Patched Conics

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1 Where we are going

A spacecraft that leaves the Earth for Mars is pulled by the Sun, the Earth, Mars, the Moon, Jupiter and everything else, all the time. Its motion is an n-body problem, and the n-body problem has no closed-form solution. Yet mission designers size the propellant tank of a Mars probe, choose its launch date and decide where its engine must fire, with nothing more than a calculator and the two-body formulas of the first half of this course.

They can do this because, at almost every moment of the journey, **one** body dominates. Near the Earth, the Earth's pull decides the motion and the Sun's is a small correction; far from every planet, the Sun decides it. The *patched-conic method* takes this literally. It splits the journey into pieces, treats each piece as a two-body problem about the body that dominates there, and joins (patches) the pieces end to end. Each piece is a conic, hence the name.

The method needs one new idea: a rule for deciding **where one body stops dominating and the next takes over**. That rule is the *sphere of influence*, and most of the physics of this note is in deriving it.

Consider the classic example of going from Earth to Mars. To eliminate the tricky bit of motion through atmosphere, we will assume that the spacecraft is in a parking orbit around Earth. Then we give it a Δv at a particular point along the circular orbit so that it escapes Earth in an hyperbolic orbit. It travels on that hyperbolic orbit to reach the edge of Earth sphere of influence. Then after it travels under Sun's gravitational pull till it reaches the Mars sphere of influence. Then again it travels in a hyperbolic trajectory in Mars system.

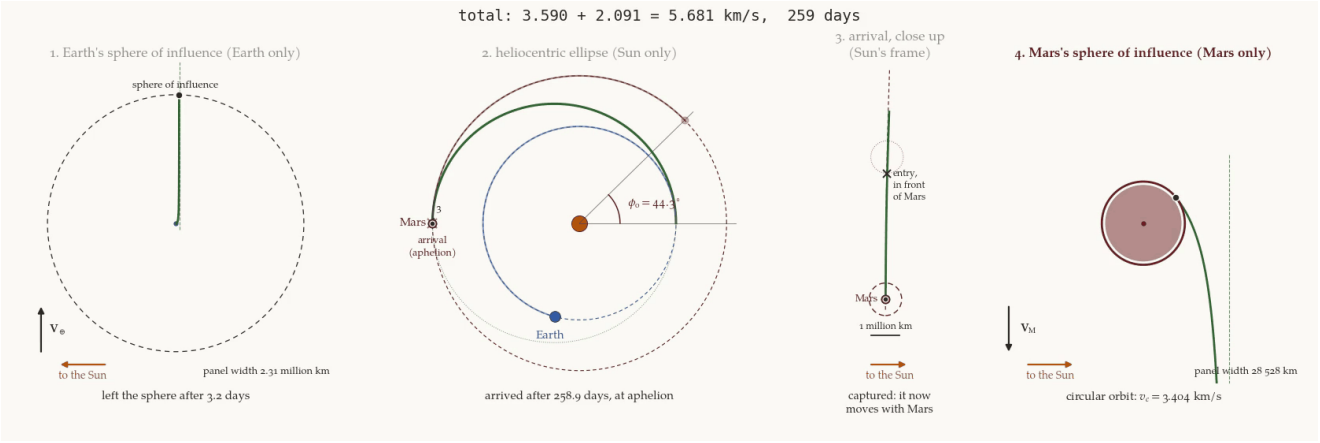

Figure 1: From the Earth to Mars, conic by conic (the numbers are worked in Section 6.1). Panel 1: inside the Earth's sphere of influence, half a turn of a 300 km parking orbit, the burn (star), and the escape hyperbola out to the edge of the sphere; the dashed line is the straight line the hyperbola approaches, parallel to the Earth's velocity. Panel 2: in the Sun's frame, the transfer ellipse with the Earth and Mars; at launch Mars is 44.3° ahead of the Earth, and it reaches the far end of the ellipse together with the spacecraft. Panel 3: the last 3.6 days, still in the Sun's frame, close up (the thin box marked 3 in panel 2, to scale). Mars is faster than the spacecraft and catches it up from behind, so its sphere of influence (dashed circle) sweeps over the spacecraft from in front; the cross marks where the spacecraft enters it. Panel 4: inside Mars's sphere of influence, the arrival hyperbola in to its closest point, the capture burn, and a 300 km circular orbit. Each panel runs on its own clock, except that panels 3 and 4 share one once the spacecraft is inside Mars's sphere; the two planet-centred panels zoom with the spacecraft. *Script: anim-pc-earth-mars.py.*

When you finish, you should be able to answer all of the following without hand-waving.

1. Why is the boundary between the Earth's region and the Sun's region **not** where their pulls are equal?
2. Where does the formula $r_{\text{SOI}} = R\,(m/M)^{2/5}$ come from, and how wrong is it?
3. What exactly is *patched* at the boundary: position, velocity, energy?

4. How do you turn a heliocentric transfer into a Δv from a parking orbit, and **where** must the engine fire?
5. How does a planetary flyby change a spacecraft's heliocentric speed, when its speed relative to the planet does not change at all?
6. What changes when the target is **out of the plane**: why does a launch site's latitude limit the departure, and why is a 180° transfer between two inclined orbits a disaster?
7. How good is the answer, compared with integrating the real equations of motion?

The note is in three stages. Section 2 to Section 4 build the tools: the prerequisites, and the sphere of influence. Section 5 states the method, and Section 6 and Section 7 work six examples with it, three in a plane and three out of it (Example 2, the Moon, has two more versions, 2a and 2b, in which the transfer is designed instead of given). Section 8 then checks the method against numerical integration.

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

2 What you need first

Everything in this note rests on the two-body problem. This section collects the results we will use, with short derivations, so that the rest of the note can use them without stopping. If you are comfortable with them, skip to Section 2.6 at the end, which chains them into one complete calculation.

2.1 The hyperbola, in the variables we will need

Every arrival at, or departure from, a planet in this note is a hyperbola about that planet. The hyperbola is usually described by a and e, but the quantities a mission designer is *given* are different: how fast the spacecraft approaches from far away, and how close it should pass. So we rewrite the hyperbola in those terms.

Energy. Let $\mu = Gm$ be the gravitational parameter of the central body, r and v the spacecraft's distance from it and speed, and a and e the semi-major axis and eccentricity of the conic. The energy per unit mass, ε, is constant on a two-body orbit and depends only on a:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}. \tag{1}$$

An ellipse has $a > 0$ and $\varepsilon < 0$; a hyperbola has $a < 0$ and $\varepsilon > 0$. Solved for v, Equation 1 is *vis-viva*: $v^2 = \mu\,(2/r - 1/a)$.

Excess speed. Far from the planet, $r \to \infty$ and Equation 1 gives $\varepsilon = v_\infty^2/2$. The speed v_∞ that remains "at infinity" is the *hyperbolic excess speed*. Comparing with $\varepsilon = -\mu/2a$:

$$v_\infty^2 = -\frac{\mu}{a}, \qquad \text{i.e.} \qquad |a| = \frac{\mu}{v_\infty^2}. \tag{2}$$

Periapsis. The closest approach is $r_p = a(1-e) = |a|(e-1)$. Given v_∞ and r_p, this fixes the eccentricity:

$$e = 1 + \frac{r_p}{|a|} = 1 + \frac{r_p\, v_\infty^2}{\mu}. \tag{3}$$

Speed at periapsis. By *vis-viva* with $a = -\mu/v_\infty^2$,

$$v_p^2 = \mu\Big(\frac{2}{r_p} + \frac{v_\infty^2}{\mu}\Big) = v_\infty^2 + \frac{2\mu}{r_p} = v_\infty^2 + v_{\mathrm{esc}}^2(r_p). \tag{4}$$

The speed at periapsis is the excess speed and the local escape speed added *in quadrature*.

Shape. From the two-body note, the orbit equation is $r = p/(1 + e\cos f)$, where f is the true anomaly, $h = |\mathbf{r} \times \mathbf{v}|$ the angular momentum per unit mass and $p = h^2/\mu = a(1 - e^2)$ the semi-latus rectum. On a hyperbola $a < 0$ and $e > 1$, so p is positive, as it must be.

Directions of the asymptotes. As $r \to \infty$ the orbit equation needs $1 + e\cos f \to 0$, so the true anomaly tends to

$$f_\infty = \arccos\Big(-\frac{1}{e}\Big), \qquad 90^\circ < f_\infty < 180^\circ. \tag{5}$$

The incoming asymptote is *parallel* to the direction $f = -f_\infty$, the outgoing one to $f = +f_\infty$. The asymptotes themselves do not pass through the planet: they cross at the centre of the hyperbola, $|a|e$ from the planet along the apse line, beyond periapsis, and each misses the planet by the distance b found below. Drawing them through the planet is a common slip.

Turning angle. Between the incoming and outgoing asymptotes the velocity relative to the planet turns through $\delta = 2f_\infty - 180^\circ$. With $\cos f_\infty = -1/e$ this simplifies: $\sin(\delta/2) = \sin(f_\infty - 90^\circ) = -\cos f_\infty = 1/e$, so

$$\delta = 2\arcsin\frac{1}{e}. \tag{6}$$

A close, slow pass (large μ, small r_p, small v_∞) has e near 1 and turns the velocity by nearly 180°; a distant, fast pass hardly turns it at all.

Impact parameter. Far away, the incoming path is a straight line that would miss the planet by a distance b, the *impact parameter* or *aim radius*. There the velocity is v_∞ and is perpendicular to the miss distance, so $h = b\,v_\infty$. At periapsis $h = r_p v_p$. Equating, and using Equation 4,

$$b = \frac{r_p v_p}{v_\infty} = r_p\sqrt{1 + \frac{2\mu}{r_p v_\infty^2}}. \tag{7}$$

b is always larger than r_p: gravity bends the path inwards, so the spacecraft must be aimed wide of the periapsis it will actually reach. This is gravitational focusing.

Two more forms of the same relations are handy when b and v_∞ are what is given. Since $h^2 = \mu p = \mu|a|(e^2 - 1)$ and $h = bv_\infty$, $b = |a|\sqrt{e^2 - 1}$, so $e = \sqrt{1 + b^2/|a|^2}$ and

$$r_p = |a|(e - 1) = \sqrt{b^2 + |a|^2} - |a|, \qquad \tan\frac{\delta}{2} = \frac{1/e}{\sqrt{1 - 1/e^2}} = \frac{|a|}{b} = \frac{\mu}{b\,v_\infty^2}, \tag{8}$$

with $|a| = \mu/v_\infty^2$. Setting r_p equal to the planet's radius R in Equation 7 gives the *grazing* aim radius $b_{\mathrm{graze}} = R\sqrt{1 + 2\mu/(Rv_\infty^2)}$: every spacecraft aimed inside it hits the planet, so a planet presents a target $(b_{\mathrm{graze}}/R)^2$ times larger than its disc.

2.2 A burn at periapsis

Leaving a circular parking orbit of radius r_p on a hyperbola of excess speed v_∞ is best done with one tangential burn at the periapsis of the hyperbola, which is then a point of the parking orbit. From Equation 4, the burn is

$$\Delta v = \sqrt{v_\infty^2 + \frac{2\mu}{r_p}} - \sqrt{\frac{\mu}{r_p}}. \tag{9}$$

Capture is the same burn, run backwards, and it cannot be avoided. A spacecraft that enters a sphere of influence of radius r_{SOI} with speed v relative to the planet stays bound only if its energy $v^2/2 - \mu/r_{\mathrm{SOI}}$ is negative, that is if $v < \sqrt{2\mu/r_{\mathrm{SOI}}}$. For Mars that is 0.385 km/s, against 2.649 km/s for the Hohmann arrival of Section 6.1: in the patched-conic model a planet never captures a spacecraft by itself. Why burn at periapsis? A small burn there changes the energy by $\Delta\varepsilon \approx v\,\Delta v$, and v is largest at periapsis: the Oberth effect of the impulsive-model note.

2.3 The Hohmann transfer

Between circular orbits of radii $r_1 < r_2$ about a body μ, the Hohmann ellipse has $a_t = (r_1 + r_2)/2$. *Vis-viva* gives its speeds at the two apsides, and the time is half its period:

$$V_{t1} = \sqrt{\mu\Big(\frac{2}{r_1} - \frac{1}{a_t}\Big)}, \qquad V_{t2} = \sqrt{\mu\Big(\frac{2}{r_2} - \frac{1}{a_t}\Big)}, \qquad t_{12} = \pi\sqrt{\frac{a_t^3}{\mu}}. \tag{10}$$

2.4 Frames that move

We will describe one spacecraft in two frames: one centred on the Sun, one centred on a planet. Both frames keep their axes parallel to the same fixed directions; the planet frame simply *moves with the planet*. For two such frames, positions and velocities add:

$$\mathbf{R}_{\mathrm{s/c}} = \mathbf{R}_{\mathrm{planet}} + \mathbf{r}, \qquad \mathbf{V}_{\mathrm{s/c}} = \mathbf{V}_{\mathrm{planet}} + \mathbf{v}. \tag{11}$$

Convention for this note. Capital letters ($\mathbf{R}$, $\mathbf{V}$) are heliocentric; small letters ($\mathbf{r}$, $\mathbf{v}$) are relative to a planet. So $\mathbf{v}_\infty = \mathbf{V}_{\mathrm{s/c}} - \mathbf{V}_{\mathrm{planet}}$ is the spacecraft's velocity relative to the planet, far from it.

2.5 Phase angle and synodic period

A Hohmann transfer only works if the target planet is at the arrival point when the spacecraft gets there. During the flight time t_{12} the target moves through $n_2 t_{12}$, with n_2 its mean motion. It must therefore *lead* the departure planet at launch by the **phase angle**

$$\phi_0 = 180^\circ - n_2\, t_{12}. \tag{12}$$

For a transfer outwards the target is slower, $n_2 t_{12} < 180^\circ$ and $\phi_0 > 0$: the target leads. For a transfer inwards the target is faster, and over the flight it moves more than half a turn; then $\phi_0 < 0$, and the target must *trail* the departure planet by $|\phi_0|$ (Earth to Venus: $t_{12} = 146.1$ days, $n_2 t_{12} = 234.0^\circ$, $\phi_0 = -54.0^\circ$).

The relative angle of two planets repeats after the **synodic period** S. Planet 1 gains on planet 2 at the rate $n_1 - n_2$, and a full lap of 360° takes

$$S = \frac{2\pi}{|n_1 - n_2|} = \frac{1}{|1/T_1 - 1/T_2|}. \tag{13}$$

2.6 One full cycle, step by step

Here is how these results chain together into a complete patched-conic calculation, for a Hohmann transfer from planet 1 to planet 2 with capture at planet 2. The transfer may go outwards ($r_1 < r_2$, Earth to Mars) or inwards ($r_1 > r_2$, Earth to Venus); the formulas are the same, and only the *directions* of the excess velocities differ. Section 5 explains why the cut into pieces is legitimate, and Section 6.1 runs the chain with numbers for a transfer outwards.

Given. The gravitational parameters $\mu_\odot$ of the Sun and μ_1, μ_2 of the two planets; the radii r_1 and r_2 of the planets' circular, coplanar heliocentric orbits; the radius r_{p1} of the circular parking orbit about planet 1; and the radius r_{p2} of the circular orbit wanted about planet 2. Both parking orbits are prograde: they turn the same way as the planets go round the Sun.

Calculate, in order.

- *Planets' heliocentric speeds* (circular orbits): $V_{P1} = \sqrt{\mu_\odot / r_1}$, $V_{P2} = \sqrt{\mu_\odot / r_2}$.
- *Transfer ellipse* about the Sun, Equation 10 with $\mu = \mu_\odot$: semi-major axis $a_t = (r_1 + r_2)/2$; speed at departure $V_{t1} = \sqrt{\mu_\odot(2/r_1 - 1/a_t)}$; speed at arrival $V_{t2} = \sqrt{\mu_\odot(2/r_2 - 1/a_t)}$; flight time $t_{12} = \pi\sqrt{a_t^3/\mu_\odot}$. Outwards, the spacecraft leaves from the perihelion of the ellipse and arrives at its aphelion; inwards, the other way round.
- *Excess speeds*, Equation 11. At both ends the two velocities are tangential and parallel, so the vector difference is a difference of speeds:

  $$v_{\infty 1} = |V_{t1} - V_{P1}|, \qquad v_{\infty 2} = |V_{t2} - V_{P2}|.$$

  The sign of each difference gives the direction. An apsis of the transfer ellipse is faster than the circular orbit there if it is the perihelion, slower if it is the aphelion. So:

	$\mathbf{v}_{\infty 1}$ at planet 1	$\mathbf{v}_{\infty 2}$ at planet 2
outwards, $r_1 < r_2$	forwards, along $\mathbf{V}_{P1}$ ($V_{t1} > V_{P1}$)	backwards, against $\mathbf{V}_{P2}$ ($V_{t2} < V_{P2}$)
inwards, $r_1 > r_2$	backwards, against $\mathbf{V}_{P1}$ ($V_{t1} < V_{P1}$)	forwards, along $\mathbf{V}_{P2}$ ($V_{t2} > V_{P2}$)

- *Departure hyperbola* about planet 1, from Equation 4, Equation 9, Equation 3 and Equation 5: parking speed $v_{c1} = \sqrt{\mu_1 / r_{p1}}$; periapsis speed $v_{p1} = \sqrt{v_{\infty 1}^2 + 2\mu_1 / r_{p1}}$; burn $\Delta v_1 = v_{p1} - v_{c1}$. This burn speeds the spacecraft up in either case: escaping always needs more speed. Going inwards, the spacecraft escapes *backwards*, so it ends up slower than the planet in the Sun's frame, and falls towards the Sun.
- *Where to burn*, Equation 20 (derived in Section 5 below): eccentricity $e_1 = 1 + r_{p1} v_{\infty 1}^2 / \mu_1$ and asymptote angle $f_{\infty 1} = \arccos(-1/e_1)$. The burn is made $f_{\infty 1}$ round the parking orbit *before* the point whose radius vector is parallel to $\mathbf{v}_{\infty 1}$. Measured from the anti-Sun line in the direction of motion, it is at

  $$\theta_{\text{burn}} = 90° - f_{\infty 1} \text{ (outwards: night side)}, \qquad \theta_{\text{burn}} = -90° - f_{\infty 1} \text{ (inwards: day side)}.$$

- *Arrival hyperbola* about planet 2, from Equation 7, Equation 4 and Equation 9. The spacecraft enters planet 2's sphere of influence on the side its $\mathbf{v}_{\infty 2}$ comes from: the leading side when going outwards (the planet catches it up), the trailing side when going inwards (it catches the planet up). The asymptote may pass on the planet's sunward side (a sunlit-side approach) or on its far side (a dark-side approach), at the same cost (Figure 10, below). Going outwards, the sunlit side gives a prograde capture orbit and the dark side a retrograde one; going inwards $\mathbf{v}_{\infty 2}$ points forwards instead of backwards, and the rule flips. Aim radius $b_2 = r_{p2}\sqrt{1 + 2\mu_2/(r_{p2}v_{\infty 2}^2)}$; periapsis speed $v_{p2} = \sqrt{v_{\infty 2}^2 + 2\mu_2/r_{p2}}$; capture orbit speed $v_{c2} = \sqrt{\mu_2/r_{p2}}$; capture burn $\Delta v_2 = v_{p2} - v_{c2}$. (For a flyby instead of a capture, there is no burn: $e_2 = 1 + r_{p2}v_{\infty 2}^2/\mu_2$, and the relative velocity turns through $\delta_2 = 2\arcsin(1/e_2)$, Equation 6.)
- *Total cost*: $\Delta v = \Delta v_1 + \Delta v_2$.
- *Timing*, Equation 12 and Equation 13: mean motions $n_1 = \sqrt{\mu_\odot/r_1^3}$, $n_2 = \sqrt{\mu_\odot/r_2^3}$; phase angle $\phi_0 = 180° - n_2 t_{12}$, positive outwards (planet 2 leads planet 1 at launch) and negative inwards (planet 2 trails by $|\phi_0|$); the opportunity recurs every $S = 2\pi/|n_1 - n_2|$.

The time spent inside the two spheres of influence is neglected in t_{12}; for the Earth–Mars transfer of Section 6.1 it is under 6 days of a 259-day trip.

3 Why one body at a time?

Consider a spacecraft just leaving the Earth, at a few Earth radii. The Earth pulls it with $\mu_\oplus/r^2$, a few m/s^2. The Sun pulls it with about $\mu_\odot/(1\,\text{AU})^2 = 5.93\,\text{mm/s}^2$, a thousand times less. But the Sun also pulls the **Earth** with almost the same acceleration, so in a frame moving with the Earth the Sun's effect nearly cancels. What survives is the *difference* between the Sun's pull on the spacecraft and on the Earth: the tidal acceleration, which is smaller still.

Now go far from the Earth, half way out to Mars's orbit: 112 days after launch on the transfer of Section 6.1, the spacecraft is 1.26 AU from the Sun and 0.31 AU from the Earth. The Earth's pull has fallen to about 0.19 μm/s^2, and the Sun's is about 3.7 mm/s^2, some twenty thousand times larger. Here it is natural to measure the motion from the Sun, and to treat the Earth's pull as the small correction.

So the *same* spacecraft, in the *same* gravitational field, is best described relative to the Earth in one region and relative to the Sun in another. Somewhere in between there must be a boundary. The question is where, and by what criterion.

4 The sphere of influence

4.1 A first guess, and why it fails

The obvious guess is the distance at which the two pulls are equal. Measure r from the planet (mass m, gravitational parameter $\mu_p = Gm$) along the line to the Sun (mass M, $\mu_\odot = GM$, at distance R from the planet). The pulls are equal when

$$\frac{\mu_p}{r^2} = \frac{\mu_\odot}{(R-r)^2} \quad\Longrightarrow\quad \frac{R-r}{r} = \sqrt{\frac{\mu_\odot}{\mu_p}} = \sqrt{\frac{M}{m}} \quad\Longrightarrow\quad r = \frac{R\sqrt{m/M}}{1+\sqrt{m/M}}.$$

For the Earth, $m/M = \mu_\oplus/\mu_\odot = 3.0035 \times 10^{-6}$ and $r \approx 259\,000$ km.

This cannot be right. The Moon is at 384 400 km, well *outside* that distance: the Sun pulls the Moon about twice as hard as the Earth does. Yet the Moon goes round the Earth, not round the Sun on an independent orbit. The guess fails because it compares the wrong things. What decides whether the Earth is a good centre is not the size of the Sun's pull, but how much that pull **disturbs the motion relative to the Earth**. And a pull that acts almost equally on the Earth and on the Moon hardly disturbs their relative motion at all.

4.2 Two descriptions of the same motion

Let us make this precise. Three bodies (Figure 2): the Sun (mass M, $\mu_\odot = GM$), a planet (mass m, $\mu_p = Gm$) and a spacecraft whose own mass is negligible. Call

- $\mathbf{R}$ the planet's position relative to the Sun,
- $\mathbf{r}$ the spacecraft's position relative to the planet,
- $\rho = \mathbf{R} + \mathbf{r}$ the spacecraft's position relative to the Sun,

and θ the angle between $\mathbf{r}$ and the Sun–planet line.

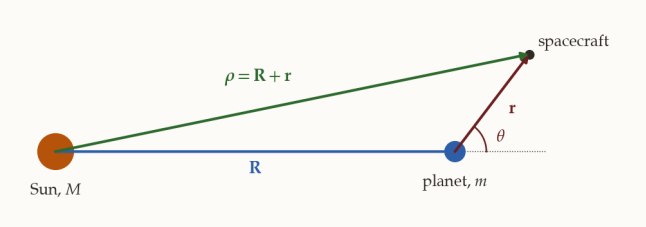

Figure 2: The three bodies of the derivation. $\mathbf{R}$: planet relative to the Sun; $\mathbf{r}$: spacecraft relative to the planet; $\rho = \mathbf{R} + \mathbf{r}$: spacecraft relative to the Sun. *Script: fig-pc-patched-conics.py.*

In an inertial frame, with positions $\mathbf{X}_\odot$, $\mathbf{X}_p$, $\mathbf{X}_s$, Newton's law gives each body's acceleration. The spacecraft is pulled by both; the planet is pulled by the Sun; the Sun is pulled by the planet (the spacecraft is too light to pull anything):

$$\ddot{\mathbf{X}}_s = -\frac{\mu_\odot}{\rho^3}\rho - \frac{\mu_p}{r^3}\mathbf{r}, \qquad \ddot{\mathbf{X}}_p = -\frac{\mu_\odot}{R^3}\mathbf{R}, \qquad \ddot{\mathbf{X}}_\odot = +\frac{\mu_p}{R^3}\mathbf{R}.$$

Relative to the planet. $\mathbf{r} = \mathbf{X}_s - \mathbf{X}_p$, so $\ddot{\mathbf{r}} = \ddot{\mathbf{X}}_s - \ddot{\mathbf{X}}_p$:

$$\ddot{\mathbf{r}} = \underbrace{-\frac{\mu_p}{r^3}\mathbf{r}}_{\text{main: the planet}} \underbrace{-\,\mu_\odot\Big(\frac{\rho}{\rho^3} - \frac{\mathbf{R}}{R^3}\Big)}_{\text{perturbation: the Sun}}. \tag{14}$$

The Sun's contribution is the *difference* between its pull on the spacecraft (ρ/ρ^3) and its pull on the planet ($\mathbf{R}/R^3$), exactly as argued in words.

