

Orbital Maneuvers — Problems

Execution errors, finite burns and real missions

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How to use these problems

Ten problems on the *Orbital Maneuvers* deck. Attempt each part before reading its answer: on the web page every answer is folded under **Answer** (use the button at the top to open them all); in the PDF the answer follows its question.

The thread running through them: the Δv you *command* is not the Δv you *get*. It is off in **magnitude** (ε), in **direction** (α), in **timing** (τ), and in **duration** (a finite burn is not a point). Each problem exposes one of these, or the design choice that follows from it.

#	type	topic	setting
1	derive	first-order burn errors	LEO → GEO departure burn
2	derive	why a long burn loses energy	any orbit; used in 5 and 6
3	derive	phasing errors grow with k	circular orbit; used in 7
4	derive	apse rotation and pointing	Example 8 of the deck
5	compute	five burns to the Moon	Chandrayaan-3
6	compute	a 22-minute injection burn	Mangalyaan
7	compute	3 hours or 2 days?	ISS rendezvous
8	compute	plane change at apogee	GTO → GEO from three launch sites
9	compute	bi-elliptic to lunar distance	the edge of the Hill sphere

#	type	topic	setting
10	compute	a Δv budget and its uncertainty	15-year GEO satellite

$\mu_{\oplus} = 398600.4418 \text{ km}^3/\text{s}^2$, $R_{\oplus} = 6378.137 \text{ km}$, $g_0 = 9.80665 \text{ m/s}^2$. Mission figures (orbits, burn durations) are rounded values from public ISRO releases; every number a problem needs is given in the problem. All motion is two-body unless a part says otherwise.

Part I — Derive

Q1 — How wrong is the orbit if the burn is wrong?

The first Hohmann burn of a LEO $\rightarrow$ GEO transfer ($r_1 = 6678.137 \text{ km}$, $r_2 = 42164 \text{ km}$) is commanded as a **tangential** impulse $\Delta v = 2.42573 \text{ km/s}$. A real burn departs from the command in three ways:

- **magnitude:** it delivers $(1 + \varepsilon) \Delta v$;
- **direction:** it is turned in the orbit plane by α from the tangent (towards the outward radial);
- **timing:** it starts τ late.

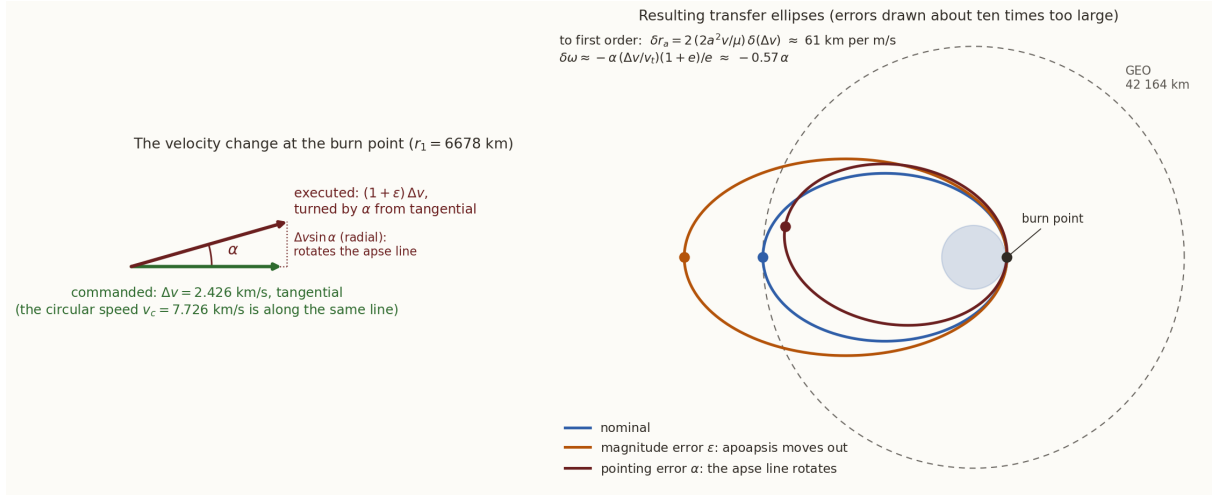


Figure 1: The velocity change at the burn point, and the transfer ellipses that result. [Script](#).

