derivations) and the Lagrange coefficients, and the numerical example with its round trip back to the elements.

Further reading

For the standard textbook treatments of the perifocal frame, the rotation and the orbit-determination problem see Bate, Mueller, and White (1971), Battin (1999), Curtis (2014) and Vallado (2013).

Bate, Roger R., Donald D. Mueller, and Jerry E. White. 1971. *Fundamentals of Astrodynamics*. New York: Dover.

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Vallado, David A. 2013. *Fundamentals of Astrodynamics and Applications*. 4th ed. Hawthorne, CA: Microcosm Press.