Relative to the Sun. $\rho = \mathbf{X}_s - \mathbf{X}_\odot$, so $\ddot{\rho} = \ddot{\mathbf{X}}_s - \ddot{\mathbf{X}}_\odot$:

$$\ddot{\rho} = \underbrace{-\frac{\mu_\odot}{\rho^3}\rho}_{\text{main: the Sun}} \underbrace{-\,\mu_p\Big(\frac{\mathbf{r}}{r^3} + \frac{\mathbf{R}}{R^3}\Big)}_{\text{perturbation: the planet}}. \tag{15}$$

The second perturbing term, $\mathbf{R}/R^3$, is there because the Sun itself is being accelerated by the planet: the "indirect" term.

Both equations are exact, and they describe the same motion. Each is a two-body problem plus a perturbation. If we must throw one perturbation away (that is what the patched-conic method does), we should use the description in which the perturbation is **relatively** smaller.

To compare the two we need sizes, so give the two parts of each equation names. Write Equation 14 and Equation 15 as

$$\ddot{\mathbf{r}} = \mathbf{a}_{\text{planet}} + \mathbf{p}_{\text{planet}}, \qquad \ddot{\rho} = \mathbf{a}_{\text{Sun}} + \mathbf{p}_{\text{Sun}},$$

where $\mathbf{a}$ is the main (two-body) acceleration and $\mathbf{p}$ the perturbing acceleration, the underbraced terms. Both are vectors. Laplace's comparison uses only their lengths, which we denote by capitals:

$$A = |\mathbf{a}|, \qquad P = |\mathbf{p}|.$$

So P_{planet}, for instance, is a scalar: the magnitude of the Sun's perturbing acceleration when the motion is measured from the planet.

4.3 The two perturbations, near the planet

Near the planet, $r \ll R$. We now find the leading term of each perturbation in the small ratio r/R.

The Sun's tidal term, in Equation 14. We need ρ/ρ^3 with $\rho = \mathbf{R} + \mathbf{r}$. First the magnitude:

$$\rho^2 = R^2 + 2\,\mathbf{R}\cdot\mathbf{r} + r^2 = R^2\Big(1 + \frac{2\,\hat{\mathbf{R}}\cdot\mathbf{r}}{R} + \frac{r^2}{R^2}\Big).$$

Raise to the power $-3/2$ and keep terms of first order in r/R, using $(1+x)^{-3/2} \approx 1 - \frac{3}{2}x$:

$$\frac{1}{\rho^3} = \frac{1}{R^3}\Big(1 - \frac{3\,\hat{\mathbf{R}}\cdot\mathbf{r}}{R} + O(r^2/R^2)\Big).$$

Now multiply by $\rho = \mathbf{R} + \mathbf{r}$, term by term, writing $\mathbf{R}/R = \hat{\mathbf{R}}$:

$$\frac{\rho}{\rho^3} = \frac{\mathbf{R}}{R^3} + \frac{\mathbf{r}}{R^3} - \frac{3\,(\hat{\mathbf{R}}\cdot\mathbf{r})\,\hat{\mathbf{R}}}{R^3} - \frac{3\,(\hat{\mathbf{R}}\cdot\mathbf{r})\,\mathbf{r}}{R^4} + O\Big(\frac{r^2}{R^4}\Big).$$

The first term is of zeroth order in r/R; the next two are of first order; the fourth is of second order and joins the remainder. Subtracting $\mathbf{R}/R^3$ cancels the zeroth-order term (that is the near-equal pull on spacecraft and planet), and what remains is

$$\frac{\rho}{\rho^3} - \frac{\mathbf{R}}{R^3} = \frac{1}{R^3}\Big[\mathbf{r} - 3\,(\hat{\mathbf{R}}\cdot\mathbf{r})\,\hat{\mathbf{R}}\Big] + O\Big(\frac{r^2}{R^4}\Big).$$

Multiplying by $-\mu_\odot$ (Equation 14) gives the perturbing acceleration itself:

$$\mathbf{p}_{\text{planet}} \approx -\frac{\mu_\odot}{R^3}\Big[\mathbf{r} - 3\,(\hat{\mathbf{R}}\cdot\mathbf{r})\,\hat{\mathbf{R}}\Big].$$

To find its length, resolve $\mathbf{r}$ along $\hat{\mathbf{R}}$ and along the unit vector $\hat{\mathbf{e}}$ perpendicular to $\hat{\mathbf{R}}$ in the plane of $\mathbf{R}$ and $\mathbf{r}$: $\mathbf{r} = r\cos\theta\,\hat{\mathbf{R}} + r\sin\theta\,\hat{\mathbf{e}}$, so $\hat{\mathbf{R}}\cdot\mathbf{r} = r\cos\theta$. The bracket is then

$$\mathbf{r} - 3\,r\cos\theta\,\hat{\mathbf{R}} = -2r\cos\theta\,\hat{\mathbf{R}} + r\sin\theta\,\hat{\mathbf{e}},$$

of length $r\sqrt{4\cos^2\theta + \sin^2\theta} = r\sqrt{1+3\cos^2\theta}$. The scalar factor $\mu_\odot/R^3$ in front is positive, so the magnitude of the tidal perturbation is

$$P_{\text{planet}} = |\mathbf{p}_{\text{planet}}| \approx \frac{\mu_\odot\, r}{R^3}\sqrt{1+3\cos^2\theta}, \tag{16}$$

and the main acceleration in this description is $A_{\text{planet}} = |\mathbf{a}_{\text{planet}}| = \mu_p/r^2$.

The factor $\mu_\odot\, r/R^3$ is worth reading physically. The Sun's pull on the planet has size $\mu_\odot/R^2$. Move a distance r *along* the Sun–planet line ($\theta = 0$) and that pull changes by $|\frac{d}{dR}(\mu_\odot/R^2)|\,r = 2\mu_\odot\, r/R^3$: this is Equation 16 with $\sqrt{4} = 2$. Move a distance r *across* the line ($\theta = 90^\circ$) and the pull keeps its size but turns through the angle r/R, a change of $(\mu_\odot/R^2)(r/R) = \mu_\odot\, r/R^3$: Equation 16 with $\sqrt{1} = 1$. Either way, the perturbation is the *gradient* of the Sun's field times the distance from the planet, which is why it is of first order in r and carries R^3 in the denominator.

The planet's pull, in Equation 15. The two perturbing terms in $\mathbf{p}_{\text{Sun}}$ have magnitudes μ_p/r^2 and μ_p/R^2; near the planet the first is larger by $(R/r)^2$, so

$$P_{\text{Sun}} \approx \frac{\mu_p}{r^2},$$

and the main acceleration is $A_{\text{Sun}} = |\mathbf{a}_{\text{Sun}}| = \mu_\odot/\rho^2 \approx \mu_\odot/R^2$.

4.4 Laplace's criterion and the radius

Form the ratio of perturbation to main acceleration in each description. The gravitational parameters enter only as $\mu_\odot/\mu_p = M/m$, since G cancels:

$$Q_{\text{planet}} = \frac{P_{\text{planet}}}{A_{\text{planet}}} = \frac{M}{m}\Big(\frac{r}{R}\Big)^3\sqrt{1+3\cos^2\theta}, \qquad Q_{\text{Sun}} = \frac{P_{\text{Sun}}}{A_{\text{Sun}}} = \frac{m}{M}\Big(\frac{R}{r}\Big)^2. \tag{17}$$

Close to the planet (r small), Q_{planet} is tiny and Q_{Sun} is huge: use the planet. Far away the opposite holds: use the Sun. **Laplace's criterion** is to use whichever description has the smaller ratio. The boundary is where the two are equal:

$$\frac{M}{m}\Big(\frac{r}{R}\Big)^3\sqrt{1+3\cos^2\theta} = \frac{m}{M}\Big(\frac{R}{r}\Big)^2 \quad\Longrightarrow\quad \Big(\frac{r}{R}\Big)^5 = \Big(\frac{m}{M}\Big)^2\frac{1}{\sqrt{1+3\cos^2\theta}}.$$

Take the fifth root:

$$r = R\Big(\frac{m}{M}\Big)^{2/5}(1+3\cos^2\theta)^{-1/10}.$$

The last factor lies between $4^{-1/10} = 0.871$ (on the Sun–planet line) and 1 (across it): the boundary is not quite a sphere. It varies by only 13%, and the boundary

is a convention anyway, so it is dropped. That leaves the **Laplace radius of the sphere of influence**:

$$\boxed{r_{\mathrm{SOI}} = R\Big(\frac{m}{M}\Big)^{2/5}} \tag{18}$$

Figure 3 plots the two ratios for the Earth. They cross at $r_{\mathrm{SOI}} = 925\,000$ km. The naive equal-pull radius is far inside it; the Hill radius ($R(m/3M)^{1/3}$, the distance of the Lagrange points L_1 and L_2) is outside it. The Hill radius is derived, from the equations of the restricted three-body problem, in the restricted three-body note, which also compares it with the sphere of influence (there): for the Earth, $r_{\mathrm{H}} = 1.497 \times 10^6$ km against $r_{\mathrm{SOI}} = 9.25 \times 10^5$ km. The two answer different questions, the Hill radius *where a body can keep a satellite in the rotating frame*, the sphere of influence *where to switch the central body in a patched-conic calculation*.

TODO: We have to derive Hill radius in the CR3BP section and reference it here.

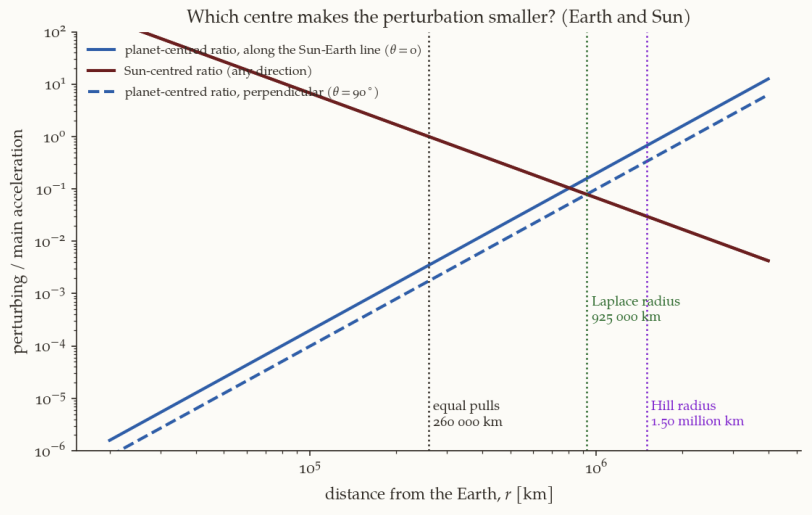

Figure 3: Laplace's two ratios for the Earth and the Sun, Equation 17. Blue: perturbation over main acceleration if the motion is measured from the Earth (solid along the Sun–Earth line, dashed across it). Maroon: the same if measured from the Sun. They cross at the Laplace radius. *Script: fig-pc-patched-conics.py.*

History. Laplace introduced the sphere of influence in his study of comets that pass close to planets, in the early 1800s. His test case was Lexell's comet, which Messier discovered on 14 June 1770, shortly before it passed within 0.015 AU of the Earth on 1 July, and was then thrown into a different orbit by a close encounter with Jupiter in 1779; it has not been seen since. To compute such an encounter, Laplace treated the comet as orbiting the Sun until it came near Jupiter, then as orbiting Jupiter, then the Sun again: the patched-conic idea, a century and a half before spaceflight.

How good is the split, at the boundary? On Laplace's exact boundary, $r = R(m/M)^{2/5}(1 + 3\cos^2\theta)^{-1/10}$, the two ratios are equal by construction. Putting this r into either expression of Equation 17 gives their common value:

$$Q_{\mathrm{planet}} = Q_{\mathrm{Sun}} = \Big(\frac{m}{M}\Big)^{1/5}(1 + 3\cos^2\theta)^{1/5}.$$

For the Earth this runs from $(m/M)^{1/5} = 0.079$ across the Sun–Earth line to $4^{1/5} \times 0.079 = 0.104$ along it: whichever centre we choose there, the neglected perturbation is still 8 to 10% of the main acceleration.

The sphere r_{SOI} drops the angle factor, and on it the two ratios are no longer equal everywhere. Putting $r = r_{\mathrm{SOI}}$ in Equation 17 gives $Q_{\mathrm{Sun}} = (m/M)^{1/5}$ in every direction, but $Q_{\mathrm{planet}} = (m/M)^{1/5}\sqrt{1 + 3\cos^2\theta}$: the same across the line, twice as large along it. Along the line the exact boundary is at $4^{-1/10} r_{\mathrm{SOI}} = 0.871\, r_{\mathrm{SOI}}$, so the sphere reaches 15% beyond it, into the region where Laplace's criterion would already prefer the Sun. That is the price of rounding the boundary: for the Earth, 16% instead of 10% at the worst point of the sphere. The patched-conic method is an approximation by design, and it is worst right at the patch.

4.5 Numbers

body	about	R	r_{SOI} [km]	in body radii	r_{SOI}/R	Q_{Sun} at r_{SOI}
Venus	Sun	0.723 AU	616 000	102	0.0057	0.075
Earth	Sun	1 AU	925 000	145	0.0062	0.079
Moon	Earth	384 400 km	66 200	38	0.17	0.41
Mars	Sun	1.524 AU	577 000	170	0.0025	0.050
Jupiter	Sun	5.203 AU	48.2 million	674	0.062	0.25

Two facts in this table are what make the patched-conic method work.

1. **Seen from the Sun, a sphere of influence is a point.** The Earth's is 0.6% of its distance from the Sun. On a heliocentric scale we may pretend the spacecraft is *at* the planet's position when it crosses the boundary.
2. **Seen from the planet, the sphere of influence is at infinity.** It is 145 Earth radii away. A hyperbola from a parking orbit has almost reached its asymptotic speed by the time it gets there (Example 1 quantifies this), so we may use v_∞ at the boundary.

The last column is Q_{Sun} of Equation 17 evaluated at the boundary, $(m/M)^{1/5}$: the size of the neglected perturbation relative to the main acceleration at the patch (Q_{planet} there is the same across the Sun–planet line and twice it along the line). It depends only on the mass ratio, not on the distance R, and it grows with it: a body that is heavier *relative to its primary* is patched less accurately. The fifth root makes the growth slow. Jupiter is 318 times heavier than the Earth, yet its ratio is only 3.2 times larger ($318^{1/5} = 3.2$); even so, 25% at Jupiter's boundary against 8% at the Earth's is a real difference. In practice the cost is smaller than this column suggests. Laplace's criterion keeps the smaller of Q_{planet} and Q_{Sun}, and that smaller ratio is largest at the boundary itself. A hyperbolic spacecraft crosses the region where both ratios are sizeable quickly, and spends most of its time either deep inside the sphere ($Q_{\text{planet}} \ll 1$) or far outside it ($Q_{\text{Sun}} \ll 1$).

The Moon is the exception: its sphere of influence is 17% of its distance from the Earth, and at its boundary the neglected acceleration is 41% of the retained one, because the Moon is 1/81 of the Earth's mass where the planets are thousandths or millionths of the Sun's. Neither approximation is good there, and Example 2 shows what that costs.

5 The patched-conic method

5.1 The idea

A transfer from planet 1 to planet 2 is split into three conics (Figure 4):

1. a **departure hyperbola** about planet 1, from the parking orbit to the edge of planet 1's sphere of influence;
2. a **heliocentric conic** about the Sun, from planet 1 to planet 2;
3. an **arrival hyperbola** about planet 2, from the edge of its sphere of influence to periapsis.

History. Laplace's spheres of influence (*Mécanique céleste*, about 1805) were made for comets. The first applications to spacecraft came in 1956, independently. In the USSR, V. A. Egorov split lunar trajectories at the Moon's sphere of influence into two-body arcs, and suggested correcting the conic elements to do better. The Italian G. A. Crocco used the same patching to plan a single trip past Mars and Venus and back to the Earth. Through the 1960s the method was refined for the Moon (Chebotarev; Crenshaw, 1963; Battin, 1964), and it became the standard first step of mission design. Who first called it the "patched-conic" method is not clear from the sources.

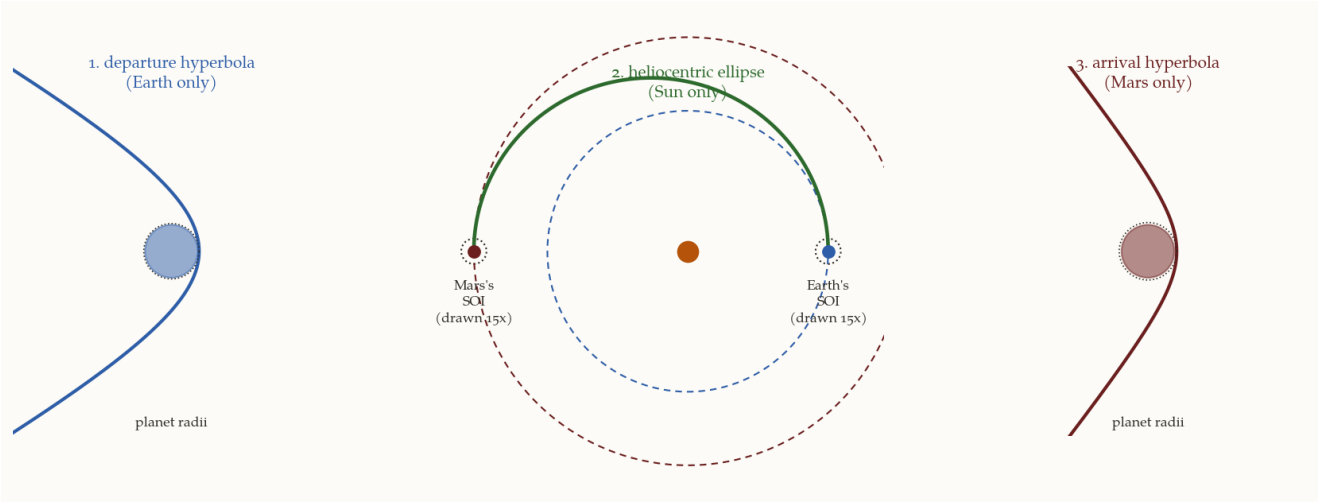

Figure 4: The three conics of an Earth–Mars transfer. The heliocentric ellipse (Sun only) is drawn to scale; the spheres of influence are drawn 15 times too large and are still small. The two hyperbolas are drawn in planet radii. *Script: fig-pc-patched-conics.py.*

At each boundary the two conics are *patched*: the spacecraft's state must be the same on both sides. With the two approximations of the last section, the patch conditions become very simple.

- **Position:** on the heliocentric conic, the spacecraft is at the planet's position (the sphere is a point).
- **Velocity:** the heliocentric velocity and the planet-relative velocity are related by Equation 11, and the planet-relative velocity there is the hyperbola's $\mathbf{v}_\infty$ (the sphere is at infinity):

$$\mathbf{v}_\infty = \mathbf{V}_{s/c} - \mathbf{V}_{\text{planet}}. \tag{19}$$

Why the planet's velocity may be held fixed. A crossing of the sphere of influence takes days: at Mars, with $v_\infty = 2.65$ km/s, about 2.4 days from the boundary to periapsis. In that time Mars moves 1.3° round the Sun, and its velocity turns by 0.53 km/s, a fifth of v_∞. That does not spoil the patch. Inside the sphere we work in the planet's own frame, which moves *with* the planet: the Sun accelerates planet and spacecraft almost equally, and only the small tidal difference of Equation 16 survives. So the hyperbola is right in the planet's frame whatever the planet does in the Sun's frame, and the only place the planet's velocity enters is the patch itself, Equation 19, evaluated at the moment of patching.

The excess-speed *vector* is the whole interface between the heliocentric and the planetary problems. The heliocentric design produces $\mathbf{v}_\infty$ at each end; each planetary design takes it as given.

5.2 The algorithm

1. **Heliocentric transfer.** Choose the transfer (for example Hohmann). Find the spacecraft's heliocentric velocity at departure and at arrival, $\mathbf{V}_1$ and $\mathbf{V}_2$.
2. **Excess velocities.** $\mathbf{v}_{\infty,1} = \mathbf{V}_1 - \mathbf{V}_{\text{planet}\,1}$ and $\mathbf{v}_{\infty,2} = \mathbf{V}_2 - \mathbf{V}_{\text{planet}\,2}$.
3. **Departure.** With $v_{\infty,1}$ and the parking-orbit radius, Equation 9 gives the burn and Equation 5 gives *where* to make it.
4. **Arrival.** With $v_{\infty,2}$ and the chosen periapsis, Equation 4 gives the periapsis speed. Then either capture (a burn) or fly by (no burn; Equation 6 gives the turn).
5. **Timing.** The phase angle (Equation 12) gives the launch date; the synodic period gives the next opportunity.

5.3 Where to burn on departure

The burn is at the periapsis of the departure hyperbola. The outgoing asymptote, along which the spacecraft leaves, is at true anomaly f_∞ from the periapsis. So the burn point is f_∞ *behind* the direction in which we want to leave, measured round the parking orbit in the direction of motion. Since f_∞ lies between 90° and 180°, the burn is always made on the far side of the planet from the departure direction.

To put a number on the burn point we need an angle on the parking orbit. A circular orbit has no periapsis, so its true anomaly is not defined; instead measure the angle θ round the parking orbit from a fixed direction, the **anti-Sun line** (from the planet's centre, away from the Sun), positive in the direction of motion. Take a prograde parking orbit, one that turns the same way as the planet goes round the Sun, and an outward Hohmann transfer, so that $\mathbf{v}_{\infty 1}$ is along the planet's velocity $\mathbf{V}_{P1}$. At the planet, $\mathbf{V}_{P1}$ is perpendicular to the Sun line and 90° ahead of the anti-Sun line, so the outgoing asymptote is at $\theta = 90°$. The burn point is $f_{\infty 1}$ behind it:

$$\theta_{\text{burn}} = 90° - f_{\infty 1}, \qquad f_{\infty 1} = \arccos\Big(-\frac{1}{e_1}\Big), \qquad e_1 = 1 + \frac{r_{p1} v_{\infty 1}^2}{\mu_1}. \tag{20}$$

Because $90° < f_{\infty 1} < 180°$, θ_{burn} lies between $-90°$ and $0°$: on the half of the parking orbit away from the Sun, between the dusk line and the anti-Sun line (Figure 5). On the Earth, where the planet also spins prograde, the anti-Sun line points at local midnight and $\theta = -90°$ at local 6 pm; each degree is 4 minutes of local time.