(a) Magnitude. Show that, to first order, a magnitude error $\delta(\Delta v)$ changes the semi-major axis of the transfer ellipse by $\delta a = \frac{2a^2 v}{\mu} \delta(\Delta v)$ and leaves the periapsis at r_1 , so that $\delta r_a = 2 \delta a$. Evaluate δr_a per m/s, and for $\varepsilon = 1\%$.

Answer. Differentiate *vis-viva*, $v^2 = \mu(2/r - 1/a)$, at fixed r :

$$2v dv = \frac{\mu}{a^2} da \quad \Rightarrow \quad \boxed{\delta a = \frac{2a^2 v}{\mu} \delta v}$$

The burn is tangential at r_1 , so after it $v_r = 0$ at r_1 : that point is still the periapsis, $r_p = r_1$, and $r_a = 2a - r_1$. Hence $\delta r_a = 2 \delta a$.

With $a_t = 24421.07 \text{ km}$ and $v_{t1} = 10.15149 \text{ km/s}$ (the speed *after* the burn):

$$\frac{\delta a}{\delta(\Delta v)} = \frac{2(24421.07)^2(10.15149)}{398600.44} = 3.038 \times 10^4 \frac{\text{km}}{\text{km/s}} = 30.4 \frac{\text{km}}{\text{m/s}},$$

$$\boxed{\frac{\delta r_a}{\delta(\Delta v)} = 60.8 \text{ km per m/s}}$$

For $\varepsilon = 1\%$, $\delta(\Delta v) = 24.3$ m/s, so $\delta r_a = \boxed{1474 \text{ km}}$: 3.5% of the GEO radius.

(b) Pointing. The thrust is turned by α towards the radial direction. Show that (i) the orbital energy changes only at **second** order in α , $\delta\varepsilon = -v_c\Delta v(1 - \cos\alpha)$, but (ii) the apse line rotates at **first** order, $\delta\omega \approx -\alpha \frac{\Delta v}{v_t} \frac{1+e}{e}$. Evaluate for $\alpha = 1^\circ$: δa , $\delta\omega$, and the along-track shift of the arrival point.

Answer. (i) Energy. The velocity after the burn has components $(v_r, v_\theta) = (\Delta v \sin\alpha, v_c + \Delta v \cos\alpha)$, so

$$\begin{aligned} |\mathbf{v}|^2 &= v_c^2 + 2v_c\Delta v \cos\alpha + \Delta v^2 \\ \Rightarrow \delta\varepsilon &= -v_c\Delta v(1 - \cos\alpha) \approx -\frac{1}{2}v_c\Delta v\alpha^2. \end{aligned}$$

The radial part of the impulse does no work ($\mathbf{v} \cdot \Delta\mathbf{v}_r = 0$), and the tangential part is shortened only by $\cos\alpha$. With $\delta a = 2a^2\delta\varepsilon/\mu$:

$$\delta a \approx -\frac{a^2}{\mu} v_c \Delta v \alpha^2 = -8.5 \text{ km} \quad (\alpha = 1^\circ), \quad \delta r_a = -17 \text{ km}.$$

The angular momentum $h = r_1(v_c + \Delta v \cos\alpha)$ is also second-order.

(ii) Aps line. Take $\hat{\mathbf{r}}$ along x . From $\mathbf{e} = \mathbf{v} \times \mathbf{h}/\mu - \hat{\mathbf{r}}$,

$$e_r = \frac{v_\theta h}{\mu} - 1, \quad e_\theta = -\frac{v_r h}{\mu}.$$

Nominally $e_\theta = 0$ and $e_r = e_t$ (periapsis at the burn point). With the error, $e_\theta = -\Delta v \sin\alpha r_1 v_t / \mu$ is **first-order**, and $\delta\omega = e_\theta / e_r$. At periapsis $r_1 v_t^2 = \mu(1+e)$, so $r_1 v_t / \mu = (1+e)/v_t$ and

$$\delta\omega \approx -\alpha \frac{\Delta v}{v_t} \frac{1+e}{e} = -\alpha (0.23895)(2.3764) = -0.568\alpha.$$

For $\alpha = 1^\circ$: $\delta\omega = -0.568^\circ$. The arrival point is 180° round the orbit at $r_a = 42164$ km, so it moves $0.568^\circ \times (\pi/180) \times 42164 = \boxed{418 \text{ km}}$ along the track. (A direct evaluation of the perturbed orbit gives -0.5679° , and $\delta e = -9 \times 10^{-5}$, second order.)

(c) Timing. The burn starts τ late on the circular orbit. With the thrust kept along the *local* velocity, show that the transfer is the nominal ellipse rotated rigidly by $n\tau$ (n = mean motion). Evaluate the arrival-point shift for $\tau = 1$ s. If instead the attitude is held **fixed in inertial space**, what is the net apse rotation?

Answer. A circular orbit looks the same from every point, so a burn τ late is the nominal burn at a point $n\tau$ further along the circle; with the thrust still tangential the transfer ellipse is the nominal one rotated about the focus by $n\tau$.

$$n = \sqrt{\frac{\mu}{r_1^3}} = 1.1569 \times 10^{-3} \text{ rad/s} = 0.0663^\circ/\text{s},$$

$$\text{arrival shift} = n\tau r_2 = \boxed{48.8 \text{ km per second}}$$

Inertial hold. The thrust still points where the velocity pointed at the nominal instant, but the velocity has turned forward by $n\tau$. The thrust is therefore misaligned by $\alpha = n\tau$ towards the outward radial. By (b) that rotates the apse line by $-0.568n\tau$, on top of the $+n\tau$ of the shifted burn point:

$$\omega = n\tau(1 - 0.568) = 0.432n\tau \Rightarrow 21 \text{ km per second}.$$

Holding attitude inertially *halves* the timing sensitivity, because the two errors partly cancel.

(d) Budget. Take 1σ errors $\varepsilon = 0.2\%$, $\alpha = 0.3^\circ$, $\tau = 1$ s (orbit-fixed attitude). Find the radial and along-track errors at arrival. Which dominates, and what does it do afterwards?

Answer.

$$\begin{aligned} \text{radial: } & 0.002 \times 2425.7 = 4.85 \text{ m/s} \Rightarrow 4.85 \times 60.8 = \boxed{295 \text{ km}} \\ \text{along-track, pointing: } & 0.3 \times 418 = 125 \text{ km} \\ \text{along-track, timing: } & 49 \text{ km} \\ \text{along-track, RSS: } & \sqrt{125^2 + 49^2} = \boxed{134 \text{ km}} \quad (0.18^\circ) \end{aligned}$$

The **magnitude** error dominates. It is also the harmful one: circularising 295 km too high gives a period error $\delta T/T = 1.5 \delta a/a = 1.0\%$, i.e. a drift of $360^\circ \times 1.0\% = 3.8^\circ/\text{day}$. The pointing and timing errors are *phase* errors, which are fixed once and stay fixed; a radius error is a *rate* error that keeps growing.

Q2 — Why one long burn loses more than several short ones

The impulsive model puts the burn at a point. A real burn of duration t_b is spread over an arc. Take constant thrust acceleration a_T **along the velocity**, with the burn centred in time on periapsis, so that the true anomaly runs from $-\theta$ to $+\theta$. Write $v_0(f)$ for the speed on the *coasting* orbit of eccentricity e , and $v_p = v_0(0)$.