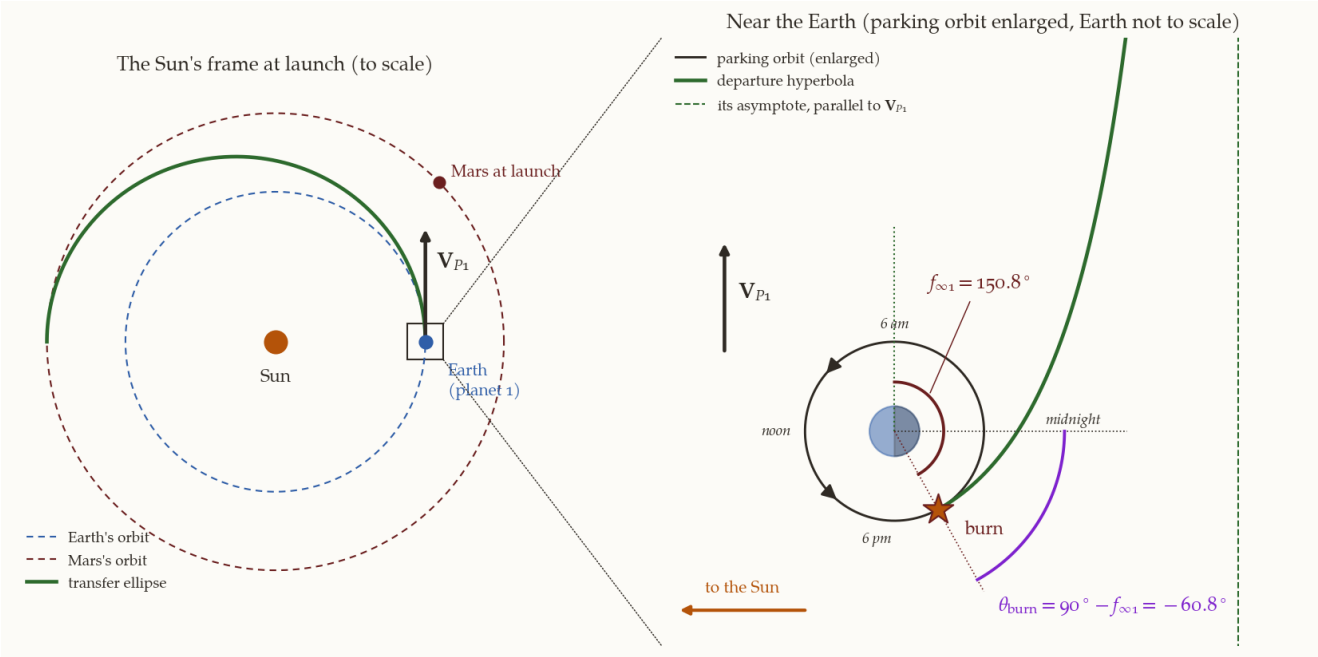

Figure 5: Where to burn, for an outward Hohmann departure (Example 1). Left: the Sun's frame at launch, to scale; the Earth's velocity $\mathbf{V}_{P1}$ is along the transfer ellipse, and Mars leads by 44.3°. Right: the boxed neighbourhood of the Earth, with the parking orbit drawn far larger than the Earth so that angles on it can be read. The departure asymptote is parallel to $\mathbf{V}_{P1}$, so its direction is $\theta = 90°$ from the anti-Sun line; the burn (periapsis of the hyperbola) is $f_{\infty 1} = 150.8°$ before it, at $\theta_{\text{burn}} = -60.8°$, about 8 pm local time. The parking orbit and the hyperbola are drawn in units of the parking-orbit radius. *Script: fig-pc-patched-conics.py.*

Why $\mathbf{v}_{\infty 1}$ is along $\mathbf{V}_{P1}$, or against it. A Hohmann transfer leaves the planet's orbit *tangentially*, and both the planet and the spacecraft go round the Sun the same way, so the spacecraft's heliocentric velocity $\mathbf{V}_1$ at departure is parallel to

the planet's $\mathbf{V}_{P1}$. By Equation 19 the excess velocity $\mathbf{v}_{\infty 1} = \mathbf{V}_1 - \mathbf{V}_{P1}$ is then on the same line, and its sign is the sign of $V_1 - V_{P1}$. For an **outward** transfer the departure point is the *perihelion* of the transfer ellipse, where the speed exceeds the circular speed (32.73 against 29.78 km/s for Earth to Mars), so $\mathbf{v}_{\infty 1}$ points **along** $\mathbf{V}_{P1}$: to climb away from the Sun the spacecraft must leave the planet *forwards* in its orbit. For an **inward** transfer it is the *aphelion*, where the speed is below the circular speed (27.29 against 29.78 km/s for Earth to Venus), so $\mathbf{v}_{\infty 1}$ points **against** $\mathbf{V}_{P1}$: to fall towards the Sun the spacecraft leaves *backwards*. (A transfer that is not tangential at departure adds a component perpendicular to $\mathbf{V}_{P1}$, and the picture below rotates by the angle of that component; the four cases are the clean ones.)

Where the burn goes. The hyperbola's periapsis, where the burn is made, is $f_{\infty 1}$ before the direction of the outgoing asymptote, measured in the sense in which the spacecraft circles the planet. So the outgoing asymptote, parallel to $\mathbf{v}_{\infty 1}$, fixes where the burn is, and the sense of the parking orbit chooses on which side of it. Figure 6 draws the four combinations of direction and sense, with the parking orbit enlarged and the night half of the planet shaded.

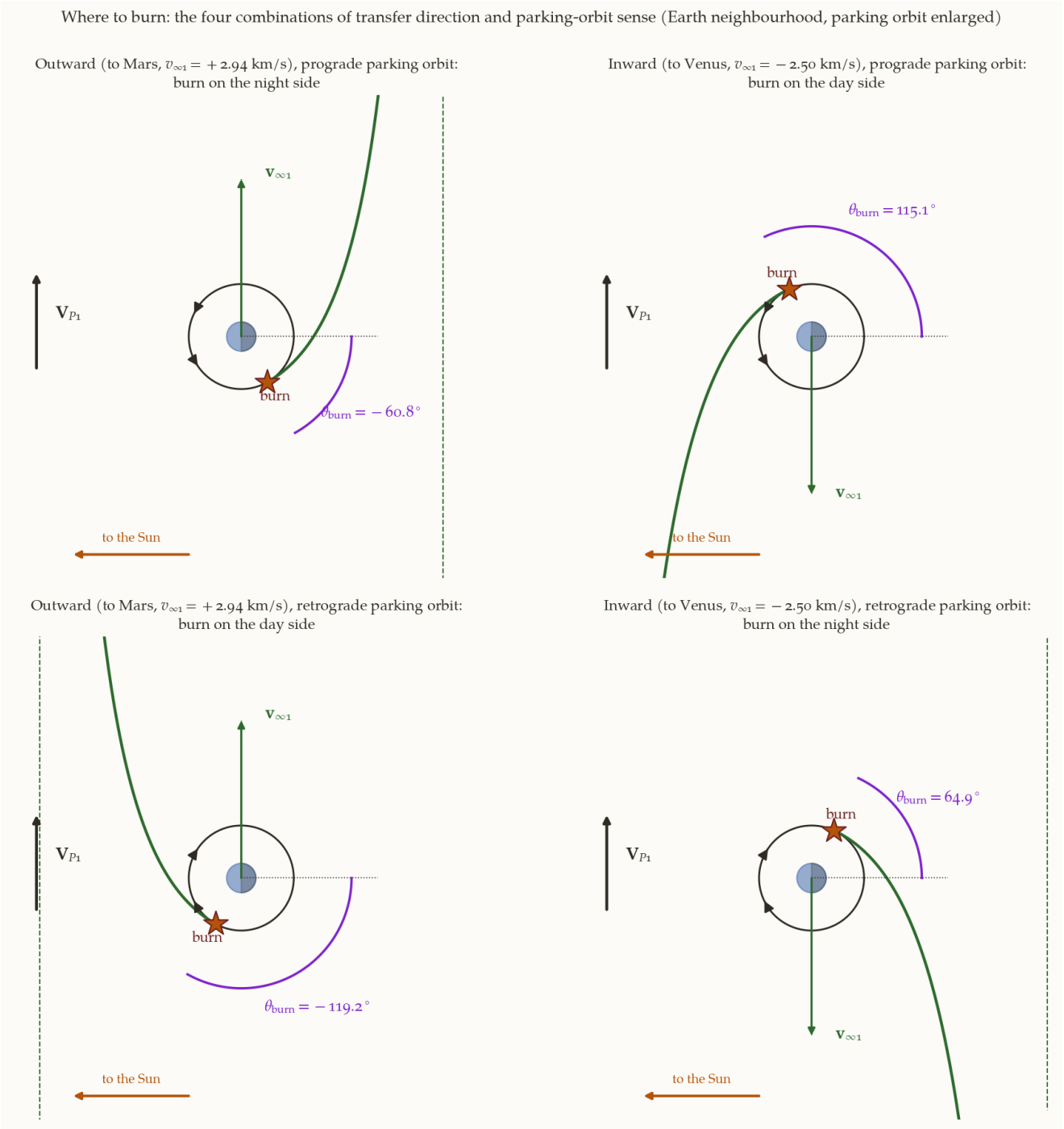

Figure 6: The four departures from a parking orbit. The planet is at the centre, moving up the page, with the Sun to the left; the night half of the planet is shaded. The dotted line is the anti-Sun direction $\theta = 0$. **Top row:** prograde parking orbits (anticlockwise, the direction in which the Earth spins), for an outward transfer (left: Example 1, $v_{\infty 1} = +2.94$ km/s, $f_{\infty 1} = 150.8°$) and an inward one (right: Earth to Venus, $v_{\infty 1} = -2.50$ km/s, $f_{\infty 1} = 154.9°$). **Bottom row:** the same transfers from retrograde parking orbits. The hyperbola (solid) leaves the parking orbit at the burn point and approaches the dashed asymptote; $\mathbf{v}_{\infty 1}$ (green) is its direction. The burn is made $f_{\infty 1}$ before the asymptote's direction in the sense of motion. *Script: fig-pc-patched-conics.py.*

With θ still measured from the anti-Sun line in the sense of the planet's orbital motion (anticlockwise in the figure), the asymptote is at $\theta = +90°$ for an outward transfer and $\theta = -90°$ for an inward one. A prograde parking orbit puts the burn at $\theta_{\text{burn}} = \theta_{\text{asymptote}} - f_{\infty 1}$ and a retrograde one at $\theta_{\text{asymptote}} + f_{\infty 1}$. The four results, with the local time they correspond to on a planet that spins prograde (4 minutes per degree, midnight on the anti-Sun line):

Table 3: The burn point for the four combinations, measured from the anti-Sun line.

	prograde parking orbit	retrograde parking orbit
outward (Earth to Mars)	$\theta_{\text{burn}} = -60.8°$, about 8 pm: **night side**	240.8° (= −119.2°), about 4 pm: **day side**
inward (Earth to Venus)	−244.9° (= 115.1°), about 7:40 am: **day side**	64.9°, about 4:20 am: **night side**

Prograde or retrograde, and what it changes. In the planet-centred two-body model *nothing* about the cost changes. A retrograde hyperbola is the mirror image of the prograde one: the same e, the same periapsis speed, the same $\Delta v = v_{p1} - v_{c1}$ (3.590 km/s to Mars, 3.481 km/s to Venus). What changes is practical.

- **The ascent.** A prograde parking orbit is reached by launching eastwards, with the help of the planet's spin: up to $\omega_\oplus R_\oplus = 0.465$ km/s at the equator (0.32 to 0.46 km/s at the latitudes from Baikonur to Kourou). A retrograde one is reached by launching westwards, against the spin, and pays that speed instead of saving it: a difference of up to 0.93 km/s on an ascent of roughly 9 to 9.5 km/s. This is the one decisive reason that almost every departure uses a prograde parking orbit, whatever the direction of the transfer.
- **Day side or night side.** For a prograde orbit the burn is on the *night* side for an outward transfer and on the *day* side for an inward one; a retrograde orbit swaps them. Burning in the planet's shadow or in sunlight, and at one local time or another, matters for the spacecraft's power, its thermal state, the attitude it needs to hold for the burn, and which ground stations can see it. These are design considerations, not Δv ones.
- **Outward against inward.** Because of the ascent, the choice of parking-orbit sense is rarely free, so the burn side follows from the direction of the transfer: prograde means night side outward and day side inward. A mission that needs the *night* side for an inward transfer would have to pay the retrograde ascent penalty to get it.

TODO: Expand this paragraph. Explain inward and outward transfer. Talk about why v∞1 is along or opposite Vp1. Add two figures for prograde orbit in the first row, for the inward and outward tranfer scenarios.Second row figures for retrograde orbits for the inward and outward transfers. Explain advantages or disadvantages (if any) of using prograde or retrograde orbits for the inward and outward transfers. DO NOT DELETE THIS task descriptions. In fact, never delete any todo lists. I will delete them once I am satisfied.

5.4 Flybys

At a flyby there is no burn. Relative to the planet, the energy $v_\infty^2/2$ is unchanged, so $|\mathbf{v}_\infty^+| = |\mathbf{v}_\infty^-|$; only the direction turns, through δ. Relative to the Sun, however,

$$\mathbf{V}^\pm = \mathbf{V}_{\text{planet}} + \mathbf{v}_\infty^\pm,$$

and since $\mathbf{v}_\infty$ has turned, $\mathbf{V}^+$ need not have the same length as $\mathbf{V}^-$. The change in the spacecraft's heliocentric energy is

$$\Delta\Big(\frac{V^2}{2}\Big) = \frac{1}{2}\big(|\mathbf{V}_P + \mathbf{v}_\infty^+|^2 - |\mathbf{V}_P + \mathbf{v}_\infty^-|^2\big) = \mathbf{V}_P \cdot (\mathbf{v}_\infty^+ - \mathbf{v}_\infty^-), \tag{21}$$

because $|\mathbf{v}_\infty^+| = |\mathbf{v}_\infty^-|$ and the V_P^2 terms cancel. The heliocentric potential does not change during the flyby (the position is the planet's, before and after), so this is the whole energy change. The spacecraft gains energy when the turn moves its relative velocity *towards* the planet's motion. The energy comes from the planet, whose orbit changes by an unmeasurably small amount.

A flyby changes the speed and the direction at once. In the Sun's frame the velocity before and after is $\mathbf{V}^\pm = \mathbf{V}_P + \mathbf{v}_\infty^\pm$ with $|\mathbf{v}_\infty^+| = |\mathbf{v}_\infty^-|$. Draw these in velocity space (bottom row of Figure 7): $\mathbf{V}_P$ is a fixed arrow from the origin, and $\mathbf{V}^-$ and $\mathbf{V}^+$ are arrows from the origin to two points on the *same circle*, of radius v_∞ and centred on the tip of $\mathbf{V}_P$. The flyby moves the point round that circle through the angle δ (Equation 6), and nothing else. The *length* $|\mathbf{V}|$ of the arrow, which is the heliocentric speed, is largest at the point of the circle in the direction of $\mathbf{V}_P$ and smallest at the opposite point. So a turn that moves $\mathbf{v}_\infty$ **towards the planet's motion** moves the point towards the far side of the circle and lengthens the arrow: the spacecraft gains speed and energy. A turn **away** from the planet's motion shortens it: energy is lost. A turn that moves the point to the mirror-image position across the $\mathbf{V}_P$ axis leaves $|\mathbf{V}|$ alone and only *rotates* the heliocentric velocity.

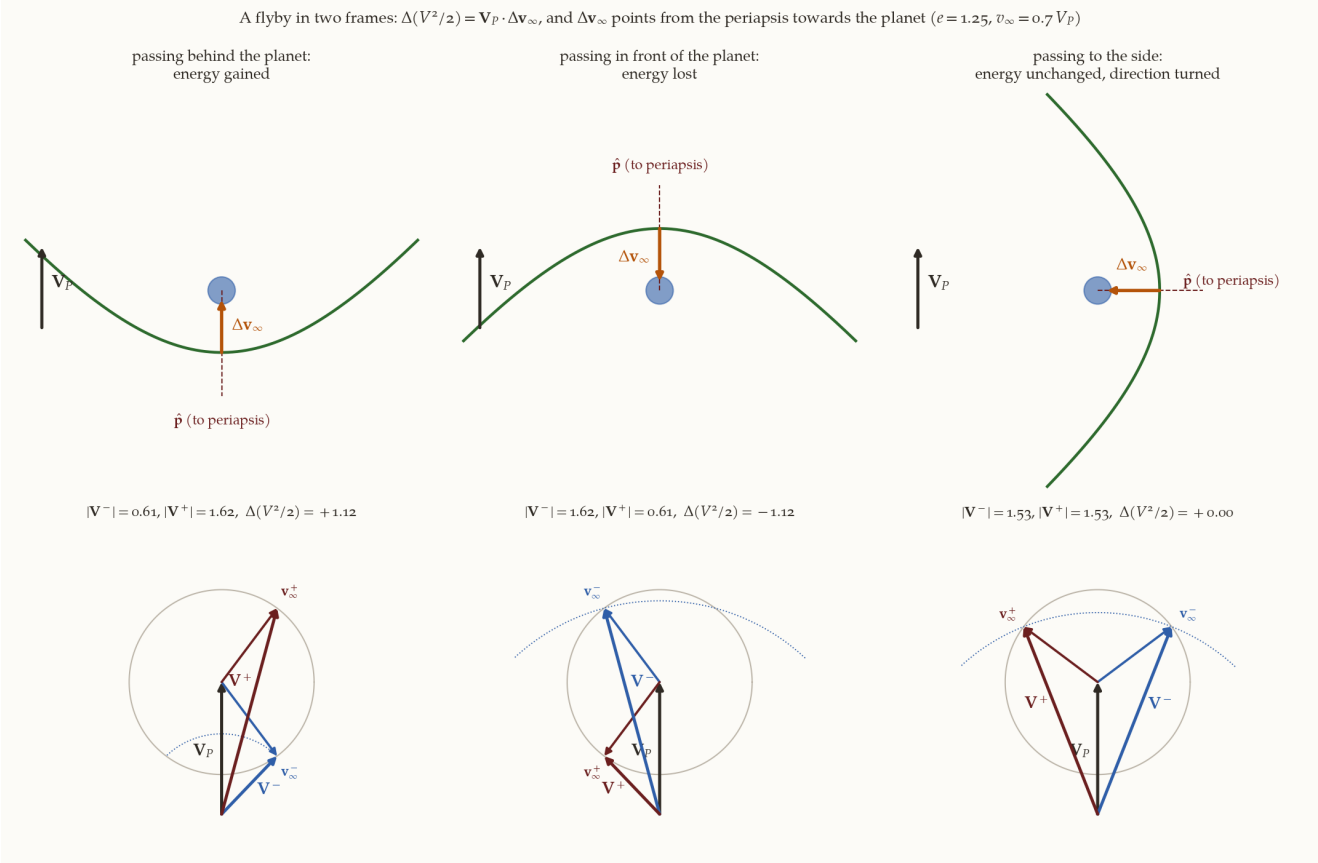

Figure 7: What a flyby does to the heliocentric velocity, for $e = 1.25$ and $v_\infty = 0.7V_P$. **Top row:** the planet's frame, with the planet moving up the page. The change $\Delta\mathbf{v}_\infty = \mathbf{v}_\infty^+ - \mathbf{v}_\infty^-$ always points from the periapsis towards the planet, since gravity is attractive. Passing behind the planet (left) makes it point along $\mathbf{V}_P$; passing in front (middle) against it; passing to the side (right) across it. **Bottom row:** the same three flybys in velocity space in the Sun's frame. The grey circle has radius v_∞ about the tip of $\mathbf{V}_P$; the dotted arc is the circle of radius $|\mathbf{V}^-|$ about the origin, so the end of $\mathbf{V}^+$ lies outside it (gain), inside it (loss) or on it (no change). *Script: fig-pc-patched-conics.py.*

Proof, and which side to pass on. The statement "a turn towards the planet's motion gains energy" is Equation 21, $\Delta(V^2/2) = \mathbf{V}_P \cdot \Delta\mathbf{v}_\infty$ with $\Delta\mathbf{v}_\infty = \mathbf{v}_\infty^+ - \mathbf{v}_\infty^-$: the energy gain is positive exactly when the change of $\mathbf{v}_\infty$ has a positive component along $\mathbf{V}_P$, that is, when the turn moves $\mathbf{v}_\infty$ towards $\mathbf{V}_P$. To see what fixes the sign, find $\Delta\mathbf{v}_\infty$ from the geometry of the hyperbola. Let $\hat{\mathbf{p}}$ point from

the planet to the periapsis and $\hat{\mathbf{q}}$ along the motion at the periapsis. The velocity at the two ends of the path points along the asymptotes, at true anomalies $\mp f_\infty$ with $\cos f_\infty = -1/e$, and is radial there, so

$$\mathbf{v}_\infty^{\mp} = v_\infty\left(\pm\tfrac{1}{e}\,\hat{\mathbf{p}} + \sin f_\infty\,\hat{\mathbf{q}}\right) \quad\Longrightarrow\quad \Delta\mathbf{v}_\infty = -\frac{2v_\infty}{e}\,\hat{\mathbf{p}}.$$

(The two asymptote directions are mirror images in the apse line, which is why only the $\hat{\mathbf{p}}$ component changes.) Its size is $2v_\infty/e = 2v_\infty \sin(\delta/2)$, and it points **from the periapsis towards the planet**: gravity pulls the spacecraft towards the planet, so its velocity in the planet's frame is changed by a vector pointing that way. Substituting into Equation 21,

$$\boxed{\Delta\left(\frac{V^2}{2}\right) = -\frac{2V_P v_\infty}{e}\,\hat{\mathbf{p}}\cdot\hat{\mathbf{V}}_P = \frac{2V_P v_\infty}{e}\,\cos\psi',} \tag{22}$$

where ψ' is the angle between $\mathbf{V}_P$ and $-\hat{\mathbf{p}}$, that is, between the planet's motion and the direction from the periapsis to the planet. If the spacecraft passes **behind** the planet (the periapsis on the side opposite to $\mathbf{V}_P$), the planet pulls it forwards, along the planet's motion, and the energy rises; at its largest, $2V_P v_\infty/e$, the pass is directly behind. If it passes **in front**, the planet pulls it back and the energy falls by as much. If it passes to the **side** ($\hat{\mathbf{p}} \perp \mathbf{V}_P$) the pull is across the motion, $\Delta(V^2/2) = 0$, and only the direction of $\mathbf{V}$ turns. The three are drawn in Figure 7. Note that e sets the strength: a closer, slower pass has a smaller e and a larger turn, and so a larger possible gain, $2V_P v_\infty \sin(\delta/2)$, which tends to $2V_P v_\infty$ as $\delta \to 180°$. The planet's own energy changes by the opposite amount, which, divided by its enormous mass, is nothing.

For Example 3 below, $2V_J v_\infty/e = 2 \times 13.058 \times 5.643/1.0899 = 135.2$ km^2/s^2 is the most that this flyby could give. It gives 124.1, which is 91.8% of that: its periapsis direction makes $\cos\psi' = 0.918$ ($\psi' = 23.4°$) with the direction straight behind Jupiter.

Gaining or losing energy, and turning. So a flyby is a tool for two jobs. To **gain** energy, pass behind: this is how Jupiter flybys speed spacecraft towards the outer planets (the trajectory of Example 3). To **lose** energy, pass in front: an inward trip, such as Mariner 10's to Mercury past Venus in February 1974, needs the spacecraft to *lose* heliocentric energy, and the flyby supplies that without propellant. To **turn** the heliocentric velocity without changing its size, pass to the side; in general both happen at once, and the geometry is the geometry of the circle in velocity space, which is why a flyby designer works with velocity triangles (as in Figure 18).

TODO: This is a small description and explanation of flyby. Especially, talk about use of flyby to gain or lose energy and also to change the direction of the velocity vector. Then with the help of diagrams, explain what you mean by energy gain when the relative velocity is moved towards the planets motion. Prove this.

History. Walter Hohmann published the minimum-energy transfer in *Die Erreichbarkeit der Himmelskörper* (1925). The gravity assist was worked out in detail by Michael Minovitch at JPL in 1961. The first spacecraft to use one planet to reach another was Mariner 10, which flew past Venus in February 1974 on its way to Mercury.

6 Examples in a plane

In these examples every orbit lies in one plane: the planets' orbits are taken as circular and coplanar, and the parking orbits lie in the same plane. The numbers use $\mu_\odot = 1.327\,124 \times 10^{11}$ km^3/s^2, 1 AU $= 1.495\,979 \times 10^8$ km, and the planetary data quoted in each example.

6.1 Example 1: from the Earth to Mars

Given. A spacecraft in a 300 km circular orbit about the Earth ($\mu_\oplus = 398\,600.4\ \mathrm{km}^3/\mathrm{s}^2$, $R_\oplus = 6378.1$ km) is to reach Mars (1.52368 AU, sidereal period 686.98 d, $\mu = 42\,828.4\ \mathrm{km}^3/\mathrm{s}^2$, $R = 3396.2$ km) on a Hohmann transfer, and be captured there.

Find. (a) The excess speeds at departure and at arrival, and the flight time. (b) The departure burn, and where to make it. (c) How long the spacecraft stays inside the Earth's sphere of influence. (d) The capture burn into a 300 km circular orbit, and into a 300 × 33 000 km ellipse; the aim radius. (e) The phase angle at departure, and how often the opportunity recurs.

symbol	orbit
$V_\oplus$, V_M	circular heliocentric speeds of the Earth and Mars
V_{t1}, V_{t2}	speeds on the heliocentric transfer ellipse, at the Earth and at Mars
$v_{\infty 1}$, $v_{\infty 2}$	excess speeds relative to the Earth and to Mars
v_c, v_{p1}	parking-orbit speed, and periapsis speed of the departure hyperbola
e_1, e_2	eccentricities of the departure and arrival hyperbolas

Step 1: the heliocentric transfer. With $r_1 = 1$ AU, $r_2 = 1.52368$ AU and $a_t = 1.26184$ AU, Equation 10 gives

$$V_\oplus = \sqrt{\mu_\odot / r_1} = 29.785\ \mathrm{km/s}, \qquad V_{t1} = 32.729\ \mathrm{km/s}$$

$$V_\mathrm{M} = 24.129\ \mathrm{km/s}, \qquad V_{t2} = 21.480\ \mathrm{km/s},$$

and $t_{12} = \pi\sqrt{a_t^3/\mu_\odot} = 258.9$ days.