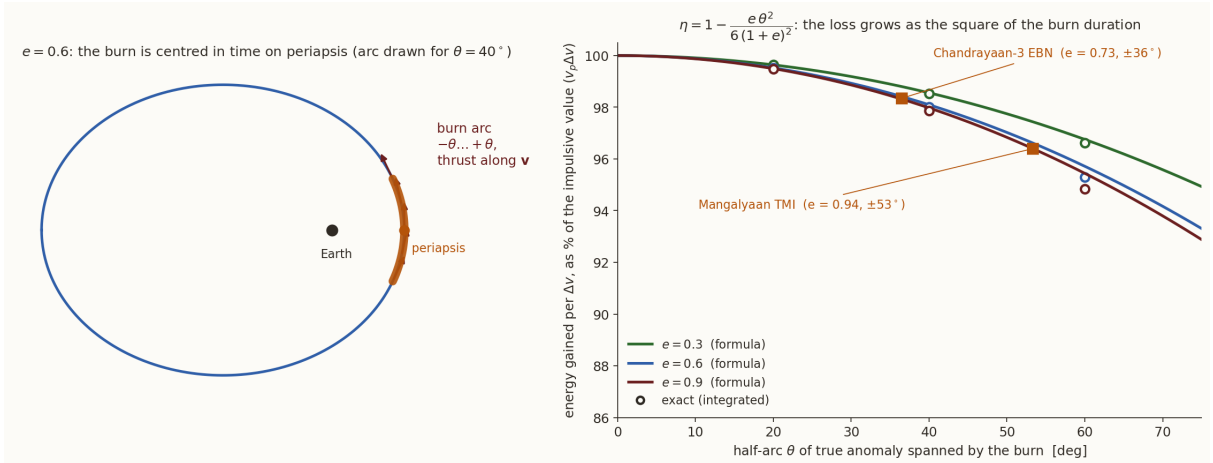


Figure 2: Left: the burn arc. Right: the fraction η of the impulsive energy $v_p \Delta v$ that a burn spanning $\pm\theta$ delivers, for three eccentricities (lines: formula; circles: exact). The two real burns of Q5 and Q6 are marked. [Script](#).

(a) Show that the thrust raises the specific energy at the rate $\dot{\varepsilon} = a_T v$, so that $d\varepsilon = v d(\Delta v)$. Hence show that, if the path is not much altered, $\Delta\varepsilon = \langle v_0 \rangle \Delta v + \frac{1}{2} \Delta v^2$, where $\langle v_0 \rangle$ is the time average of v_0 over the burn. Compare with the impulsive value $v_p \Delta v + \frac{1}{2} \Delta v^2$.

Answer. With $\varepsilon = v^2/2 - \mu/r$ and $\mathbf{a} = -\mu\mathbf{r}/r^3 + \mathbf{a}_T$,

$$\begin{aligned} \dot{\varepsilon} &= \mathbf{v} \cdot \mathbf{a} + \frac{\mu \dot{r}}{r^2} \\ &= -\frac{\mu \mathbf{v} \cdot \mathbf{r}}{r^3} + \frac{\mu \dot{r}}{r^2} + \mathbf{v} \cdot \mathbf{a}_T = \mathbf{v} \cdot \mathbf{a}_T = a_T v, \end{aligned}$$

because $\mathbf{v} \cdot \mathbf{r} = r\dot{r}$: gravity is conservative and drops out. With $d(\Delta v) = a_T dt$, $d\varepsilon = v d(\Delta v)$.

During the burn $v = v_0(t) + u(t)$, where $u = \int a_T dt$ is the Δv delivered so far (taking the path as unaltered). Then

$$\Delta\varepsilon = \int_0^{\Delta v} (v_0 + u) d(\Delta v) = \langle v_0 \rangle \Delta v + \frac{1}{2} \Delta v^2.$$

For the impulsive burn $v_0 \rightarrow v_p$ and the same $\frac{1}{2}\Delta v^2$ appears. The finite burn therefore falls short by exactly $(v_p - \langle v_0 \rangle)\Delta v$.

(b) For small θ show that the efficiency is $\eta \equiv \langle v_0 \rangle / v_p = 1 - \frac{e\theta^2}{6(1+e)^2}$. Check it for $e = 0.5$, $\theta = 30^\circ$.