Step 2: the excess speeds. At departure the transfer velocity and the Earth's velocity are parallel (both tangential, at perihelion), so $\mathbf{v}_{\infty 1}$ is along the Earth's motion with

$$v_{\infty 1} = V_{t1} - V_\oplus = 2.945\ \mathrm{km/s}.$$

At arrival the spacecraft is at the aphelion of the transfer ellipse, and Mars is at the same point of its circular orbit. Both velocities are again tangential, so they are again parallel, and the excess velocity, Equation 19, is a subtraction along a single line (Figure 8, left):

$$\mathbf{v}_{\infty 2} = \mathbf{V}_{t2} - \mathbf{V}_\mathrm{M}, \qquad v_{\infty 2} = |\mathbf{V}_{t2} - \mathbf{V}_\mathrm{M}| = V_\mathrm{M} - V_{t2} = 24.129 - 21.480 = 2.649\ \mathrm{km/s}.$$

The order of the two speeds is not an accident of the numbers. Put $a_t = (r_1 + r_2)/2$ in Equation 10:

$$V_{t2} = \sqrt{\mu_\odot \left(\frac{2}{r_2} - \frac{2}{r_1 + r_2} \right)} = \sqrt{\frac{\mu_\odot}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} = V_\mathrm{M} \sqrt{\frac{2r_1}{r_1 + r_2}},$$

and the last square root is less than 1 because $r_1 < r_2$: the aphelion of a transfer outward is the slowest point of the ellipse, slower than a circular orbit at the same radius. So $\mathbf{v}_{\infty 2}$ points *backwards*, opposite to Mars's motion. At departure it was the other way round: the spacecraft at perihelion was faster than the Earth, and $\mathbf{v}_{\infty 1}$ pointed forwards.

Why "from in front". In the Sun's frame, at the moment of arrival, Mars and the spacecraft move in the same direction along the same line, and Mars is faster by 2.649 km/s. Mars is catching up with the spacecraft from behind, as a fast car catches up with a slower one on a road. Now ride on Mars. The spacecraft moves backwards, at $\mathbf{v}_{\infty 2}$. Something that moves backwards relative to Mars and ends up at Mars must have started ahead of it. The spacecraft therefore enters Mars's sphere of influence on its *leading* side, the half that faces the direction in which Mars is moving, and its incoming asymptote runs backwards from there (Figure 8, right; panel 3 of Figure 1 shows the same thing in the Sun's frame). This is the frame in which the arrival hyperbola is drawn, and the aim radius b of Step 5 is measured from Mars's centre to that incoming asymptote.

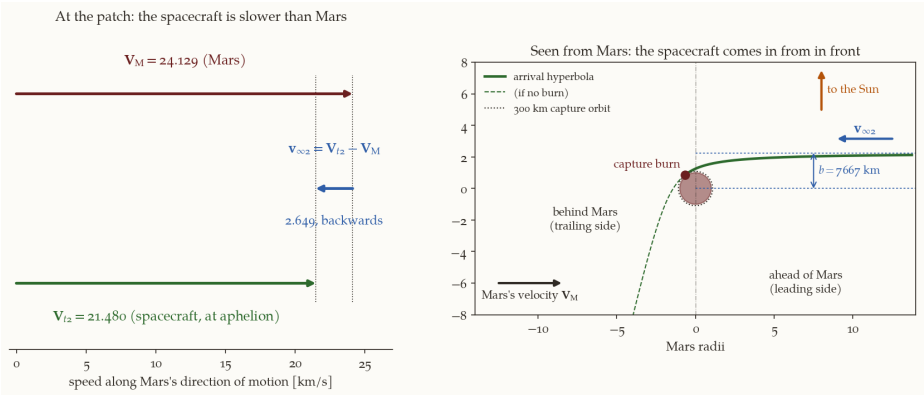

Figure 8: Example 1: arrival at Mars. Left: at the patch the transfer velocity $\mathbf{V}_{t2}$ and Mars's velocity $\mathbf{V}_\mathrm{M}$ are parallel, and the spacecraft is slower, so $\mathbf{v}_{\infty 2} = \mathbf{V}_{t2} - \mathbf{V}_\mathrm{M}$ points backwards. Right: the same arrival seen from Mars, to scale. The spacecraft drifts backwards relative to Mars, so it comes in from the side Mars is moving towards, along an asymptote $b = 7667$ km from Mars's centre, and is captured at periapsis into the 300 km orbit. *Script: fig-pc-patched-conics.py.*

Step 3: the departure burn. At $r_p = 6678.1$ km, the parking orbit has $v_c = \sqrt{\mu_\oplus / r_p} = 7.726$ km/s. The hyperbola's periapsis speed, Equation 4, is

$$v_{p1} = \sqrt{2.945^2 + \frac{2(398\,600)}{6678.1}} = \sqrt{8.67 + 119.37} = 11.316 \text{ km/s},$$

so the burn is $\Delta v_1 = 11.316 - 7.726 = 3.590$ km/s.

Where. $e_1 = 1 + r_p v_{\infty 1}^2 / \mu_\oplus = 1.1453$, so $f_\infty = \arccos(-1/1.1453) = 150.8°$. The burn is made 150.8° before the point of the parking orbit whose radius vector points along the Earth's heliocentric velocity. By Equation 20, $\theta_\text{burn} = 90° - 150.8° = -60.8°$: 60.8° before the anti-Sun line, on the night side at about 8 pm local time (Figure 9, Figure 5).

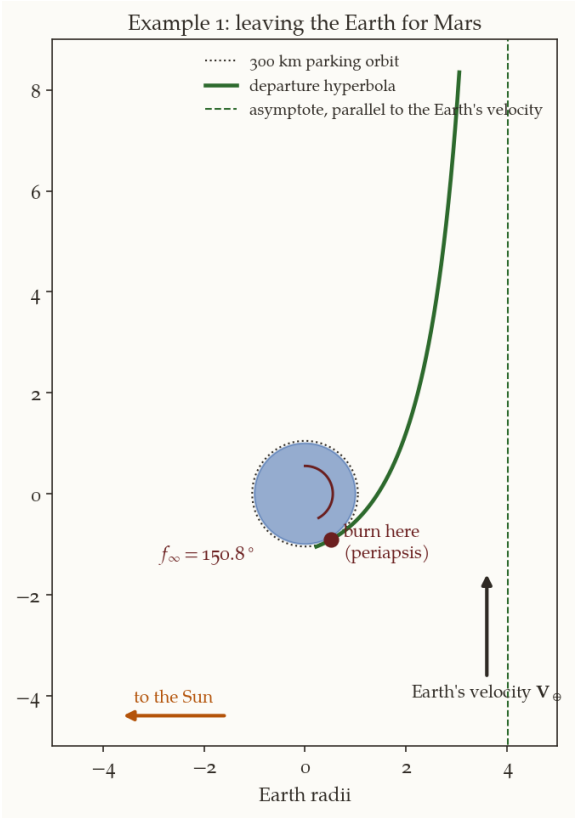

Figure 9: Example 1: the departure hyperbola, to scale. The burn is at periapsis, 150.8° round the parking orbit before the direction of the asymptote; the asymptote is parallel to the Earth's velocity. *Script: fig-pc-patched-conics.py.*

Step 4: time inside the Earth's sphere of influence. The hyperbola has $|a| = \mu_\oplus / v_{\infty 1}^2 = 45\,960$ km. On a hyperbola, $r = |a|(e\cosh H - 1)$ and the time from periapsis is $t = \sqrt{|a|^3/\mu}\,(e\sinh H - H)$, with H the hyperbolic anomaly of the two-body note. At $r = r_{\text{SOI}} = 925\,000$ km, $\cosh H = (1 + r/|a|)/e$ gives $H = 3.61$, and then $t = 3.2$ days. The spacecraft spends 3 days of a 259-day trip inside the Earth's sphere of influence: about 1% of the time, over a distance of only 0.6% of the Sun–Earth distance. And there its speed is $\sqrt{v_{\infty 1}^2 + 2\mu_\oplus / r_{\text{SOI}}} = 3.088$ km/s, already within 5% of $v_{\infty 1}$.

Step 5: capture at Mars. At $r_p = 3396.2 + 300 = 3696.2$ km the arrival hyperbola has $v_{p2} = \sqrt{2.649^2 + 2(42\,828)/3696.2} = 5.495$ km/s.

- Into the 300 km circle (3.404 km/s): $\Delta v = 2.091$ km/s.
- Into a 300 × 33 000 km ellipse (periapsis speed 4.587 km/s, period 23.9 h): $\Delta v = 0.908$ km/s.

The elliptical capture costs less than half as much, for the reason of Problem 4 of the Indian missions set: only a little speed has to be removed at a periapsis reached at high speed. The spacecraft must be aimed at $b = r_p\sqrt{1 + 2\mu/(r_p v_{\infty 2}^2)} = 7667$ km from Mars's centre, more than twice the periapsis radius.

Which side of Mars: sunlit or dark. The aim radius fixes how far from Mars's centre the incoming asymptote must pass, but not on which side. In the plane of the transfer there are two choices (Figure 10): the asymptote can pass b from the centre on the side facing the Sun, as in Figure 8 (a *sunlit-side approach*), or b from it on the side facing away (a *dark-side approach*). Both come in from in front, along the same $\mathbf{v}_{\infty 2}$; the second hyperbola is the mirror image of the first in the line

through Mars's centre along its velocity. A mirror image has the same lengths and speeds, so e_2, r_p, v_{p2}, the capture burn of 2.091 km/s and the 2.4 days inside the sphere of influence are all unchanged. Three things do change.

- *The sense of the capture orbit.* On the sunlit side the hyperbola swings round Mars anticlockwise, the way Mars goes round the Sun, and the capture orbit is **prograde**. On the dark side it swings clockwise, and the capture orbit is **retrograde**. The capture burn only slows the spacecraft along its velocity, so it cannot reverse the sense: that is settled on the way in, by the side chosen.
- *Where the burn is made.* With $e_2 = 1 + r_p v_{\infty 2}^2/\mu_M = 1.6056$, the asymptote angle of Equation 5 is $f_{\infty 2} = \arccos(-1/e_2) = 128.5°$, so periapsis lies 128.5° round from the direction in which Mars is moving, measured in the direction of the spacecraft's motion. That puts it on Mars's trailing half in both cases: 38.5° from the point directly under the Sun (day side) on the sunlit approach, and 38.5° from the point directly opposite (night side) on the dark one.
- *Nothing else, in the patched-conic budget.* The side is chosen by the small trajectory-correction burns made on the way in. Switching sides moves the aim point by $2b = 15\,334$ km, and a sideways velocity change δv made a time τ before arrival moves it by about $\delta v\,\tau$. Made $\tau = 30$ days out, the switch costs about $15\,334/(30 \times 86\,400)$ km/s = 5.9 m/s.

So which side is flown depends on the orbit the mission needs, not on the cost of getting there. Mars spins the same way it goes round the Sun (its equator is tilted 25.2° to its orbit), so a prograde orbit moves with the rotating atmosphere: an aerobraking pass, or a lander released from orbit, meets the air up to 0.241 km/s more slowly, the rotation speed of Mars's equator. Retrograde orbits are flown as well: MOM's orbit is inclined 150° to Mars's equator (Section 7.4), and is retrograde. In three dimensions the aim point may lie anywhere on a circle of radius b about the asymptote, and Section 7.4 shows how that choice sets the inclination of the capture orbit. The sunlit-side and dark-side approaches are the two points of that circle that lie in the plane of the transfer.

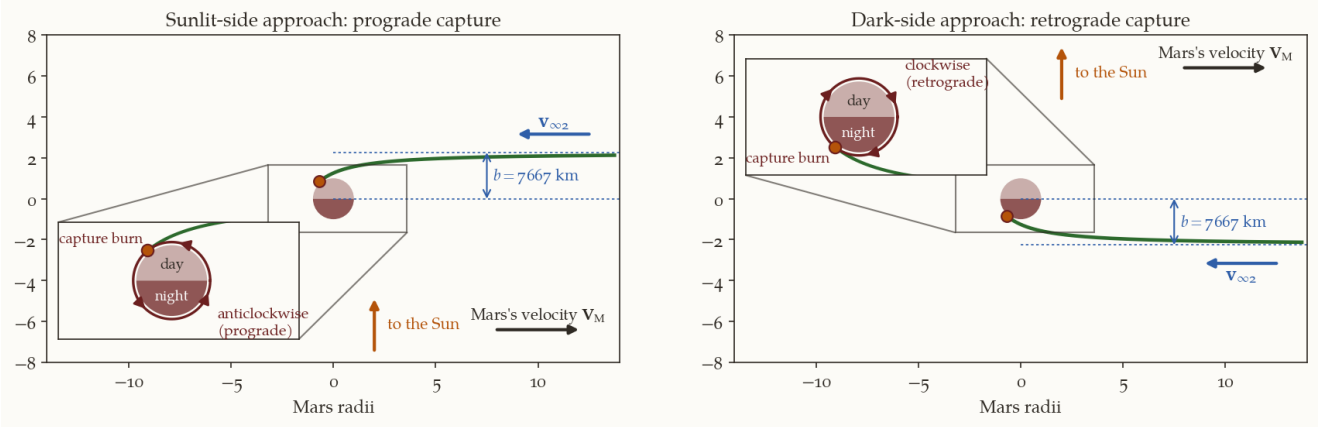

Figure 10: Example 1: the two ways to arrive at Mars in the plane of the transfer, to scale, in the axes of Figure 8 (Mars moves to the right, the Sun is up). Both come in from in front with the same $\mathbf{v}_{\infty 2}$ and the same aim radius b, and they are mirror images of each other. Left: passing on the Sun's side, the spacecraft swings round anticlockwise and is captured on the day side into a prograde orbit. Right: passing on the far side, it swings round clockwise and is captured on the night side into a retrograde orbit. The insets enlarge Mars, to scale, so that the 300 km capture orbit and its sense of motion can be seen. *Script: fig-pc-patched-conics.py.*

Step 6: timing. Mars moves at $n_M = 360°/686.98$ d, so in 258.9 days it moves 135.7°. At launch it must lead the Earth by $\phi_0 = 180° - 135.7° = 44.3°$. The geometry repeats every $S = 1/(1/365.256 - 1/686.98) = 780$ days, about 2.1 years.

The animation of Figure 1, in the introduction, is this example. Panels 1, 2 and 4 are the three conics, in the same non-rotating axes, and each runs on its own clock, as the method does: 3.2 days inside the Earth's sphere of influence, 258.9 days on the ellipse, 2.4 days inside Mars's. Panel 3 is the end of the ellipse seen close up in the Sun's frame, the picture behind "from in front": Mars, 2.649 km/s faster, closes on the spacecraft and its sphere of influence sweeps over it. Outside the sphere the spacecraft there moves about the Sun alone; inside, its position is Mars's plus the arrival hyperbola's, so panels 3 and 4 run on the same clock. The two planetocentric panels zoom with the spacecraft, so the parking orbit and the sphere of influence, 140 times larger, can both be seen. Inside the spheres the frames are equally spaced in the hyperbolic anomaly H of Step 4, not in time; on the ellipse they are equally spaced in time.

What the numbers say. The whole interplanetary problem was reduced to two numbers, $v_{\infty 1}$ and $v_{\infty 2}$, and everything else followed from two-body formulas. The Earth's part of the trip is short and the burn is big; the heliocentric part is long and costs nothing.

6.2 Example 2: to the Moon

The Moon is the hardest case for the method, because its sphere of influence is not small compared with its distance (Section 4). The patch can no longer be at "the Moon's position"; it must be at a definite point on the Moon's sphere of influence. This example shows how that is done, and how far the answer is from the truth.

Given. A spacecraft is injected at perigee of a geocentric ellipse, at $r_0 = 6678.1$ km (300 km altitude), with speed $v_0 = 10.84$ km/s and zero flight-path angle. It is to cross the Moon's sphere of influence at the point where the line from the Moon makes an angle $\lambda_1 = 60°$ with the Moon–Earth line, on the side that the Moon is moving towards (its leading side). The Moon is on a circle of radius $r_{\text{Moon}} = 384\,400$ km, $\mu_{\text{Moon}} = 4902.8$ km^3/s^2, $R_{\text{Moon}} = 1737.4$ km (the lower-case r_{Moon} is the radius of the Moon's orbit, the capital R_{Moon} the Moon's own radius).

Find. The geocentric state at the patch, the velocity relative to the Moon there, the perilune, the flight time, and where the Moon must be at injection.

TODO: Replace D with r_{Moon} in this example.

symbol	orbit
$\varepsilon_0, h_0, e_0, a_0$	geocentric transfer ellipse
$r_1, v_1, \gamma_1, f_1, \eta_1$	geocentric state at the patch, and the angle at the Earth between the Moon and the patch
v_{Moon}	the Moon's orbital speed
$\mathbf{v}_2, \varepsilon_2, h_2, e_2$	selenocentric hyperbola

Step 1: the Moon's sphere of influence and speed. $r_{\text{SOI}} = r_{\text{Moon}}(\mu_{\text{Moon}}/\mu_\oplus)^{2/5} = 66\,183$ km, and $v_{\text{Moon}} = \sqrt{(\mu_\oplus + \mu_{\text{Moon}})/r_{\text{Moon}}} = 1.0245$ km/s.

Step 2: the geocentric ellipse. $\varepsilon_0 = v_0^2/2 - \mu_\oplus/r_0 = -0.9346$ km^2/s^2, so $a_0 = -\mu_\oplus/2\varepsilon_0 = 213\,253$ km; $h_0 = r_0 v_0 = 72\,391$ km^2/s, $p_0 = h_0^2/\mu_\oplus = 13\,147$ km and $e_0 = p_0/r_0 - 1 = 0.9687$. Its apogee would be at 419 828 km, beyond the Moon.

Step 3: the patch point. In the triangle Earth–Moon–patch point, two sides (r_{Moon} and r_{SOI}) and the angle between them (λ_1) are known. The cosine rule gives the distance from the Earth,

$$r_1 = \sqrt{r_{\text{Moon}}^2 + r_{\text{SOI}}^2 - 2 r_{\text{Moon}} r_{\text{SOI}} \cos\lambda_1} = 355\,953 \text{ km},$$

and the sine rule the angle at the Earth between the Moon and the patch point, $\sin\eta_1 = (r_{\text{SOI}}/r_1)\sin\lambda_1$, so $\eta_1 = 9.27°$. Both angles are marked in panel (a) of Figure 11, which also repeats these definitions and the formulas.

Step 4: the geocentric state at the patch. Energy and angular momentum are conserved on the ellipse:

$$v_1 = \sqrt{2\Big(\varepsilon_0 + \frac{\mu_\oplus}{r_1}\Big)} = 0.6087 \text{ km/s}, \qquad \cos\gamma_1 = \frac{h_0}{r_1 v_1} \;\Rightarrow\; \gamma_1 = 70.48°.$$

The flight-path angle (defined in panel (c) of Figure 11) is large: the spacecraft is still climbing steeply when it reaches the Moon's neighbourhood. Its true anomaly there is $f_1 = \arccos((p_0/r_1 - 1)/e_0) = 173.83°$, and Kepler's equation ($E_1 = 2.3334$ rad) gives the flight time from injection, $t_1 = 70.75$ h.

Step 5: the velocity relative to the Moon. This is the patch: Equation 19, with the Moon as the planet. What is needed is the angle between $\mathbf{v}_1$ and $\mathbf{v}_{\text{Moon}}$, and it follows from three statements about angles, drawn in Figure 11.

1. *The same η_1 at the patch point.* The lines from the Earth to the Moon and to the patch point meet at the Earth with angle η_1. Draw through the patch point P a line parallel to the Earth–Moon line. It meets the line EP, extended, at the same angle η_1: these are corresponding angles of two parallel lines cut by the transversal EP (panel a).
2. *A quarter turn changes nothing.* The Moon moves on a circle about the Earth, so its velocity is perpendicular to the Earth–Moon line. The local horizontal at P is perpendicular to EP by definition. These two lines are the two lines of statement 1, each turned through 90° in the same sense. Turning a pair of lines together leaves the angle between them unchanged (angles with mutually perpendicular sides are equal), so the Moon's velocity, drawn from P, makes the angle η_1 with the local horizontal (panel b).
3. *The difference.* The spacecraft's velocity $\mathbf{v}_1$ is turned from the local horizontal by γ_1 (its definition), and the Moon's by η_1, both towards the outward radial direction. They are on the same side of the horizontal, so the angle between them is the difference (panel c): $\gamma_1 - \eta_1 = 70.48° - 9.27° = 61.21°$.

With the patch point ahead of the Moon, as here, η_1 is counted positive. For a patch point behind the Moon, η_1 is negative and the same formula holds with its sign. The velocity relative to the Moon is $\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_{\text{Moon}}$, the third side of the triangle in panel (d), and the cosine rule gives

$$v_2 = \sqrt{v_1^2 + v_{\text{Moon}}^2 - 2v_1 v_{\text{Moon}}\cos(\gamma_1 - \eta_1)} = 0.9053 \text{ km/s}.$$

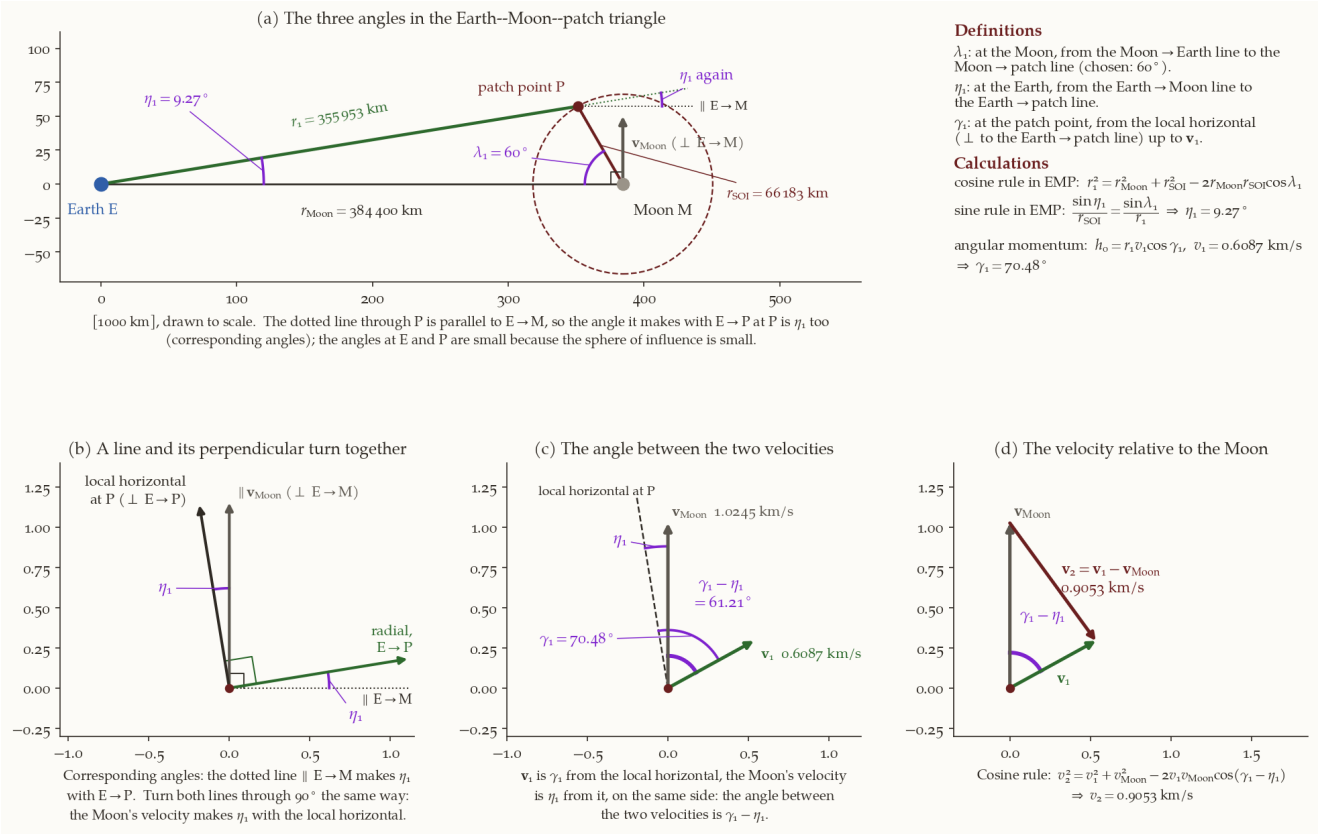

Figure 11: Example 2: the three angles and the angle between the two velocities. (a) The triangle Earth–Moon–patch point, drawn to scale. λ_1 is the angle at the Moon and η_1 the angle at the Earth; the dotted line through the patch point P is parallel to the Earth–Moon line, so it makes the same angle η_1 with EP at P (corresponding angles). The panel on the right lists the definitions and the calculations. (b) At P: the Moon's velocity (perpendicular to the Earth–Moon line) and the local horizontal (perpendicular to EP) are the two lines of panel (a) each turned through 90°, so they too make the angle η_1. (c) The spacecraft's velocity $\mathbf{v}_1$ is turned from the local horizontal by γ_1, the Moon's velocity by η_1, on the same side: the angle between the two velocities is $\gamma_1 - \eta_1 = 61.21°$. (d) The velocity triangle, closed by $\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_{\text{Moon}}$, and the cosine rule for its length. *Script: fig-pc-patched-conics.py.*

Step 6: the selenocentric hyperbola. At the patch point the spacecraft is r_{SOI} from the Moon with speed v_2:

$$\varepsilon_2 = \frac{v_2^2}{2} - \frac{\mu_{\text{Moon}}}{r_{\text{SOI}}} = 0.3357 \text{ km}^2/\text{s}^2 > 0.$$

The energy is positive, so the selenocentric orbit is a hyperbola: without a burn the spacecraft cannot stay. To find its shape we need its angular momentum about the Moon, $\mathbf{h}_2 = \rho \times \mathbf{v}_2$, where ρ is the vector from the Moon to the patch point. Both vectors lie in the plane of the Moon's orbit, so we write them in components in the frame of Figure 11: x from the Earth towards the Moon, y along the Moon's velocity, and z along the orbit normal ($\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$).

The vector ρ. The patch point is at the distance r_{SOI} from the Moon, at the angle λ_1 from the direction towards the Earth ($-x$), on the Moon's leading side ($+y$):

$$\begin{aligned}\rho &= r_{\text{SOI}}(-\cos\lambda_1,\ \sin\lambda_1) \\ &= 66\,183\,(-0.5000,\ 0.8660) = (-33\,091,\ 57\,316) \text{ km}.\end{aligned}$$

The vector $\mathbf{v}_2$. At the patch point P, at the polar angle η_1 about the Earth, the unit vectors along EP and along the local horizontal (EP turned through +90°) are $\hat{\mathbf{r}} = (\cos\eta_1,\ \sin\eta_1)$ and $\hat{\mathbf{t}} = (-\sin\eta_1,\ \cos\eta_1)$. The geocentric velocity is

$\mathbf{v}_1 = v_1(\sin\gamma_1\,\hat{\mathbf{r}} + \cos\gamma_1\,\hat{\mathbf{t}})$, and expanding with $\sin(\gamma_1 - \eta_1) = \sin\gamma_1\cos\eta_1 - \cos\gamma_1\sin\eta_1$ and the matching cosine gives

$$\begin{aligned}\mathbf{v}_1 &= v_1(\sin(\gamma_1 - \eta_1),\ \cos(\gamma_1 - \eta_1))\\ &= 0.6087\,(0.8764,\ 0.4815) = (0.53346,\ 0.29309)\ \text{km/s},\end{aligned}$$

which is the statement of Step 5 in components: $\mathbf{v}_1$ is turned by $\gamma_1 - \eta_1 = 61.21°$ from the y axis. The Moon's velocity is along y, $\mathbf{v}_{\text{Moon}} = (0,\ 1.0245)$ km/s, so

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{v}_1 - \mathbf{v}_{\text{Moon}} = (0.53346,\ 0.29309 - 1.02455)\\ &= (0.53346,\ -0.73146)\ \text{km/s}.\end{aligned}$$

Its length, $\sqrt{0.53346^2 + 0.73146^2} = 0.9053$ km/s, is the value that the cosine rule gave in Step 5.

The product. Both vectors have no z component, so $\rho \times \mathbf{v}_2$ points along z and its only component is

$$\begin{aligned}h_2 &= \rho_x v_{2y} - \rho_y v_{2x} = (-33\,091)(-0.73146) - (57\,316)(0.53346)\\ &= 24\,205 - 30\,576 = -6371\ \text{km}^2/\text{s}.\end{aligned}$$

The sign says that the spacecraft goes round the Moon clockwise seen from $+z$, the opposite way to the Moon's motion round the Earth; only $|h_2| = 6371$ km^2/s enters what follows. As a check, $b = |h_2|/v_2 = 7037$ km is the distance by which a straight line along $\mathbf{v}_2$ would miss the Moon's centre; the Moon's pull bends the path inwards, to the perilune below. Then $e_2 = \sqrt{1 + 2\varepsilon_2 h_2^2/\mu_{\text{Moon}}^2} = 1.4607$ and

$$r_p = \frac{h_2^2/\mu_{\text{Moon}}}{1 + e_2} = 3364\ \text{km},$$

a perilune altitude of 1627 km. On the hyperbola ($|a_2| = \mu_{\text{Moon}}/(2\varepsilon_2) = 7302$ km) the time from the patch to perilune is 18.2 h, so the whole flight takes 88.9 h.

Step 7: where the Moon must be at injection. In $t_1 = 70.75$ h the Moon moves through $n_{\text{Moon}} t_1 = 38.9°$. At the patch it is at geocentric angle $f_1 - \eta_1 = 164.56°$ from the injection point, so at injection it must be $164.56° - 38.90° = 125.66°$ ahead of it.

TODO: Show the velocity vector of Earth and also the motion of mooon about earth. So on the circular orbit, draw the arrows showing the direction of motion of moon.

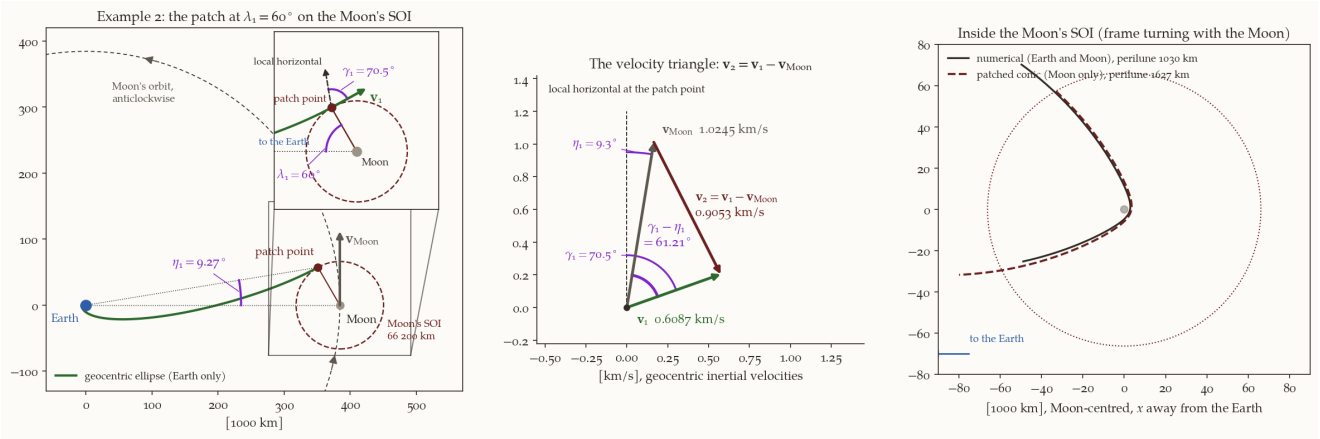

Figure 12: Example 2. Left: the geocentric ellipse up to the patch point on the Moon's sphere of influence, drawn as in Figure 11 (the Earth–Moon line along the horizontal, the Moon moving up the page, the patch point on its leading side), with the Moon's velocity $\mathbf{v}_{\text{Moon}}$ and arrows along its orbit showing the anticlockwise motion; the injection point at perigee is therefore 164.56° behind the Moon, as in Step 7. The angles of Steps 3 and 4 are marked: λ_1 at the Moon, between the Moon–Earth line and the line to the patch point; η_1 at the Earth, between the Moon and the patch point; γ_1 at the patch point, between the local horizontal (perpendicular to the Earth–spacecraft line) and the geocentric velocity $\mathbf{v}_1$. The enlargement shows the patch point. Middle: the velocity triangle of Step 5, drawn with the local horizontal at the patch point pointing up. Both the spacecraft's geocentric velocity $\mathbf{v}_1$ and the Moon's $\mathbf{v}_{\text{Moon}}$ (perpendicular to the Earth–Moon line, so turned by η_1 from that horizontal) start at one point; $\mathbf{v}_1$ lies γ_1 from the horizontal, so the angle between $\mathbf{v}_1$ and $\mathbf{v}_{\text{Moon}}$ is $\gamma_1 - \eta_1$, and the velocity relative to the Moon is $\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_{\text{Moon}}$, the third side. Right: inside the sphere, in a frame turning with the Moon. Dashed: the patched-conic hyperbola (Moon only); solid: a numerical integration of the same injection with the Earth and the Moon both acting. *Script: fig-pc-patched-conics.py.*

Step 8: how sensitive is it? Repeating Steps 3 to 6 for nearby λ_1 shows that the perilune altitude changes by about 600 km per degree of λ_1. A perilune of 200 km needs $\lambda_1 = 57.49°$. A lunar trajectory is extremely sensitive to the injection conditions, which is why every lunar mission flies trajectory-correction burns.

Step 9: the truth. Integrating the same injection numerically, with the Earth and the Moon both pulling throughout (Figure 12), gives a perilune altitude of about 1030 km at 87.0 h. The patched conic is 600 km off in perilune and 2 hours off in time. For the Moon, the patched conic is a good *first guess* to start a numerical search, not an answer.

Example 2 was given its ellipse. The next two examples design it, a Hohmann transfer and a bi-elliptic one, and find where it meets the Moon's sphere of influence.

6.3 Example 2a: a Hohmann transfer to the Moon

Example 2 was *given* its geocentric ellipse: an injection speed, with nothing said about where it came from. Here the ellipse is **designed**: it is the Hohmann transfer to the Moon's orbit. And the patch point is not given either; it is **found**, as the point where the ellipse meets the Moon's sphere of influence. The steps after that are those of Example 2.

Problem. A spacecraft is in a circular parking orbit 300 km above the Earth. It is to be sent to the Moon on a Hohmann transfer ellipse, with its perigee at the injection from the parking orbit and its apogee A at the radius of the Moon's orbit. At the Moon it must pass at a perilune altitude of 100 km and be captured there, by one burn at perilune, into a circular lunar orbit. Find:

1. where the Moon must be at the moment of injection;
2. where the ellipse meets the Moon's sphere of influence, and the geocentric state of the spacecraft there;
3. the spacecraft's velocity relative to the Moon there, and the perilune of its selenocentric orbit;
4. the capture burn, the total Δv and the flight time.

Data.

parking orbit	circular, $r_0 = R_\oplus + 300$ km $= 6678.137$ km
Earth	$\mu_\oplus = 398\,600.4$ km^3/s^2
the Moon's orbit	a circle about the Earth: $r_{\text{Moon}} = 384\,400$ km, $v_{\text{Moon}} = 1.0245$ km/s, $n_{\text{Moon}} = v_{\text{Moon}}/r_{\text{Moon}} = 2.6653 \times 10^{-6}$ rad/s
the Moon	$\mu_{\text{Moon}} = 4902.8$ km^3/s^2, $R_{\text{Moon}} = 1737.4$ km
its sphere of influence	$r_{\text{SOI}} = 66\,183$ km (Example 2, Step 1)

What is *not* given is where the Moon is when the spacecraft is injected. That is item 1 of the problem, and it also decides where the ellipse meets the sphere. So the calculation is done for a trial position of the Moon, and the position is adjusted until the perilune is 100 km.

Plan. Step 1 designs the ellipse. Step 2 writes the Moon's position as a function of time, with its unknown starting angle ϕ_0. Steps 3 to 6 are then a complete calculation for any trial ϕ_0: where the ellipse meets the sphere (Step 3), and from there the state, the relative velocity and the selenocentric orbit (Steps 4 to 6). We do them first for the *textbook timing* of a Hohmann transfer, which turns out to hit the Moon, and then in Step 7 adjust ϕ_0 until the requirement is met. Steps 8 and 9 finish the problem and test the answer.

symbol	meaning
$a_{\text{H}}, e_{\text{H}}, p_{\text{H}}, h_{\text{H}}, \varepsilon_{\text{H}}, T_{\text{H}}$	the Hohmann ellipse about the Earth, and its period
$\theta_{\text{Moon}}(t)$, ϕ_0	the Moon's polar angle at time t after injection, and its value at injection
$t_1, r_1, v_1, \gamma_1, f_1, \eta_1, \lambda_1$	time from injection, geocentric state and angles at the patch point (as in Example 2)
$\mathbf{v}_2, \varepsilon_2, h_2, e_2, a_2, r_p$	selenocentric orbit

Step 1: the Hohmann ellipse. Perigee r_0 and apogee r_{Moon} give

$$a_{\text{H}} = \frac{r_0 + r_{\text{Moon}}}{2} = 195\,539.1 \text{ km},$$
$$e_{\text{H}} = \frac{r_{\text{Moon}} - r_0}{r_{\text{Moon}} + r_0} = 0.96585,$$
$$p_{\text{H}} = r_0(1 + e_{\text{H}}) = 13\,128.2 \text{ km}.$$

Then $h_{\text{H}} = \sqrt{\mu_\oplus p_{\text{H}}} = 72\,338.8$ km^2/s and $\varepsilon_{\text{H}} = -\mu_\oplus/(2a_{\text{H}}) = -1.0192$ km^2/s^2. The injection speed is $v_0 = \sqrt{2(\varepsilon_{\text{H}} + \mu_\oplus/r_0)} = 10.8322$ km/s; the circular speed at r_0 is 7.7258 km/s, so $\Delta v_1 = 3.1064$ km/s. The speed at the apogee is $h_{\text{H}}/r_{\text{Moon}} = 0.1882$ km/s, and the half period $T_{\text{H}}/2 = \pi\sqrt{a_{\text{H}}^3/\mu_\oplus} = 119.52$ h (4.980 d) is the time from perigee to A.

Compare Example 2: its 10.84 km/s is 7.8 m/s above the Hohmann speed, and its apogee, 419 828 km, overshoots the Moon's orbit by 35 428 km. Example 2 was, in effect, a Hohmann transfer that is about 35 000 km too long at its apogee.

Step 2: the Moon as a function of time. Measure polar angles from the perigee direction of the ellipse, counter-clockwise, the way both the spacecraft and the Moon move: the injection point is at 0° and the apogee A at 180°. The Moon moves on its circle at the rate n_{Moon}, so at time t after injection its polar angle is

$$\theta_{\text{Moon}}(t) = \phi_0 + n_{\text{Moon}}\, t,$$

where $\phi_0 = \theta_{\text{Moon}}(0)$ is the angle by which the Moon leads the injection point at the moment of injection. It is the unknown of the problem, and it is the same phase angle ϕ_0 as in the planetary transfers (Equation 12). Figure 13 shows the definitions.

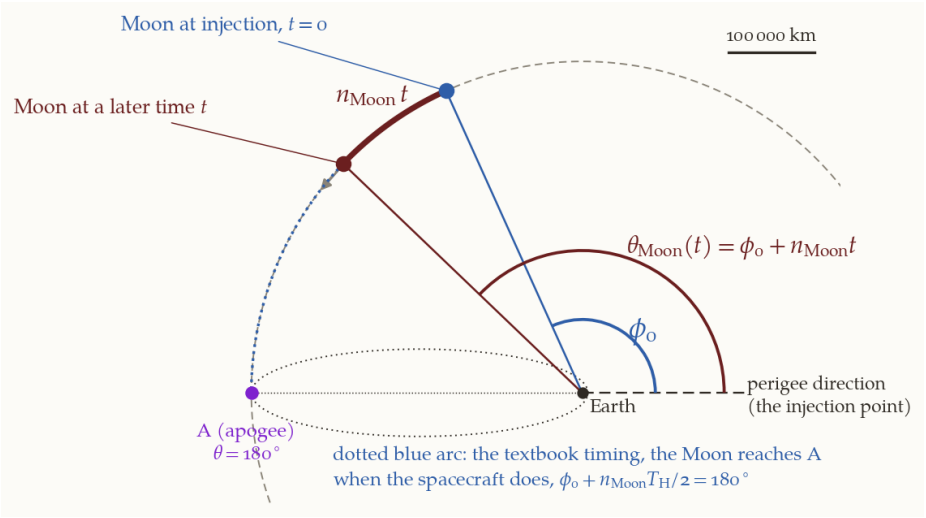

Figure 13: The Moon's polar angle in Example 2a. Angles are measured counter-clockwise from the perigee direction of the Hohmann ellipse (dashed line to the right: the direction of the injection point), so the apogee A is at 180°. At injection the Moon is at the angle ϕ_0 (blue). After a time t it has moved through the angle $n_{\text{Moon}}t$ along its circle (heavy line) and is at $\theta_{\text{Moon}}(t) = \phi_0 + n_{\text{Moon}}t$ (maroon). The picture is drawn for the textbook timing of this step, in which the Moon reaches A just as the spacecraft does: the heavy and the dotted arcs together make up $n_{\text{Moon}} T_{\text{H}}/2 = 65.71°$. *Script: fig-pc-transfers.py, with _pc_transfer.py.*

The **textbook timing** of a Hohmann transfer is to launch so that the Moon reaches A when the spacecraft does, at $t = T_{\text{H}}/2$: $\theta_{\text{Moon}}(T_{\text{H}}/2) = 180°$, so

$$\phi_0 = 180° - n_{\text{Moon}} \frac{T_{\text{H}}}{2} = 180° - 65.71° = 114.29°,$$

with $n_{\text{Moon}} T_{\text{H}}/2 = 2.6653 \times 10^{-6} \times 119.52 \times 3600 = 1.1468$ rad $= 65.71°$. We take this as the first trial.

Step 3: where the ellipse meets the sphere of influence (trial $\phi_0 = 114.29°$). The spacecraft is on the sphere of influence when its distance from the Moon's centre equals r_{SOI}, at the *same* time t for both. On the ellipse, in axes with x towards the perigee, the spacecraft is at $(a_{\text{H}}(\cos E - e_{\text{H}}),\ a_{\text{H}}\sqrt{1 - e_{\text{H}}^2}\sin E)$ at the time $t = (E - e_{\text{H}} \sin E)/n_{\text{H}}$, with $n_{\text{H}} = \sqrt{\mu_\oplus / a_{\text{H}}^3}$ (Kepler's equation); the Moon is at $r_{\text{Moon}}(\cos\theta_{\text{Moon}}(t), \sin\theta_{\text{Moon}}(t))$. The condition is

$$|\mathbf{r}_{\text{sc}}(t) - \mathbf{r}_{\text{Moon}}(t)| = r_{\text{SOI}}.$$

It is one equation in one unknown, t, and it has no closed-form solution; it is solved by bisection (see the script). We want the *first* time at which the gap closes, so the search starts at injection. Table 8 gives the distance d between the spacecraft and the Moon's centre at six times (Figure 14, below, shows the same instants):

Table 8: The distance $d = |\mathbf{r}_{sc} - \mathbf{r}_{Moon}|$ between the spacecraft and the Moon's centre, and its excess over r_{SOI}, at six times after injection.

t [h]	0	25	50	75	97	98
d [km]	387 195	255 006	195 125	130 068	67 270	64 326
$d - r_{SOI}$ [km]	+321 012	+188 823	+128 942	+63 885	+1087	−1857

The distance falls through r_{SOI} between 97 and 98 h; bisection gives $t_1 = 97.370$ h ($E_1 = 162.91°$, true anomaly $f_1 = 177.731°$), and there $r_1 = 376\,060$ km. The triangle Earth–Moon–patch point has the sides r_{Moon}, r_{SOI} and r_1. Example 2 gave λ_1 and computed r_1 from it; **here r_1 is known and λ_1 is the unknown**, and the same triangle gives it: the cosine rule

$$\cos\lambda_1 = \frac{r_{Moon}^2 + r_{SOI}^2 - r_1^2}{2\,r_{Moon} r_{SOI}} \;\Rightarrow\; \lambda_1 = 77.834°,$$

and the sine rule $\sin\eta_1 = (r_{SOI}/r_1)\sin\lambda_1$ gives $\eta_1 = 9.906°$. The patch point is on the Moon's leading side, as it must be for a spacecraft that the Moon overtakes.

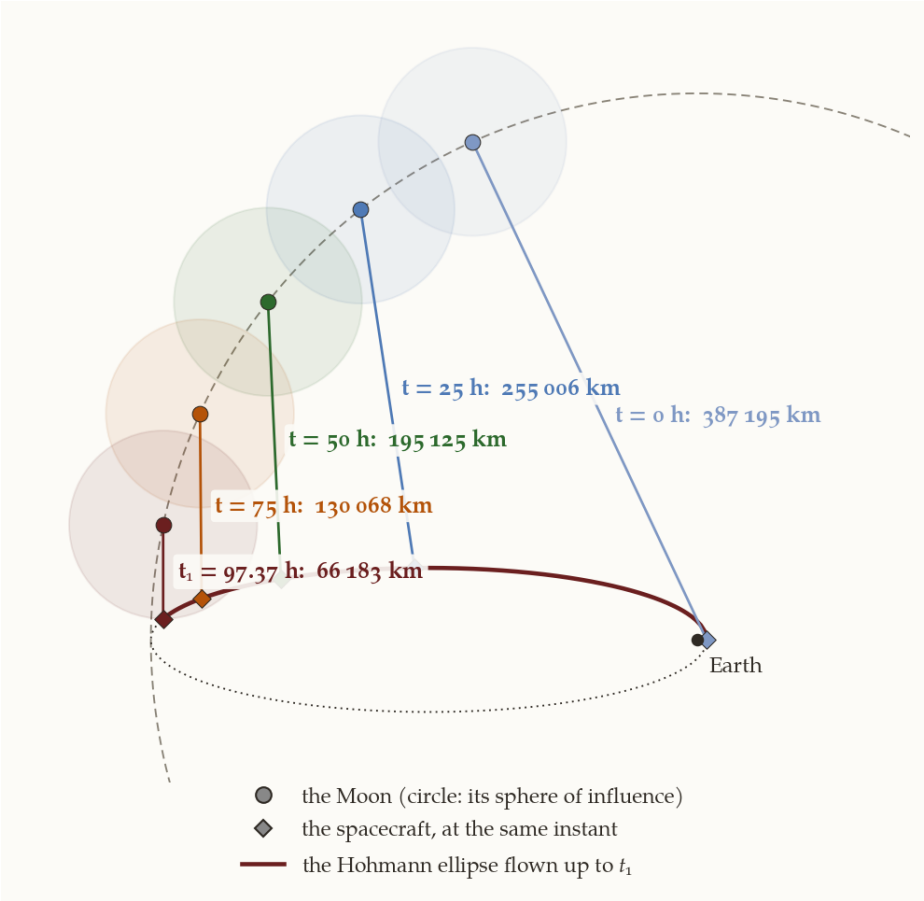

Figure 14: Example 2a, textbook timing ($\phi_0 = 114.29°$): the spacecraft (diamonds) and the Moon (circles, with its sphere of influence drawn to scale) at the same instants. The number at each pair is the distance between them. The distance shrinks, and at $t_1 = 97.37$ h it equals r_{SOI}: the spacecraft is on the sphere. *Script: fig-pc-transfers.py.*

Check the root. From the table, the distance minus r_{SOI} is +1087 km at 97 h and −1857 km at 98 h. Interpolate linearly for the zero. Then check that the Moon's polar angle at t_1, namely ϕ_0 plus the $n_{\text{Moon}}t_1$ that it has moved, equals $f_1 - \eta_1$.

What you should find

$t_1 \approx 97 + 1087/(1087 + 1857) = 97.369$ h, which agrees with the bisection to the digits shown. The Moon has moved $n_{\text{Moon}}t_1 = 53.53°$, so it is at $114.29° + 53.53° = 167.82°$, and $f_1 - \eta_1 = 177.731° - 9.906° = 167.825°$. The spacecraft is at the polar angle f_1 and the Moon is η_1 behind it: the geometry closes.

Step 4: the geocentric state at the patch. Energy and angular momentum on the ellipse, as in Example 2:

$$v_1 = \sqrt{2\left(\varepsilon_{\text{H}} + \frac{\mu_\oplus}{r_1}\right)} = 0.28532 \text{ km/s}, \qquad \cos\gamma_1 = \frac{h_{\text{H}}}{r_1 v_1} \;\Rightarrow\; \gamma_1 = 47.609°,$$

positive because the spacecraft is still climbing.

Step 5: the velocity relative to the Moon. The angle between the two velocities is $\gamma_1 - \eta_1 = 37.703°$ (the argument of Example 2, Step 5) and the cosine rule gives $v_2 = \sqrt{v_1^2 + v_{\text{Moon}}^2 - 2v_1 v_{\text{Moon}}\cos(\gamma_1 - \eta_1)} = 0.8176$ km/s.

A trap: freezing the Moon. It is tempting to put the Moon at A, draw its sphere of influence there, and intersect the ellipse with that circle. That treats the sphere as if it were at A when the spacecraft meets it, and it is not. The spacecraft reaches the sphere 22.15 h before it would reach A (97.37 h against 119.52 h), and in 22.15 h the Moon moves $n_{\text{Moon}} \times 22.15$ h $= 12.18°$, about 81 700 km, more than r_{SOI}. Where the sphere is when the spacecraft arrives depends on the Moon's phase: the intersection is that of two *moving* things.