Answer. On the orbit $v_0^2 = (\mu/p)(1 + 2e \cos f + e^2)$ and $v_p^2 = (\mu/p)(1 + e)^2$, so

$$\left(\frac{v_0}{v_p}\right)^2 = 1 - \frac{2e(1 - \cos f)}{(1 + e)^2} \approx 1 - \frac{ef^2}{(1 + e)^2}$$

$$\Rightarrow \frac{v_0}{v_p} \approx 1 - \frac{ef^2}{2(1 + e)^2}.$$

Near periapsis $\dot{f} = h/r^2 \propto (1 + e \cos f)^2$ changes only at order f^2 , so uniform in time $\approx$ uniform in f , and $\langle f^2 \rangle = \frac{1}{2\theta} \int_{-\theta}^{\theta} f^2 df = \theta^2/3$:

$$\eta = 1 - \frac{e\theta^2}{6(1+e)^2}$$

Check: $e = 0.5$, $\theta = 0.5236$ rad: $\eta = 1 - 0.5(0.2742)/(6 \times 2.25) = 0.9898$. Integrating the orbit numerically gives 0.9896.

(c) At periapsis $\dot{f}_p = v_p/r_p$; use it to estimate $\theta \approx v_p t_b/(2r_p)$. For $r_p = 7000$ km, $e = 0.5$, find θ and the loss for $t_b = 818$ s and $t_b = 1803$ s, using Kepler's equation for the exact θ .

Answer. $v_p = \sqrt{\mu(1+e)/r_p} = 9.2420$ km/s, so $\dot{f}_p = 1.3203 \times 10^{-3}$ rad/s and $a = 14000$ km, $T = 16486$ s, $n = 3.811 \times 10^{-4}$ rad/s.

t_b	$\theta \approx \dot{f}_p t_b/2$	exact θ (Kepler)	η (formula)	loss
818 s	30.9°	30.0°	0.9898	1.0%
1803 s	68.2°	60.0°	0.9594	4.1%

For 818 s: $M = n(t_b/2) = 0.1559$, $E = 0.3070$ rad, $\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \Rightarrow f = 30.0^\circ$. For 1803 s: $M = 0.3436$, $E = 0.6435$, $f = 60.0^\circ$. The estimate $\dot{f}_p t_b/2$ overshoots because $\dot{f}$ falls as the craft leaves periapsis, so use Kepler for long burns.

(d) Split the burn into n equal burns on n successive periapsis passes. Show the total loss falls as $1/n^2$. Compare with direct integration for the 1803 s burn.

Answer. Each burn lasts t_b/n , so $\theta \rightarrow \theta/n$ and each loses a fraction $\propto (\theta/n)^2$ of its own $\Delta v/n$. The loss, as a fraction of the total Δv , is that of one short burn:

$$\frac{\text{loss}(n)}{\text{loss}(1)} \approx \frac{1}{n^2}.$$

Integrating the orbit for $t_b = 1803$ s: $n = 1$: 4.39%; $n = 2$ (902 s each, $\theta = 32.9^\circ$): 1.25%; $n = 3$ (601 s, 22.3°): 0.57%. The ratios (3.5, 7.7) fall a little short of 4 and 9 because θ shrinks slightly slower than t_b .

(e) What limits how many burns one can use?

Answer. Each burn needs its own periapsis pass: the orbit period (hours to days for the raising orbits of Q5), one more engine restart, one more tracking arc and ground contact, and one more chance of a failure. The loss falls as $1/n^2$ while the time grows as n . The optimum is where the extra Δv is no longer worth the extra days: Chandrayaan-3 used five burns over ten days.

Q3 — Phasing: errors grow with the number of revolutions

A chaser trails a target by $\Delta\theta$ on a common circular orbit (radius r , speed v , period T). It closes the gap in k revolutions with two equal and opposite tangential burns (phasing slides of the deck). Each burn has a **relative** error ε and an **absolute** error σ (accelerometer bias, minimum impulse bit).