Step 6: the selenocentric orbit. At the patch point $\varepsilon_2 = v_2^2/2 - \mu_{\text{Moon}}/r_{\text{SOI}} = 0.2602\ \text{km}^2/\text{s}^2 > 0$, so the orbit is a hyperbola. For the angular momentum we need components in the frame of Example 2, Step 6 (x from the Earth towards the Moon, y along the Moon's velocity). The vector from the Moon to the patch point is $\rho = r_{\text{SOI}}(-\cos\lambda_1,\ \sin\lambda_1) = (-13\,947,\ 64\,697)$ km, and the velocities are

$$\begin{aligned}\mathbf{v}_1 &= v_1(\sin(\gamma_1 - \eta_1),\ \cos(\gamma_1 - \eta_1))\\ &= 0.28532\,(0.6116,\ 0.7912) = (0.17450,\ 0.22575)\ \text{km/s},\\ \mathbf{v}_2 &= \mathbf{v}_1 - \mathbf{v}_{\text{Moon}} = (0.17450,\ 0.22575 - 1.02455)\\ &= (0.17450,\ -0.79880)\ \text{km/s}.\end{aligned}$$

Then

$$\begin{aligned}h_2 &= \rho_x v_{2y} - \rho_y v_{2x} = (-13\,947)(-0.79880) - (64\,697)(0.17450)\\ &= 11\,141 - 11\,289 = -148\ \text{km}^2/\text{s},\end{aligned}$$

$$e_2 = \sqrt{1 + \frac{2\varepsilon_2 h_2^2}{\mu_{\text{Moon}}^2}} = 1.0002, \qquad r_p = \frac{h_2^2/\mu_{\text{Moon}}}{1 + e_2} = 2.24\ \text{km}.$$

The perilune of the hyperbola is 2.2 km from the Moon's centre, which is *inside* the Moon ($R_{\text{Moon}} = 1737.4$ km). The angular momentum is almost zero: the spacecraft is aimed at the Moon's centre. **With the textbook timing the patched conic says that the spacecraft hits the Moon**, which is the right answer to the question "what happens if I aim at the centre?", and not what the problem asks for.

Step 7: choosing the Moon's phase. To pass at 100 km the Moon's phase ϕ_0 has to be changed, and with it the intersection of Step 3. Table 9 repeats Steps 3 to 6 for other values of ϕ_0.

Table 9: The perilune altitude of the Hohmann transfer for different phases of the Moon at injection. "Impact" means that the patched conic's perilune is inside the Moon. The bold rows are the two phases that give a 100 km perilune.

ϕ_0 [deg]	t_1 [h]	λ_1 [deg]	perilune altitude [km]
130	67.27	39.92	12 351
125	75.27	54.70	2266
123.07	78.86	59.85	100
120	85.01	67.23	impact
114.29 (textbook)	97.37	77.83	impact (2 km from the centre)
110	107.01	82.76	impact
106.39	115.11	84.78	100
100	129.33	83.65	10 380

There are two values of ϕ_0 that give a perilune altitude of 100 km: 123.07° ($\lambda_1 = 59.85°$, nearly the 60° of Example 2) and 106.39° ($\lambda_1 = 84.78°$, an entry nearly abreast of the Moon). We take the first. This answers item 1: **the Moon must be** 123.07° **ahead of the injection point at injection.** The table also shows how wide the band of impacts is: the patched conic predicts an impact for every ϕ_0 from 110° to 120°, because the spacecraft arrives nearly along the Moon's own track, so a change of phase moves the encounter mostly along the track and not across it. Outside the band the perilune altitude changes by about 1100 km per degree of ϕ_0 (from the table, $(2266 - 100)/(125 - 123.07)$).

For $\phi_0 = 123.07^\circ$ Steps 3 to 6 give the right-hand column of Table 10. Every entry was found by the step named in its row, in the same way as for the textbook timing.

Table 10: Steps 3 to 6 for the textbook timing and for the Moon's phase of the design.

quantity	textbook, $\phi_0 = 114.29^\circ$	design, $\phi_0 = 123.07^\circ$
t_1 [h] (Step 3)	97.370	78.859
r_1 [km] (Step 3)	376 059.5	355 791.0
λ_1 (Step 3)	77.834°	59.850°
η_1 (Step 3)	9.906°	9.256°
v_1 [km/s] (Step 4)	0.28532	0.44964
γ_1 (Step 4)	47.609°	63.116°
v_2 [km/s] (Step 5)	0.8176	0.8417
h_2 [km^2/s] (Step 6)	−148.1	4461.9
e_2 (Step 6)	1.0002	1.2100
r_p [km] (Step 6)	2.2	1837.4
perilune altitude [km]	−1735 (inside the Moon)	100.0

Step 8: the capture burn and the total. For the design the energy at the patch point is $\varepsilon_2 = v_2^2/2 - \mu_{\text{Moon}}/r_{\text{SOI}} = 0.2802$ km^2/s^2, so $|a_2| = \mu_{\text{Moon}}/(2\varepsilon_2) = 8750$ km, and the time from the patch to the perilune on the hyperbola is 18.94 h. The flight time is 78.86 + 18.94 = 97.80 h, 4.075 **days**. The circular speed at r_p is $\sqrt{\mu_{\text{Moon}}/r_p} = 1.6335$ km/s and the speed at perilune is $\sqrt{2\varepsilon_2 + 2\mu_{\text{Moon}}/r_p} = 2.4284$ km/s, so the capture burn is 0.7949 km/s and the total

$$\Delta v = \Delta v_1 + \Delta v_{\text{capture}} = 3.1064 + 0.7949 = 3.9013 \text{ km/s}.$$

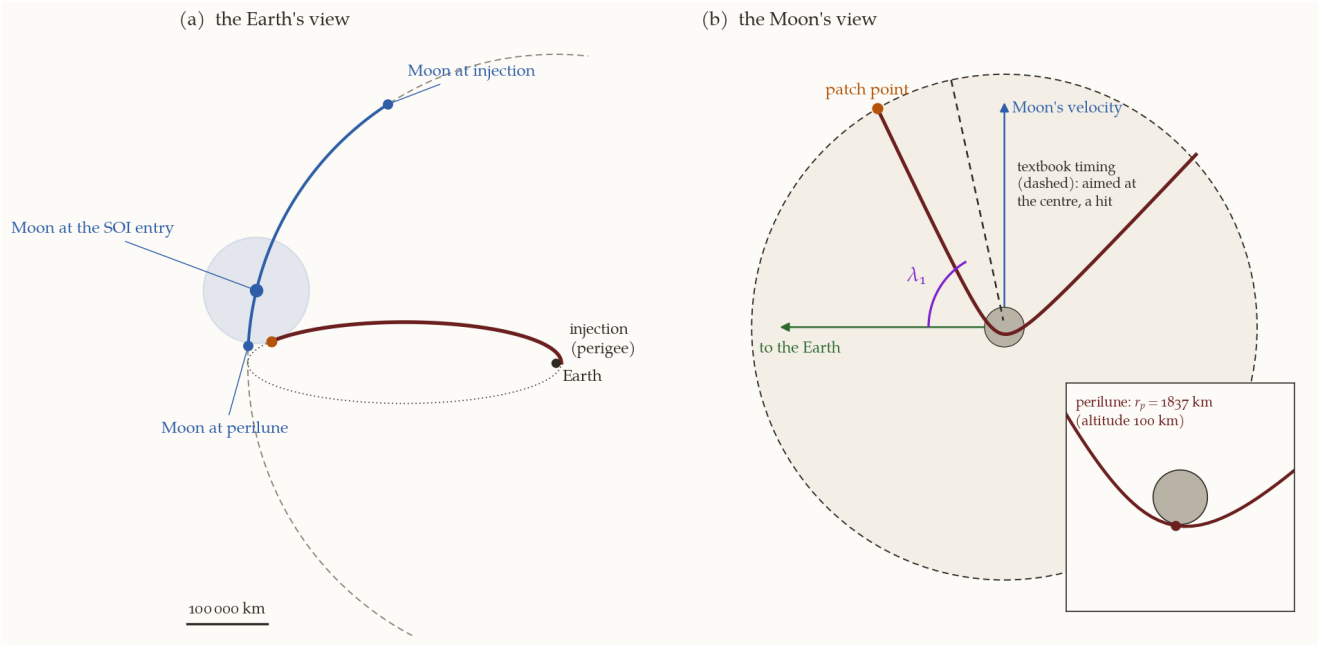

Figure 15: Example 2a, the design ($\phi_0 = 123.07°$). (**a**) The Earth's view, to scale. The Hohmann ellipse (dotted: the part not flown) ends at the patch point (amber) on the Moon's sphere of influence (shaded), which the Moon (blue) has reached at $t_1 = 78.86$ h; the Moon's path (blue line) is drawn from injection to perilune. (**b**) The Moon's view: the Moon at the centre, the Earth to the left and the Moon's velocity upwards. The patched-conic hyperbola (solid) enters at the patch point, with λ_1 the angle at the Moon from the Earth direction; the dashed line is the path of the same spacecraft with the textbook timing, aimed at the centre. The inset enlarges the perilune (100 km altitude; the Moon is drawn to scale). *Script: fig-pc-transfers.py, with _pc_transfer.py.*

Step 9: the truth. Integrating the same injection with the Earth and the Moon both acting (the Moon on its circle, with the phase $\phi_0 = 123.07°$), the spacecraft reaches the sphere of influence at 77.67 h (patched conic: 78.86 h) and passes the Moon at a perilune altitude of 668 km at 95.62 h. The patched conic is 568 km low in the perilune and 2.2 h late: the same quality as in Example 2 (600 km, 2 h). It is a first guess for a numerical search.

6.4 Example 2b: a bi-elliptic transfer to the Moon

The Moon's orbit is $r_{\text{Moon}}/r_0 = 57.56$ times the radius of the parking orbit. For a transfer between two *circular orbits about the Earth* a bi-elliptic transfer is cheaper than the Hohmann transfer when this ratio is large enough: above 11.94 if the intermediate apoapsis may be taken to infinity, and above 15.58 whatever the apoapsis (any $r_b > r_{\text{Moon}}$); both numbers are computed in the verify script. The Moon is far past both. It is reasonable to guess that a bi-elliptic transfer to the Moon is cheaper, and slower. This example designs one, finds its intersection with the Moon's sphere of influence as in Example 2a, and tests the guess.

A bi-elliptic transfer has three burns. Δv_1 at r_0 puts the spacecraft on **ellipse I** (periapsis r_0, apoapsis r_b). At the apoapsis Δv_2 raises the periapsis to r_{Moon}: **ellipse II** (periapsis r_{Moon}, apoapsis r_b). At the periapsis of II the classical transfer makes the third burn, to circularise at r_{Moon}. For the Moon the third burn is the capture at the Moon.

Problem. The problem of Example 2a, for a bi-elliptic transfer: a spacecraft in the 300 km parking orbit is sent to the Moon with an intermediate apoapsis $r_b = 2r_{\text{Moon}} = 768\,800$ km, to pass the Moon at a perilune altitude of 100 km and be captured there into a circular orbit. Find where the Moon must be at injection, where the transfer orbit meets the Moon's sphere of influence and the state there, the selenocentric orbit, the capture burn, the total Δv and the flight time; and then: is it worth it?

Data. Those of Example 2a, and r_b. (The Earth's sphere of influence about the Sun is 924 647 km, 2.405 r_{Moon}, so r_b must stay below that for the Earth's two-body orbit to mean anything.) As in Example 2a, the Moon's phase at injection, ϕ_0, is the unknown, and the calculation is done for trial values of it.

symbol	orbit or point
a_{I}, e_{I}; a_{II}, e_{II}	ellipse I; ellipse II
t_1, r_1, v_1, γ_1, η_1, λ_1	the patch point, now on ellipse II
$\mathbf{v}_2$, ε_2, h_2, e_2, a_2, r_p	selenocentric orbit

Step 1: the two ellipses. Ellipse I has $a_{\text{I}} = (r_0 + r_b)/2 = 387\,739.1$ km and $e_{\text{I}} = 0.98278$; the injection speed is $v_0 = 10.8787$ km/s, so $\Delta v_1 = 3.1530$ km/s, and the speed at the apoapsis is 0.09450 km/s. Ellipse II has $a_{\text{II}} = (r_{\text{Moon}} + r_b)/2 = 576\,600$ km and $e_{\text{II}} = 1/3$; the speed at its apoapsis is 0.58792 km/s, so

$$\Delta v_2 = 0.58792 - 0.09450 = 0.4934 \text{ km/s}.$$

The speed at the periapsis of II is 1.17584 km/s, 0.1513 **km/s more than the Moon's speed**. The half periods are 13.905 d for I and 25.216 d for II: the spacecraft reaches B at $t_{\text{B}} = 39.121$ d.

Step 2: the Moon's phase. As in Example 2a, with polar angles measured from the perigee direction of ellipse I (the apoapsis is at 180°, and the periapsis B of ellipse II at $0° \equiv 360°$), $\theta_{\text{Moon}}(t) = \phi_0 + n_{\text{Moon}} t$. The textbook timing is to have the Moon reach B when the spacecraft does, at $t_{\text{B}} = 39.121$ d: $\theta_{\text{Moon}}(t_{\text{B}}) \equiv 0°$, so $\phi_0 = -n_{\text{Moon}} t_{\text{B}} = -516.18° \equiv 203.82°$. This is the first trial.

Step 3: where the transfer orbit meets the sphere of influence. The equation is that of Example 2a, $|\mathbf{r}_{\text{sc}}(t) - \mathbf{r}_{\text{Moon}}(t)| = r_{\text{SOI}}$, now for the two-ellipse trajectory, and the search starts at $t = 0$ so that the *first* entry is the one found. For the textbook timing it is on ellipse II at $t_1 = 823.88$ h (34.33 d), 115.03 h before B, with $r_1 = 449\,302$ km and $\lambda_1 = -167.78°$: the spacecraft comes in from beyond the Moon's orbit and meets the sphere on the far side of the Moon. Steps 4 and 5 below, done for it, give a perilune radius of 1646 km, which is 91 km *inside* the Moon: a hit again. Adjusting ϕ_0 as in Step 7 of Example 2a, four values give a 100 km perilune: $\phi_0 = 210.25°$, 207.18°, 204.01° and 199.43°, with $\lambda_1 = -95.6°$, $-133.1°$, $-166.3°$ and $+162.9°$. We take the first, which answers the first question: **the Moon must be** 210.25° **ahead of the injection point**. From here on the numbers are for $\phi_0 = 210.25°$. Ellipse I crosses the radius r_{Moon} on its way out at 2.57 d, but the Moon is nowhere near: the spacecraft never comes within 332 000 km of the Moon's centre during the whole of ellipse I. The first entry is on ellipse II at $t_1 = 892.95$ h = 37.206 d, which is 45.96 h *before* the periapsis, with $r_1 = 396\,332$ km. The same triangle gives

$$\lambda_1 = -95.566°, \qquad \eta_1 = -9.567°.$$

The signs matter. The spacecraft is *faster* than the Moon ($v_1 = 1.1490$ km/s against 1.0245 km/s), so it catches up with the Moon from behind and enters the sphere on the Moon's **trailing** side: λ_1 and η_1 are negative. And it is descending towards the periapsis of II: from the energy and angular momentum of ellipse II, $v_1 = 1.1490$ km/s and $\gamma_1 = -6.991°$, negative. The formulas of Example 2 hold unchanged with these signs.

Step 4: the relative velocity. The angle between the two velocities is $\gamma_1 - \eta_1 = -6.991° + 9.567° = 2.576°$, a small angle: the two move almost together. The cosine rule gives $v_2 = 0.1336$ km/s.

Step 5: the selenocentric orbit, and a warning. $\varepsilon_2 = v_2^2/2 - \mu_{\text{Moon}}/r_{\text{SOI}} = -0.06515\ \text{km}^2/\text{s}^2$. It is **negative**: $v_2 = 0.134$ km/s is below $\sqrt{2\mu_{\text{Moon}}/r_{\text{SOI}}} = 0.385$ km/s, the escape speed from the sphere's boundary, so the energy is negative and the orbit about the Moon is an *ellipse*: $a_2 = \mu_{\text{Moon}}/(2|\varepsilon_2|) = 37\,628$ km. With the components of Example 2a, $\mathbf{v}_2 = (0.0516,\ 0.1233)$ km/s and $\rho = r_{\text{SOI}}(-\cos\lambda_1, \sin\lambda_1) = (6420,\ -65\,871)$ km, $h_2 = 4192\ \text{km}^2/\text{s}$, and

$$
\begin{aligned}
e_2 &= 0.9512, \\
r_p &= \frac{h_2^2/\mu_{\text{Moon}}}{1+e_2} = 1837.4\ \text{km (altitude 100 km)}, \\
r_a &= a_2(1+e_2) = 73\,418\ \text{km}.
\end{aligned}
$$

The time from the patch point to perilune is 55.63 h, and the whole flight takes $892.95 + 55.63 = 948.58$ h, 39.52 **days**.

Step 6: the capture and the total. The speed at perilune is $\sqrt{2\varepsilon_2 + 2\mu_{\text{Moon}}/r_p} = 2.2817$ km/s (valid for either sign of ε_2) against a circular speed of 1.6335 km/s, so the capture burn is 0.6482 km/s and

$$\Delta v = \Delta v_1 + \Delta v_2 + \Delta v_{\text{capture}} = 3.1530 + 0.4934 + 0.6482 = 4.2946\ \text{km/s}.$$

Step 7: is it worth it? Table 12 sets the two transfers side by side. The bi-elliptic transfer costs 0.393 km/s *more* and takes 9.7 times longer. So the guess is wrong for a transfer to the Moon. It is right for the classical problem, which is why it was reasonable.

Table 12: The two designs of this section, for the same 100 km perilune.

	Hohmann (2a)	bi-elliptic, $r_b = 2r_{\text{Moon}}$ (2b)
Δv_1 at the parking orbit [km/s]	3.1064	3.1530
Δv_2 at the apoapsis [km/s]	none	0.4934
capture at the 100 km perilune [km/s]	0.7949	0.6482
total [km/s]	**3.9013**	**4.2946**
speed relative to the Moon at the sphere, v_2 [km/s]	0.8417	0.1336
time to perilune	4.075 d	39.52 d

The classical problem. To go from the 300 km circular orbit to a circular orbit of radius r_{Moon} about the Earth, the Hohmann transfer needs $3.1064 + 0.8301 = 3.9365$ km/s. The bi-elliptic transfer of this example needs $3.1530 + 0.4934 + 0.1575 = 3.8039$ km/s, with a third burn of only 0.1575 km/s because the spacecraft arrives at the periapsis of II only 0.151 km/s faster than circular. It saves 0.133 km/s, at the price of 39 days instead of 5. (This is panel (a) of Figure 17: the bi-elliptic cost stays below the Hohmann one for every r_b between 1.05 and $3r_{\text{Moon}}$, and the saving grows with r_b.)

The lunar capture. Here the third burn is not a circularisation at r_{Moon}. It brings the spacecraft down into the Moon's potential well, to a circular orbit 100 km above its surface, and most of that burn is the depth of the well, not the arrival speed. The capture burn is $\sqrt{2\varepsilon_2 + 2\mu_{\text{Moon}}/r_p} - \sqrt{\mu_{\text{Moon}}/r_p}$ and ε_2 enters it only through $2\varepsilon_2$,

Not "captured". The selenocentric ellipse has its apolune, 73 418 km, beyond the sphere's radius, 66 183 km. Entering at r_{SOI} it must leave at r_{SOI} again, after the perilune. In the patched-conic model the spacecraft simply flies past the Moon, slowly, and a burn at perilune is still needed; it is only smaller than for a hyperbola. A true "ballistic capture" needs the Earth's pull *inside* the Moon's neighbourhood, a three-body effect that the patched conic leaves out; the note on the circular restricted three-body problem is the place for it.

which is small next to $2\mu_{\text{Moon}}/r_p = 5.337\ \text{km}^2/\text{s}^2$. The most that a slow arrival can save is the case $v_2 \to 0$, $\varepsilon_2 \to -\mu_{\text{Moon}}/r_{\text{SOI}}$: a capture burn of 0.6443 km/s against 0.7949 km/s for the Hohmann arrival, 0.151 **km/s**. Against that the bi-elliptic transfer pays for Δv_2 on the Earth's side: $\Delta v_1 + \Delta v_2 - \Delta v_{1,\text{Hohmann}} = 0.540$ km/s at $r_b = 2r_{\text{Moon}}$. Larger r_b lowers Δv_2, but not enough:

Table 13: The bi-elliptic transfer for four apoapsis radii; $r_b = 2.4r_{\text{Moon}}$ is at the Earth's sphere of influence. All Δv in km/s. The fifth column uses the *smallest conceivable* capture burn (0.6443 km/s), so the true excess is larger; it does not depend on which phase shift is chosen. The last column is the half period of I plus the half period of II, the time to B.

r_b/r_{Moon}	Δv_1	Δv_2	extra Earth-side cost	lunar total, at least this much above Hohmann	$T_{\text{I}}/2+T_{\text{II}}/2$
1.25	3.1250	0.7079	0.7264	+0.576	23.30 d
1.5	3.1374	0.6178	0.6488	+0.498	28.25 d
2	3.1530	0.4934	0.5400	+0.389	39.12 d
2.4	3.1608	0.4253	0.4797	+0.329	48.66 d

So for a transfer to the Moon the bi-elliptic transfer loses by at least about 0.33 km/s for any r_b inside the Earth's sphere of influence. It is worse, not better, and the *numbers* of the guess were not wrong: the Moon's orbit is far beyond the break-even ratios. What was wrong was the picture of the third burn.

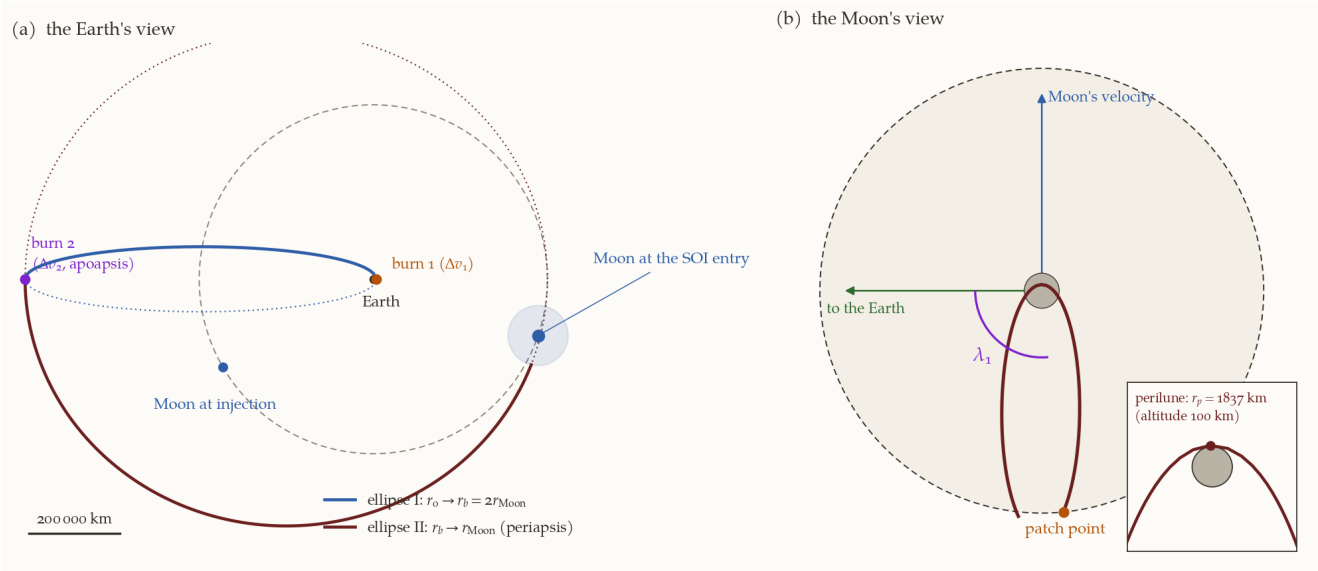

Figure 16: Example 2b. (**a**) The Earth's view, to scale: ellipse I (blue) from the injection at the right to the apoapsis at the left, ellipse II (maroon) from there round to the Moon's sphere of influence (shaded), which the Moon (blue) has reached at $t_1 = 37.2$ d. Dotted: the parts not flown. (**b**) The Moon's view, with the Earth to the left and the Moon's velocity upwards. The spacecraft enters the sphere at the bottom, behind the Moon ($\lambda_1 < 0$), passes the Moon at 100 km altitude (inset) and, on its elliptical selenocentric orbit, leaves the sphere again. *Script: fig-pc-transfers.py.*

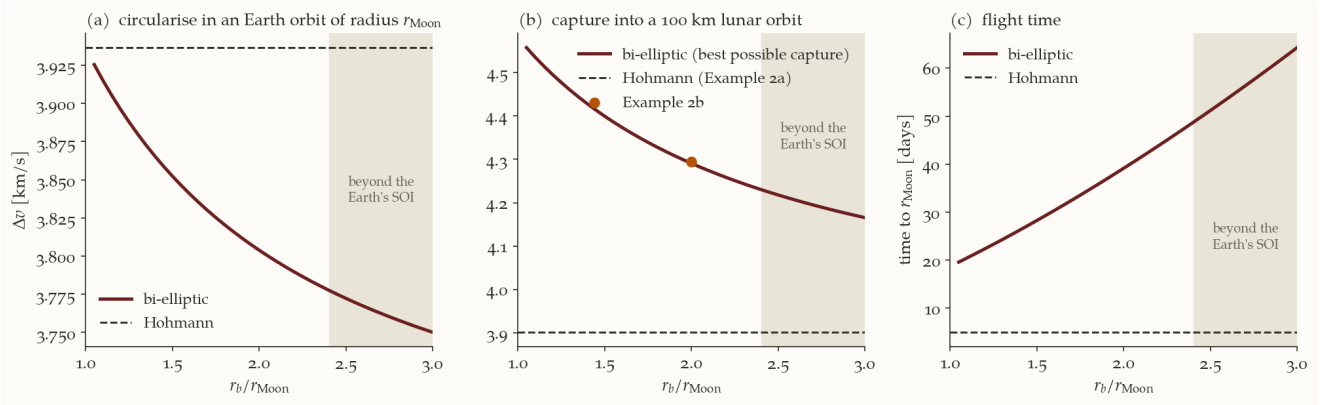

Figure 17: The bi-elliptic transfer against its apoapsis radius r_b. **(a)** Circularising in an Earth orbit of radius r_{Moon}: the bi-elliptic transfer (maroon) is cheaper than the Hohmann one (dashed) for every r_b. **(b)** Capture into a 100 km lunar orbit: the bi-elliptic curve is the *lower bound* (the least possible capture burn is used), and it is above the Hohmann total of Example 2a (dashed) everywhere. The amber point is Example 2b. **(c)** The flight time to r_{Moon}. The shaded region is beyond the Earth's sphere of influence about the Sun, where the Earth's two-body orbit is not meaningful. *Script: fig-pc-transfers.py*.

Step 8: how far to trust it. The Hohmann transfer of Example 2a lasts four days, inside the region where the Earth and the Moon dominate, and its patched conic is off by hundreds of kilometres. The bi-elliptic transfer spends weeks near r_b. There the Sun's tidal pull, $2\mu_\odot r_b/(1\text{ AU})^3 = 6.1 \times 10^{-8}$ km/s^2, is 9% of the Earth's pull, $\mu_\oplus/r_b^2 = 6.7 \times 10^{-7}$ km/s^2, and the Moon's pull is also not negligible. A numerical integration with only the Earth and the Moon, as in Step 9 of Example 2a, would therefore not be a "truth" for this transfer, and none is given: the comparison above is between two patched-conic estimates. What makes the conclusion safe is the size of the gap, 0.4 km/s or 10% of the total, against the small effect that the approximations have on the capture burn, which stays between 0.64 and 0.80 km/s over the whole range of v_2; it does not depend on the last digit.

A classmate says: "The bi-elliptic transfer saves 0.133 km/s over the Hohmann transfer, so we should use it for the Moon." What is wrong with the argument? What would you need to check before the saving could apply?

Answer

The 0.133 km/s is for circularising at r_{Moon} about the *Earth*. The Moon mission ends in a low *lunar* orbit, so the third burn is a capture into the Moon's well, which depends only weakly on the arrival speed (0.6443 to 0.7949 km/s over the whole possible range of v_2). The saving at the Moon is at most 0.151 km/s, and the bi-elliptic transfer pays 0.48 to 0.73 km/s more on the Earth's side for the second burn. To make the saving apply the mission would have to *end* in a circular orbit about the Earth at the Moon's distance, which no lunar mission does.

6.5 Example 3: a flyby of Jupiter

Given. A spacecraft reaches Jupiter (5.2026 AU, $\mu_J = 1.266\,865 \times 10^8$ km^3/s^2, $R_J = 71\,492$ km) at the aphelion of a Hohmann transfer from the Earth. It flies past Jupiter with periapsis $r_p = 5R_J$, passing behind Jupiter (on the side Jupiter is moving away from).

Find. The turning angle, the heliocentric velocity after the flyby, the energy gained, and the new heliocentric orbit. Can any flyby of this kind send the spacecraft out of the solar system?

Step 1: arrival. $a_t = (1 + 5.2026)/2$ AU; at aphelion $V^- = V_{t2} = 7.415$ km/s, while Jupiter moves at $V_J = 13.058$ km/s. The spacecraft is overtaken by Jupiter,

and relative to Jupiter it moves backwards with $v_\infty = V_J - V^- = 5.643$ km/s. The trip took 2.73 years.

Step 2: the turn. $e = 1 + r_p v_\infty^2/\mu_J = 1 + \dfrac{357\,460 \times 31.85}{1.2669 \times 10^8} = 1.0899$, so $\delta = 2\arcsin(1/e) = 133.1°$.

Step 3: the velocity triangle. Take Jupiter's velocity along $+y$, with the Sun towards $-x$ (Jupiter goes round the Sun anticlockwise), so $\mathbf{v}_\infty^- = (0, -5.643)$ km/s. Passing behind Jupiter turns $\mathbf{v}_\infty$ by 133.1° towards $+x$:

$$\mathbf{v}_\infty^+ = v_\infty(\sin\delta, -\cos\delta) = (4.118,\ 3.859)\ \text{km/s}, \qquad \mathbf{V}^+ = \mathbf{V}_J + \mathbf{v}_\infty^+ = (4.118,\ 16.917)\ \text{km/s}.$$

The heliocentric speed has gone from 7.41 to $|\mathbf{V}^+| = 17.41$ km/s (Figure 18). The relative speed is 5.643 km/s before and after; only its direction changed.

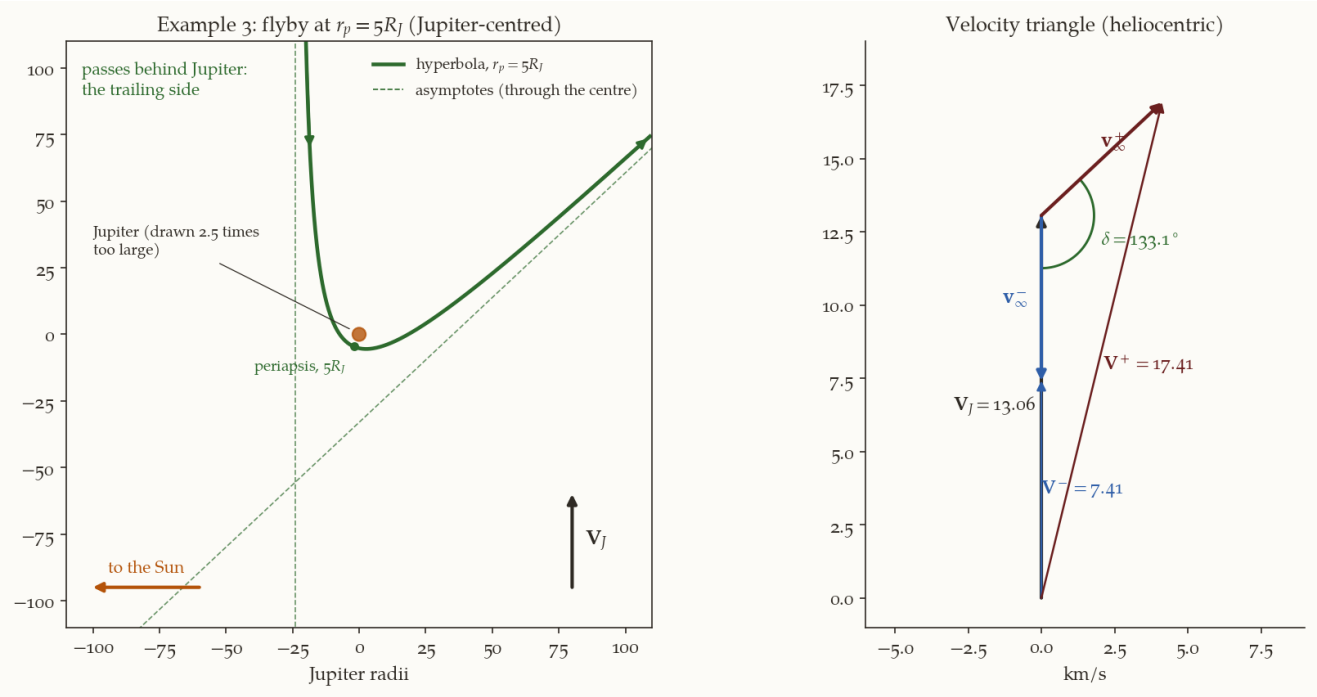

Figure 18: Example 3. Left: the flyby in Jupiter's frame, passing behind Jupiter at $5R_J$, to scale except for Jupiter itself. Jupiter moves up the page and the Sun is to the left. The spacecraft comes in along one asymptote and leaves along the other; the asymptotes cross at the centre of the hyperbola, not at Jupiter. Right: the velocity triangle in the Sun's frame. The relative velocity keeps its length and turns through δ; added to Jupiter's velocity, it makes a much longer heliocentric velocity. *Script: fig-pc-patched-conics.py.*

TODO: The second panel is not clear. Create a panel where you show velocities in wrt jupyter and show the rotation. In that figure show the direction of jupyter velocity. In a third panel, show the V- and V+ vectors

Step 4: the energy. By Equation 21, $\Delta(V^2/2) = \mathbf{V}_J \cdot (\mathbf{v}_\infty^+ - \mathbf{v}_\infty^-) = 13.058 \times (3.859 + 5.643) = 124.1\ \text{km}^2/\text{s}^2$. The new heliocentric energy is $V^{+2}/2 - \mu_\odot/r = -18.95\ \text{km}^2/\text{s}^2$, still negative: the spacecraft is still bound to the Sun. Its new orbit has $a = 23.4$ AU and $e = 0.79$: perihelion 4.87 AU, aphelion 42 AU, beyond Neptune.

Step 5: can it escape? Solar escape speed at Jupiter's distance is $\sqrt{2\mu_\odot/r} = 18.47$ km/s. With $|\mathbf{V}^+|^2 = V_J^2 + v_\infty^2 - 2V_J v_\infty \cos\delta$, reaching it needs $\cos\delta = -0.941$, $\delta = 160.2°$, and so $e = 1/\sin(80.1°) = 1.0151$ and $r_p = (e-1)\mu_J/v_\infty^2 = 60\,100$ km. That is **inside** Jupiter ($R_J = 71\,492$ km). Even a grazing flyby ($r_p = R_J$) turns

the velocity by only 158.4° and gives 18.42 km/s, just short. A Hohmann arrival at Jupiter is too slow to be flung out of the solar system; the probes that left the solar system after a Jupiter flyby, starting with Pioneer 10 in 1973, arrived faster than Hohmann (Pioneer 10 took 21 months, against the Hohmann transfer's 2.7 years).

What the numbers say. No propellant was spent, yet the heliocentric speed more than doubled. The rule of Equation 21 explains the side: passing behind the planet turns $\mathbf{v}_\infty$ towards the planet's own motion, and $\mathbf{V}_J \cdot \Delta\mathbf{v}_\infty > 0$.

TODO: In the Jupiter flyby example, add a figure that shows the initial ellipse if the spacecraft didn't encounter Jupiter. Then add the ellipse after the jupyter encounter. Show the a of nepture, earth, jupyter. Both these ellipses would be around Sun as the central body of course.

6.6 Behind, in front or to the side: what the arrival decides

Example 3 came in on a Hohmann transfer and could only gain energy. The section on flybys described three passes: behind the planet, in front of it and to the side. Starting from a transfer, can we choose between them? The answer has two parts. **The transfer decides which passes are possible, and the aim chooses among those.** (So the three panels of Figure 7 are three different *arrivals*, with the incoming asymptote pointing a different way in each. They are not three aims at one arrival.)

Step 1: the periapsis is tied to the arrival. The proof of Equation 22 gave the velocity at the two ends of the path, $\mathbf{v}_\infty^{\mp} = v_\infty(\pm\hat{\mathbf{p}}/e + \sin f_\infty\,\hat{\mathbf{q}})$. Dot the arrival velocity with $\hat{\mathbf{p}}$, and write $\hat{\mathbf{v}} = \mathbf{v}_\infty^-/v_\infty$:

$$\hat{\mathbf{p}}\cdot\hat{\mathbf{v}} = \frac{1}{e} = \sin\frac{\delta}{2}.$$

The direction of the periapsis is not free. It makes the fixed angle $90° - \delta/2$ with the arrival direction and lies on a cone about it, and the line of the cone that is flown is chosen by the *aim*. Write

$$\hat{\mathbf{p}} = \sin\frac{\delta}{2}\,\hat{\mathbf{v}} + \cos\frac{\delta}{2}\,\hat{\mathbf{n}}, \qquad \hat{\mathbf{n}} \perp \hat{\mathbf{v}}. \tag{23}$$

The unit vector $\hat{\mathbf{n}}$ has a meaning: it is the direction in which the incoming asymptote is displaced from the planet. The asymptotes of a hyperbola pass through its centre, which lies at the distance $|a|e$ from the planet on the side of the periapsis, and the part of that displacement perpendicular to $\hat{\mathbf{v}}$ is $|a|e\cos\frac{\delta}{2}\,\hat{\mathbf{n}} = |a|\sqrt{e^2-1}\,\hat{\mathbf{n}} = b\,\hat{\mathbf{n}}$, with b the aiming radius. The spacecraft is aimed to miss the planet by b, at a point of the plane perpendicular to $\hat{\mathbf{v}}$ (the "B-plane", met again in Example 6) that lies on a circle of radius b. The aim is one angle, the position on that circle.

Now $\Delta\mathbf{v}_\infty = -(2v_\infty/e)\hat{\mathbf{p}}$, and with $2\sin^2\frac{\delta}{2} = 1-\cos\delta$ and $2\sin\frac{\delta}{2}\cos\frac{\delta}{2} = \sin\delta$,

$$\begin{aligned}\Delta\mathbf{v}_\infty &= -v_\infty[(1-\cos\delta)\,\hat{\mathbf{v}} + \sin\delta\,\hat{\mathbf{n}}],\\ \mathbf{v}_\infty^+ &= \mathbf{v}_\infty^- + \Delta\mathbf{v}_\infty = v_\infty(\cos\delta\,\hat{\mathbf{v}} - \sin\delta\,\hat{\mathbf{n}}).\end{aligned} \tag{24}$$

The velocity turns through δ in the plane of $\hat{\mathbf{v}}$ and $\hat{\mathbf{n}}$, away from the side on which the aim point lies. (In Example 3, $\hat{\mathbf{v}} = -\hat{\mathbf{y}}$, and an aim on the Sun's side is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$; Equation 24 gives $v_\infty(\sin\delta, -\cos\delta)$, the result of its Step 3.)

Step 2: the energy. Put Equation 24 into Equation 21, $\Delta(V^2/2) = \mathbf{V}_P\cdot\Delta\mathbf{v}_\infty$. Let ξ be the angle between the arrival velocity $\mathbf{v}_\infty^-$ and the planet's velocity $\mathbf{V}_P$, so

that $\hat{\mathbf{v}} \cdot \hat{\mathbf{V}}_P = \cos\xi$. Let φ be the angle of $\hat{\mathbf{n}}$ round $\hat{\mathbf{v}}$, measured from the direction that is perpendicular to $\hat{\mathbf{v}}$, in the plane of $\hat{\mathbf{v}}$ and $\mathbf{V}_P$, and points towards $\mathbf{V}_P$; then $\hat{\mathbf{n}} \cdot \hat{\mathbf{V}}_P = \sin\xi\cos\varphi$. So

$$\Delta\frac{V^2}{2} = -V_P v_\infty\Big[(1-\cos\delta)\cos\xi + \sin\delta\,\sin\xi\,\cos\varphi\Big]. \tag{25}$$

Each angle has its own job. The angle ξ belongs to the **transfer**: $\mathbf{v}_\infty^- = \mathbf{V}^- - \mathbf{V}_P$ is what the heliocentric orbit delivers. The angle φ is the **aim**. At $\varphi = 0$ the aim point is on the planet's leading side and the spacecraft passes in front of it; at $\varphi = 180°$ it is on the trailing side and the spacecraft passes behind. With $1-\cos\delta = 2\sin^2\frac{\delta}{2}$ and $\sin\delta = 2\sin\frac{\delta}{2}\cos\frac{\delta}{2}$ the bracket at these two aims is $2\sin\frac{\delta}{2}\sin(\xi \pm \frac{\delta}{2})$, and $2\sin\frac{\delta}{2} = 2/e$, so for a given transfer the aim moves the energy change between

$$-\frac{2V_P v_\infty}{e}\,\sin\Big(\xi+\frac{\delta}{2}\Big) \;\le\; \Delta\frac{V^2}{2} \;\le\; \frac{2V_P v_\infty}{e}\,\sin\Big(\xi-\frac{\delta}{2}\Big), \tag{26}$$

the lower limit at $\varphi = 0$ (in front) and the upper at $\varphi = 180°$ (behind). The largest possible gain, $2V_P v_\infty/e$ of Equation 22, is reached only if $\xi = 90° + \delta/2$.

What a transfer allows. From Equation 26:

- a **gain** is possible if $\xi > \delta/2$, and a **loss** if $\xi < 180° - \delta/2$;
- if both are possible, $\delta/2 < \xi < 180° - \delta/2$, then so is $\Delta(V^2/2) = 0$, by continuity: the pass **to the side**, at $\cos\varphi_0 = -\tan(\delta/2)/\tan\xi$ (the bracket of Equation 25 set to zero), whose periapsis direction is perpendicular to $\mathbf{V}_P$;
- a **Hohmann** transfer is tangential at both ends, so $\mathbf{v}_\infty^-$ is along $\mathbf{V}_P$ or against it: $\xi = 180°$ for an outward transfer (the spacecraft is slower than the planet and is overtaken) and $\xi = 0$ for an inward one. Then $\sin\xi = 0$, the aim drops out of Equation 25, and **every aim gives the same energy change**, $+V_P v_\infty(1-\cos\delta)$ outward and $-V_P v_\infty(1-\cos\delta)$ inward.

Figure 19 draws this. In the left panel the band between the two limits of Equation 26 pinches to a point at the two Hohmann arrivals and opens as the arrival is turned away from the line of $\mathbf{V}_P$. The right panel draws Equation 25 against the aim: flat for the Hohmann arrivals, a cosine for the others.

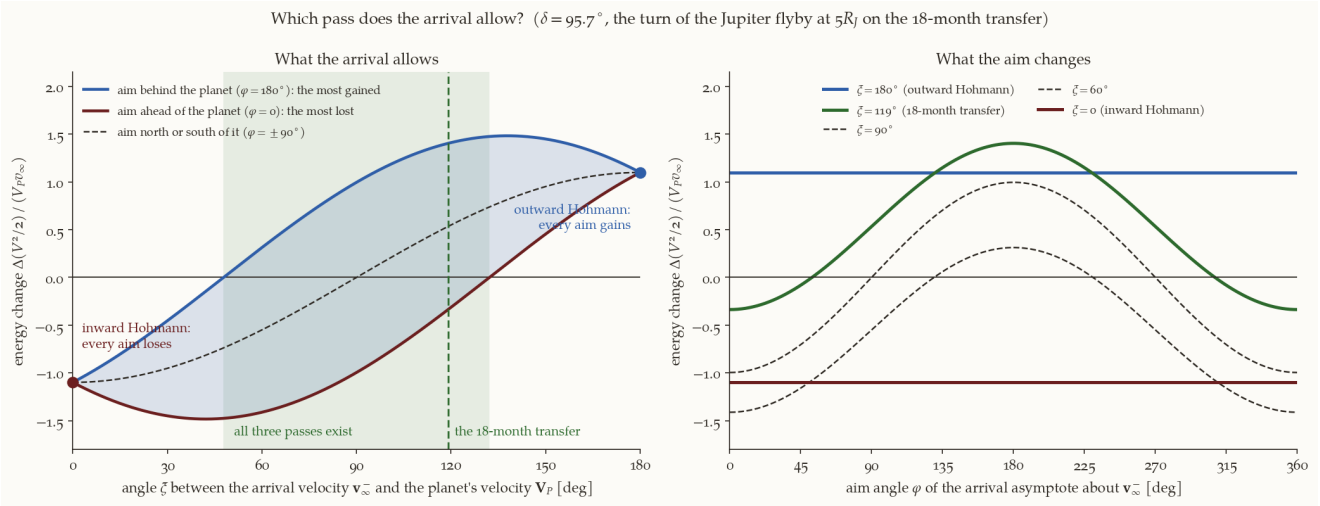

Figure 19: Which passes an arrival allows. Both panels show $\Delta(V^2/2)/(V_P v_\infty)$ for $\delta = 95.7°$, the turning angle of the flyby at $5R_J$ of the 18-month transfer below. **Left:** against the angle ξ between $\mathbf{v}_\infty^-$ and $\mathbf{V}_P$. The shaded band is what the aim can reach, from the pass in front (lowest) to the pass behind (highest), Equation 26; it pinches to a point at the two Hohmann arrivals, $\xi = 180°$ (outward) and $\xi = 0$ (inward), and the green strip is the set of arrivals for which all three passes, including the one with no change of energy, exist. **Right:** against the aim angle φ, for several arrivals. *Script: fig-pc-patched-conics.py.*

A Hohmann arrival, outward: Jupiter again. For Example 3, $\xi = 180°$ and $\Delta(V^2/2) = V_J v_\infty(1 - \cos\delta) = 124.1$ km^2/s^2 whatever the aim. Since $\xi = 180°$, $\hat{\mathbf{p}}\cdot\hat{\mathbf{V}}_P = -\sin\frac{\delta}{2}$ whatever the aim: the angle ψ' of Equation 22 is $90° - \delta/2 = 23.4°$, and the periapsis is always on the trailing side. (The spacecraft is overtaken by Jupiter, so it comes in from Jupiter's leading side and its periapsis is on the trailing side: "passing behind" refers to the periapsis, the side on which the spacecraft turns round.) The aim does decide the *direction* of $\mathbf{V}^+$. By Equation 24 it keeps its component 16.917 km/s along $\mathbf{V}_J$ and gains a component of 4.118 km/s across it, in the direction of $-\hat{\mathbf{n}}$; the four aims on the plane perpendicular to $\mathbf{V}_J$ give

aim of the incoming asymptote	$\Delta(V^2/2)$ [km^2/s^2]	the 4.118 km/s of $\mathbf{V}^+$ across $\mathbf{V}_J$ points	the new heliocentric orbit ($a = 23.41$ AU in all four)
on the Sun's side (Example 3)	124.1	away from the Sun	$e = 0.792$, perihelion 4.87 AU, aphelion 41.95 AU; outbound, 31.1° after perihelion
on the far side	124.1	towards the Sun	the same orbit, but inbound: perihelion is reached first
north or south of Jupiter	124.1	out of the ecliptic	$e = 0.778$, $i = 13.68°$, perihelion 5.20 AU (the encounter itself), aphelion 41.62 AU

So a Hohmann arrival does let the spacecraft pass to the side of the planet, above or below it, but this is not the pass of no energy change: it rotates $\mathbf{V}^+$ about $\mathbf{V}_J$ and tilts the orbit by up to 13.7°, with the same gain.

A Hohmann arrival, inward: Venus. Take a transfer from the Earth to Venus's orbit (0.72333 AU), which reaches it at the perihelion of the transfer with 37.727 km/s, while Venus moves at 35.021 km/s. The spacecraft overtakes Venus: $v_\infty = 2.707$ km/s along $\mathbf{V}_V$, $\xi = 0$. A flyby at $r_p = 2R_V$ ($e = 1.273$, $\delta = 103.55°$) loses $V_V v_\infty(1 - \cos\delta) = 117.0$ km^2/s^2 whatever the aim. In the plane, $|\mathbf{V}^+| = 34.49$ km/s and the new orbit has $a = 0.702$ AU, perihelion 0.644 AU and aphelion

0.760 AU. Passing behind Venus is not possible from this arrival, and the loss is not enough to bring the perihelion near the orbit of Mercury (0.31 to 0.47 AU).

Beyond Hohmann: all three from one arrival. Leave the Earth tangentially, as before, but on a larger ellipse: $a = 4.724$ AU, $e = 0.788$ (aphelion 8.45 AU). It needs $v_\infty = 10.05$ km/s at the Earth, against 8.79 km/s for the Hohmann transfer to Jupiter, and it reaches Jupiter's orbit after 18 months instead of 2.73 years. There the spacecraft is at true anomaly 146.4° with an outward speed $V_r = 9.727$ km/s and a transverse speed $V_t = 7.656$ km/s, so, relative to Jupiter, $\mathbf{v}_\infty^- = (9.727,\ -5.402)$ km/s in the axes of Example 3, with $v_\infty = 11.127$ km/s. It is no longer head-on: $\xi = 119.05°$. A flyby at $r_p = 5R_J$ has $e = 1.349$ and $\delta = 95.65°$, so $\delta/2 = 47.8° < \xi < 132.2° = 180° - \delta/2$, and by Equation 26 the energy change can be anywhere from −48.9 to +203.9 km^2/s^2. The aiming radius is $b = r_p\sqrt{(e+1)/(e-1)} = 927\,000$ km. At three points of the circle of that radius in the plane perpendicular to $\mathbf{v}_\infty^-$, the same arrival gives three different heliocentric orbits:

pass	aim φ	$\Delta(V^2/2)$ [km^2/s^2]	$\lvert\mathbf{V}^+\rvert$ [km/s]	the new heliocentric orbit
behind	180°	+203.9	23.69	hyperbolic ($a = -4.03$ AU): 23.69 km/s is above the solar escape speed at Jupiter's distance, 18.47 km/s, so the spacecraft leaves the solar system
to the side	52.2° (out of the ecliptic)	0	12.38, as on arrival	$a = 4.724$ AU as before, but $e = 0.357$, $i = 48.8°$, perihelion 3.04 AU, aphelion 6.41 AU
in front	0	−48.9	7.44	$a = 3.11$ AU, $e = 0.922$, perihelion 0.243 AU, aphelion 5.97 AU: it falls inside the orbit of Mercury

A transfer that is only moderately faster than Hohmann still gains whatever the aim: at ξ near 180° the lower limit of Equation 26 is still positive. For the same ellipses with the perihelion at 1 AU and the same $r_p = 5R_J$, the weakest pass of the 24-month transfer still gains 61.2 km^2/s^2, and that of the 21-month transfer 14.6. The pass in front, and with it the pass to the side, first appear at about 20.2 months. This is the content of the green strip in Figure 19: it is not the aim but the arrival that must be turned away from the line of $\mathbf{V}_P$.

What the numbers say. The planet supplies, to a given arrival, one circle of aims. Along the circle the energy change is a cosine of the aim angle, and the amplitude of the cosine, $V_P v_\infty \sin\delta \sin\xi$, is set by how far the arrival is from the line of the planet's velocity. A Hohmann arrival is exactly on that line, so the amplitude is zero: a Hohmann transfer can give a gain or a loss, depending on its direction, but never the choice between them.

History: Ulysses. The spacecraft Ulysses (launched in October 1990) went to Jupiter on a fast transfer and used the flyby of February 1992 to turn its orbit out of the plane of the ecliptic, to an inclination of about 80°, so as to pass over the poles of the Sun. A pass that rotates the heliocentric velocity out of the plane, as in the middle row of the table, is what such a mission needs.

7 Examples out of the plane

7.1 What changes in three dimensions

In three dimensions $\mathbf{v}_\infty$ is a *direction in space*, not just a speed, and that direction need not lie in any convenient plane. Two new questions arise.

TODO: Add a note in margin with figure that explains declination graphically.

Which planetary orbits contain a given asymptote? Every hyperbola leaving along a direction $\hat{\mathbf{s}}$ lies in a plane that contains $\hat{\mathbf{s}}$. There is a whole family of such planes, obtained by turning one of them about $\hat{\mathbf{s}}$. Let δ_∞ be the declination of $\hat{\mathbf{s}}$ above the planet's equator. To find the inclinations of the planes of the family, build two unit vectors perpendicular to $\hat{\mathbf{s}}$:

$$\hat{\mathbf{e}}_1 = \frac{\hat{\mathbf{K}} \times \hat{\mathbf{s}}}{|\hat{\mathbf{K}} \times \hat{\mathbf{s}}|}, \qquad \hat{\mathbf{e}}_2 = \hat{\mathbf{s}} \times \hat{\mathbf{e}}_1,$$

with $\hat{\mathbf{K}}$ the planet's pole. Since $|\hat{\mathbf{K}}\times\hat{\mathbf{s}}| = \cos\delta_\infty$, $\hat{\mathbf{K}}\cdot\hat{\mathbf{e}}_1 = 0$ and $\hat{\mathbf{K}}\cdot\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1\cdot(\hat{\mathbf{K}}\times\hat{\mathbf{s}}) = \cos\delta_\infty$. Any orbit normal of the family is $\hat{\mathbf{h}} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$, and its inclination is

$$\cos i = \hat{\mathbf{h}} \cdot \hat{\mathbf{K}} = \cos\delta_\infty \, \sin\theta. \tag{27}$$

As θ varies, $\cos i$ ranges between $\pm\cos\delta_\infty$:

$$|\delta_\infty| \;\le\; i \;\le\; 180^\circ - |\delta_\infty|. \tag{28}$$

An orbit plane can only contain an asymptote whose declination is no larger than its inclination. This is the same relation that ties a launch site's latitude to the smallest inclination it can reach directly (the Orbital Elements note): there the direction is the radius to the launch site, here it is the asymptote.

How does the heliocentric transfer reach a target out of the ecliptic? The heliocentric conic must lie in a plane through the Sun, the departure point and the arrival point. If those three are nearly in a line (a transfer of nearly 180°) the plane is badly determined, and a target slightly out of the ecliptic forces a steeply inclined transfer. Example 5 shows how expensive that is, and the usual remedy.

7.2 Example 4: the departure asymptote and the launch site

Given. The Hohmann departure of Example 1, $v_{\infty 1} = 2.945$ km/s, with $\mathbf{v}_{\infty 1}$ along the Earth's heliocentric velocity (in the ecliptic). The Earth's equator is inclined to the ecliptic by $\varepsilon = 23.44^\circ$. The parking orbit is reached by launching from Sriharikota, latitude 13.72° N.

Find. (a) The declination δ_∞ of the departure asymptote, relative to the Earth's equator, as a function of the date. (b) Its value for a departure on about 30 November ($\lambda_\odot = 248^\circ$) and on about 20 March ($\lambda_\odot = 0^\circ$). (c) Whether a parking orbit launched due east from Sriharikota can contain the asymptote; if not, the launch azimuth that can, or the cost of a combined plane change.

Step 1: where the Earth's velocity points. On the day when the Sun's ecliptic longitude is $\lambda_\odot$, the Earth is at heliocentric longitude $\lambda_\odot + 180^\circ$, and its velocity is 90° ahead of that, at ecliptic longitude $\lambda_v = \lambda_\odot + 270^\circ = \lambda_\odot - 90^\circ$ and ecliptic latitude 0.

Step 2: its declination. A direction in the ecliptic at longitude λ_v has declination $\sin\delta = \sin\varepsilon\,\sin\lambda_v$ (the ecliptic is the equator tilted by ε about the equinox line). So

$$\sin\delta_\infty = \sin\varepsilon\,\sin(\lambda_\odot - 90^\circ) = -\sin\varepsilon\cos\lambda_\odot.$$

Figure 20 plots it over the year. It swings between $\pm 23.44^\circ$: in March the asymptote points far south, in September far north, and near the solstices it lies in the equator.

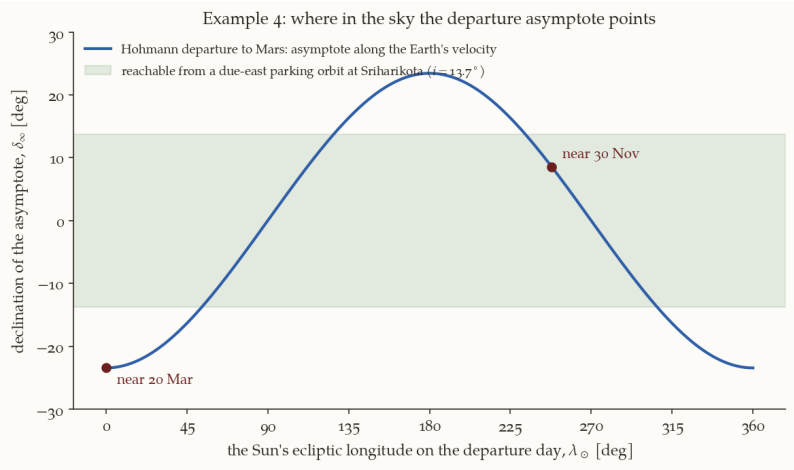

Figure 20: Example 4. Declination of a Hohmann departure asymptote to Mars over the year, and the band that a due-east parking orbit from Sriharikota can contain. *Script: fig-pc-patched-conics.py.*

Step 3: the two dates.

- 30 November, $\lambda_\odot = 248°$: $\sin\delta_\infty = -0.3978\cos 248° = 0.149$, so $\delta_\infty = 8.57°$.
- 20 March, $\lambda_\odot = 0°$: $\delta_\infty = -23.44°$.

Step 4: can the parking orbit contain the asymptote? By Equation 28, the parking orbit needs $i \geq |\delta_\infty|$. A due-east launch from Sriharikota gives $i = 13.72°$.

- In November, $8.57° < 13.72°$: yes. The departure plane is the parking orbit's plane, and the burn of Example 1 is all that is needed.
- In March, $23.44° > 13.72°$: no. Either launch into a more inclined parking orbit, with azimuth β from $\cos i = \cos\phi\sin\beta$: $\sin\beta = \cos 23.44°/\cos 13.72° = 0.9444$, so $\beta = 70.8°$ or $109.2°$ (a little north or south of east); or keep the due-east orbit and turn the velocity out of its plane during the departure burn.

Step 5: the cost of turning during the burn. An impulse changes the velocity but not the position, so the new orbit still passes through the burn point: the burn can turn the orbit plane only about the radius vector of that point. The departure asymptote, however, lies $f_{\infty 1} = 150.8°$ (Example 1) round from that radius, only 29.2° from the line about which the plane turns. Turning the plane through α therefore moves the asymptote out of the old plane by an angle ψ with $\sin\psi = \sin f_{\infty 1}\sin\alpha$, much less than α. The asymptote has to leave the band of declinations that the parking orbit can reach by $23.44° - 13.72° = 9.72°$, so

$$\sin\alpha = \frac{\sin 9.72°}{\sin 150.8°} = 0.346, \qquad \alpha = 20.3°,$$

twice the 9.72° one might guess. (A search over every burn point and tilt finds the same smallest tilt.) Figure 21 shows the geometry.

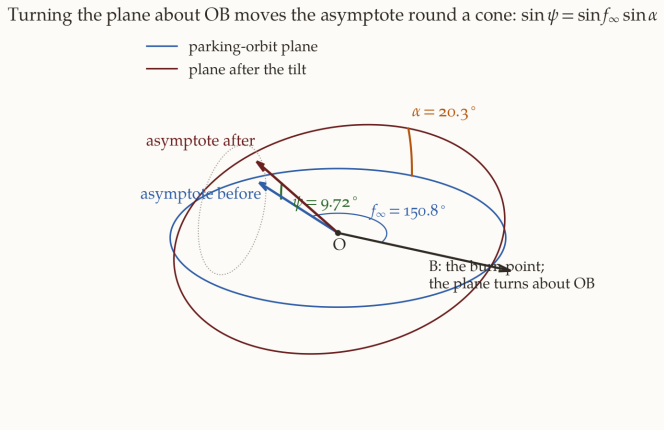

Figure 21: The tilt in Example 4. The burn can turn the parking-orbit plane (blue) only about the burn radius OB. The departure asymptote, $f_\infty = 150.8°$ round from OB, then slides round a cone about OB (dotted), so a tilt $\alpha = 20.3°$ of the plane lifts it out of the old plane by only $\psi = 9.72°$: $\sin\psi = \sin f_\infty \sin\alpha$. *Script: fig-pc-departure-tilt.py.*

A burn that takes the speed from $v_c = 7.726$ to $v_{p1} = 11.316$ km/s and turns the velocity through α costs, by the cosine rule,

$$\Delta v = \sqrt{v_c^2 + v_{p1}^2 - 2v_c v_{p1}\cos\alpha} = 4.869 \text{ km/s},$$

1.279 km/s (36%) more than the coplanar 3.590 km/s. Launching on the 70.8° azimuth instead costs only the part of the Earth's rotation that is no longer used: the site moves east at $\omega_\oplus R_\oplus \cos 13.72° = 0.452$ km/s, and the speed the launcher must add rises from $7.726 - 0.452 = 7.274$ to 7.301 km/s, about 27 m/s. In March, launch north of east.

What the numbers say. The *heliocentric* problem fixes $\mathbf{v}_\infty$ as a vector. Its direction, seen from the Earth, depends on the date, and the parking orbit must be chosen to contain it: turning the plane during the departure burn is about fifty times dearer than choosing the plane at launch, because the burn can only turn the plane about a line almost along the asymptote. Launch sites near the equator can reach a wider band of declinations without paying.

7.3 Example 5: a target out of the ecliptic

Given. The Earth–Mars Hohmann transfer of Example 1, but now Mars's orbit is inclined to the ecliptic by 1.85°, and on the arrival date Mars is at its greatest ecliptic latitude, $\beta = 1.85°$. The Earth's orbit is the ecliptic.

Find. (a) Why a transfer of exactly 180° cannot reach Mars. (b) The cost of tilting the whole transfer so that a transfer of slightly less than 180° reaches Mars. (c) The cost of a *broken-plane* transfer instead.

Step 1: the 180° **problem.** The transfer plane must contain the Sun, the Earth's position at departure and Mars's position at arrival. In a Hohmann transfer those three are in a line, and the plane through a line and a point *on* that line is undetermined. If Mars is slightly off the line (above the ecliptic), the only plane through all three is the one containing the line and the small offset, and that plane is nearly **perpendicular** to the ecliptic. Flying it would mean leaving the Earth almost at right angles to its orbit: hopeless.

Step 2: tilting the transfer. Shorten the transfer to an angle $\Theta < 180°$, with its line of nodes through the departure point. A point Θ round an orbit inclined

at i_t from its node has latitude given by $\sin\beta = \sin i_t \sin\Theta$ (the *Orbital Elements* relation $\sin\delta = \sin i \sin u$). So

$$\sin i_t = \frac{\sin\beta}{\sin\Theta}.$$

To keep the arithmetic simple, keep the Hohmann speeds and only tilt the departure velocity. The spacecraft must leave at $V_{t1} = 32.729$ km/s, inclined at i_t to the Earth's $V_\oplus = 29.785$ km/s, and by the cosine rule

$$v_{\infty 1} = \sqrt{V_{t1}^2 + V_\oplus^2 - 2V_{t1}V_\oplus \cos i_t}.$$

Θ	i_t	$v_{\infty 1}$ [km/s]	Δv from 300 km [km/s]
150°	3.70°	3.57	3.77
160°	5.42°	4.17	3.97
170°	10.71°	6.53	5.00
175°	21.74°	12.14	8.61

Near 180° the cost explodes. Even at $\Theta = 170°$ the departure burn grows by 1.4 km/s, because a 10.7° tilt of a 33 km/s velocity is a big vector change.

Step 3: the broken-plane transfer. Leave in the ecliptic on the Hohmann ellipse, and change the plane on the way, at a point 90° before arrival. A plane change of Δi made at a point 90° before the target reaches a latitude Δi at the target (again $\sin\beta = \sin\Delta i \sin 90°$), so $\Delta i = \beta = 1.85°$ suffices. That point is at true anomaly 90° on the transfer ellipse, at $r = p_t = 1.2075$ AU, where the transverse velocity is $V_\theta = h_t/r = 27.10$ km/s. Turning the plane about the radius vector rotates only the transverse component:

$$\Delta v_{\text{break}} = 2V_\theta \sin\frac{\Delta i}{2} = 0.875 \text{ km/s}.$$

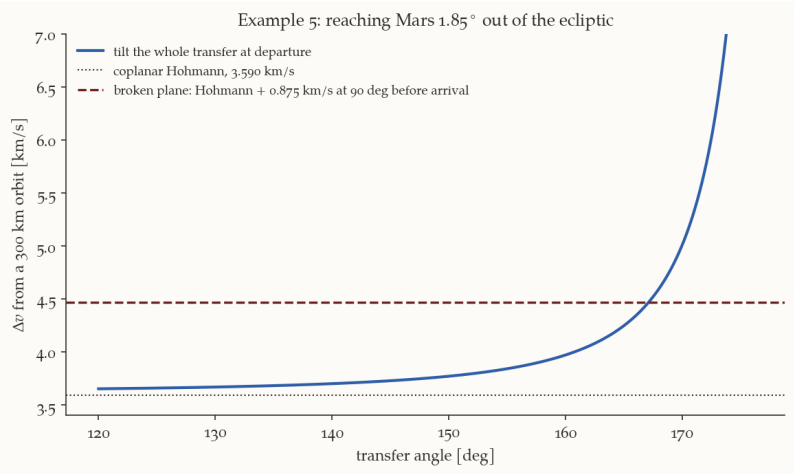

Figure 22: Example 5. The departure burn needed to reach Mars 1.85° above the ecliptic by tilting the whole transfer, against the transfer angle; and the total for the broken-plane transfer. *Script: fig-pc-patched-conics.py.*

What the numbers say. The broken-plane transfer costs 3.590 + 0.875 = 4.47 km/s in all. That is better than tilting the transfer at departure whenever Θ

exceeds about 167° (Figure 22), and it keeps the near-Hohmann geometry. The cheapest option of all is to **choose the dates** so that Mars is near a node of its orbit on arrival ($\beta \approx 0$): then no plane change is needed. This is one reason why real transfers are rarely exactly 180° (Problem 4 of the Indian missions set: Mangalyaan's transfer swept 218°).

7.4 Example 6: arriving at Mars out of its equator

Given. The Mars Orbiter Mission's arrival, reconstructed in Problem 4 of the Indian missions set: $v_\infty = 3.155$ km/s, along a direction whose declination relative to Mars's equator is $\delta_\infty = -30.25°$ (from the Lambert arc between the actual dates and Mars's pole direction). The capture orbit was 421.7 × 76 993.6 km with inclination 150.0° to Mars's equator.

Find. (a) The inclinations that a capture orbit can have without a plane change. (b) Whether MOM's orbit is one of them. (c) The cost of an equatorial capture orbit instead: turning the plane during the insertion burn, or later at apoapsis.

Step 1: the family of planes. By Equation 27, $\cos i = \cos 30.25° \sin\theta = 0.864 \sin\theta$, so $30.25° \le i \le 149.75°$ (Figure 23). Which member is flown is chosen by *aiming*: the angle θ fixes the orbit normal, and with it the miss vector, which is perpendicular to both the normal and the asymptote and so lies in the plane perpendicular to the asymptote (the "B-plane"). It is set by the trajectory corrections on the way in. The sunlit-side and dark-side approaches of Section 6.1 (Figure 10) are the two members of this family whose plane is the plane of the transfer.

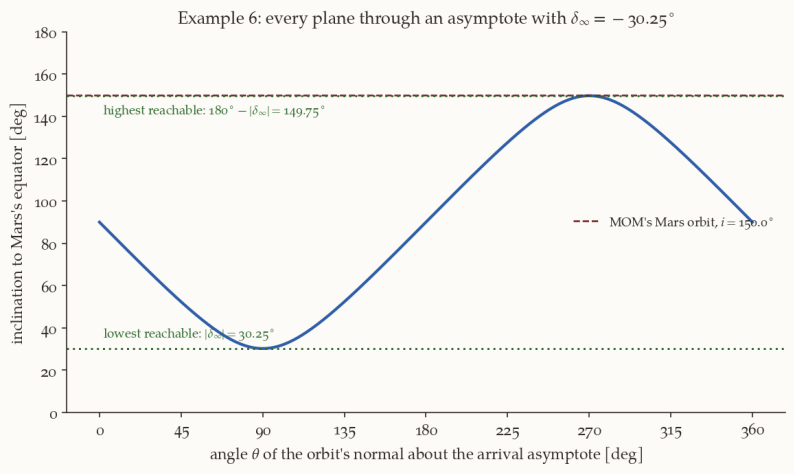

Figure 23: Example 6. Inclination to Mars's equator of every orbit plane that contains MOM's arrival asymptote, against the angle of the orbit normal about the asymptote. *Script: fig-pc-patched-conics.py.*

Step 2: MOM's orbit. 150.0° lies at the upper edge of the range, within the accuracy of the reconstruction: MOM's orbit is the most nearly equatorial (retrograde) orbit that this arrival direction allows without a plane change. That edge is reached when $\sin\theta = -1$, i.e. when the orbit normal is as close as possible to Mars's south pole.

Step 3: an equatorial orbit instead. Reaching $i = 180°$ needs a further plane change. As in Example 4, an impulse turns the orbit plane only about the radius vector of the burn point, so that point must lie on the line where the old and the new planes cross. For an equatorial target, the burn must be made where the orbit crosses Mars's equator.

- *During the insertion burn,* at periapsis. Then the periapsis itself must lie on the equator. The arrival hyperbola has $e = 1 + r_p v_\infty^2/\mu_M = 1.887$ and $f_\infty = 122.0°$, and by the argument of Example 4 the member of the family whose periapsis is on the equator is tilted from it by $\sin\alpha = \sin 30.25°/\sin 122.0° = 0.594$, $\alpha = 36.44°$ (its inclination is 143.56°). The burn must take the hyperbola's 5.691 km/s to the orbit's 4.628 km/s and turn it through 36.44°: $\sqrt{5.691^2 + 4.628^2 - 2(5.691)(4.628)\cos 36.44°} = 3.381$ km/s, against 1.063 km/s coplanar.
- *Later, after a coplanar capture.* On MOM's own plane ($i = 149.75°$) the periapsis is at latitude −15.5°, 32.0° past a node, so the apoapsis is *not* on the equator and a turn there cannot reach $i = 180°$. The 30.25° turn has to be made at the node 148.0° past periapsis, where $r = 31\,855$ km. There the orbit is climbing, and only the transverse part of the velocity, 0.555 km/s, has to turn: $2(0.555)\sin 15.125° = 0.289$ km/s.
- *Better aim, then apoapsis.* Aim instead for the $i = 143.56°$ member, whose apse line lies in the equator, capture into it with the same coplanar burn, and turn it through 36.44° at apoapsis, where the orbit moves at only 0.220 km/s: $2(0.220)\sin 18.22° = 0.137$ km/s.

What the numbers say. The arrival direction limits the capture inclinations exactly as the launch site's latitude limits the parking orbit's. When a different plane is wanted, a high-apoapsis capture orbit lets the plane change be made where the spacecraft is slow: 0.137 km/s, against 2.32 km/s extra if the turn is folded into the insertion burn. The aiming matters as well: the plane change must be made where the two planes cross, so the capture plane should be chosen to put the apoapsis there.

8 The method against the truth

The patched-conic method neglects, by construction, the perturbation at the edge of each sphere of influence: 8% at the Earth's, 5% at Mars's and 41% at the Moon's (Section 4). Two numerical checks show what that does.

Earth to Mars (Example 1). Integrating the departure from the 300 km orbit with both the Sun and the Earth pulling all the time, with the patched-conic burn of 3.590 km/s, gives an aphelion of 1.5271 AU, not 1.5237 AU: about 510 000 km beyond Mars's orbit. The burn that does reach 1.5237 AU in the integration is 3.586 km/s. The patched conic overestimates the burn by 3.8 m/s (0.1%), and that tiny error misses the target by half a million km. The method is excellent for the Δv **budget**, and inadequate for **aiming**.

Earth to the Moon (Example 2). The patched conic's perilune is 1627 km; the integration gives 1030 km. The Moon's sphere of influence is not small compared with the Earth–Moon distance, and the Earth keeps pulling hard inside it.

So the patched-conic method is the first step of mission design, not the last. It gives the launch window, the Δv budget, the burn locations and a first guess at the trajectory. That guess is then refined by numerical integration of the full equations, and in flight by trajectory-correction manoeuvres.

9 Summary

1. The motion of a spacecraft can be written about the planet or about the Sun; each is a two-body problem plus a perturbation (Equation 14, Equation 15). The Sun's perturbation of motion about the planet is *tidal*: the difference between its pulls on spacecraft and planet.
2. Laplace's criterion picks the description with the smaller relative perturbation. The boundary is $r_{SOI} = R(m/M)^{2/5}$: 925 000 km for the Earth, 66 200

km for the Moon. At the boundary the neglected perturbation is still about $(m/M)^{1/5}$ of the main pull.

3. Patched conics: one two-body problem per region. At each patch the position is the planet's and the velocity obeys $\mathbf{v}_\infty = \mathbf{V} - \mathbf{V}_{\text{planet}}$. The vector $\mathbf{v}_\infty$ is the whole interface between the heliocentric and the planetary designs.
4. Departure and capture are periapsis burns, $\Delta v = \sqrt{v_\infty^2 + 2\mu/r_p} - v_{\text{orbit}}$; the departure burn is $f_\infty = \arccos(-1/e)$ before the asymptote.
5. A flyby keeps $|\mathbf{v}_\infty|$ and turns it through $\delta = 2\arcsin(1/e)$; the heliocentric energy changes by $\mathbf{V}_P \cdot \Delta\mathbf{v}_\infty$. Which passes (behind, in front, to the side) are possible is decided by the arrival: a Hohmann arrival, with $\mathbf{v}_\infty$ along or against $\mathbf{V}_P$, always gains (outward) or always loses (inward), whatever the aim (Section 6.6).
6. Out of the plane: an orbit can contain an asymptote only if $i \geq |\delta_\infty|$; near-180° transfers to inclined targets are ruinous, and a broken-plane transfer or a better choice of dates avoids them.
7. The method gives budgets to 0.1%, but aiming errors of 10^5 km at a planet and hundreds of km at the Moon: it is the starting point for numerical refinement.

Every number in this note is recomputed by patched-conics-verify.py, including the two numerical integrations of Section 8. The figures are drawn by fig-pc-patched-conics.py.