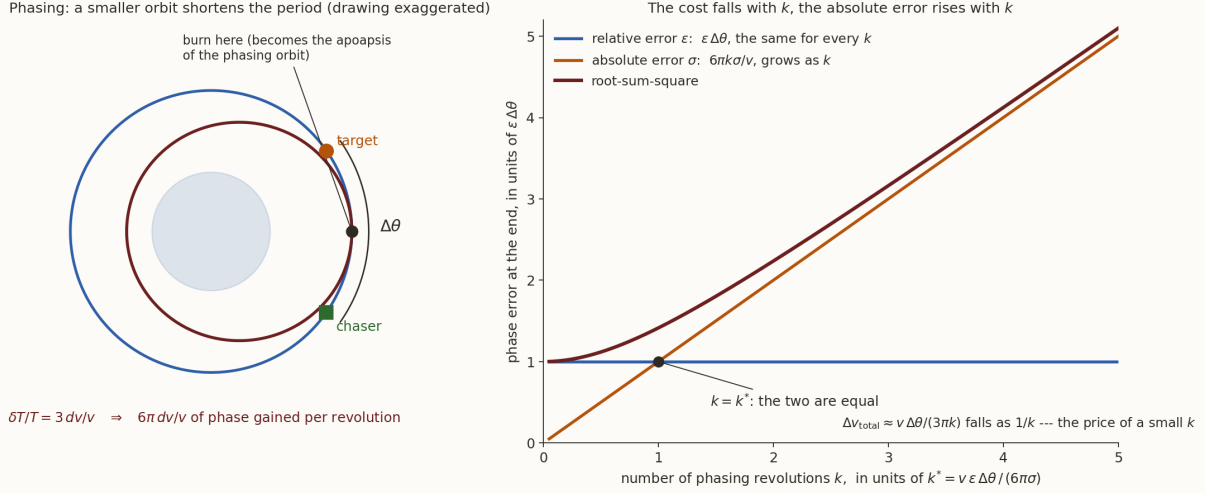


Figure 3: Left: a phasing orbit (drawn exaggerated). Right: the phase error of the manoeuvre against the number of revolutions. [Script](#).

(a) Show that a tangential burn dv on a circular orbit changes the period by $\delta T/T = 3 dv/v$, so that each revolution on the new orbit shifts the phase by $6\pi dv/v$.

Answer. $T = 2\pi\sqrt{a^3/\mu} \Rightarrow \delta T/T = \frac{3}{2} \delta a/a$. On a circle $\mu = v^2 a$, so $\delta a = 2a^2 v dv/\mu$ gives $\delta a/a = 2 dv/v$ and

$$\boxed{\frac{\delta T}{T} = \frac{3 dv}{v}} \Rightarrow \text{phase shift per revolution} = 2\pi \frac{\delta T}{T} = \boxed{6\pi \frac{dv}{v}}.$$

(b) Derive the closing condition, and show that for small $\Delta\theta$ $\Delta v_{\text{total}} \approx v\Delta\theta/(3\pi k)$ and the phasing perigee drops by $2r\Delta\theta/(3\pi k)$. Compare with the exact values for $r = 6778.137$ km, $\Delta\theta = 20^\circ$, $k = 2$.

Answer. The chaser completes k revolutions in the time the target covers $2\pi k - \Delta\theta$: $kT_{ph} = kT(1 - \Delta\theta/2\pi k)$, so $\delta T/T = -\Delta\theta/(2\pi k)$. By (a),

$$dv = -\frac{v}{3} \frac{\Delta\theta}{2\pi k} \Rightarrow \Delta v_{\text{total}} = 2|dv| = \boxed{\frac{v\Delta\theta}{3\pi k}}.$$

The phasing orbit has its apoapsis at the burn point, $a_{ph} = r(1 + \delta T/T)^{2/3} \approx r(1 - \Delta\theta/3\pi k)$, so $r_p = 2a_{ph} - r \approx r(1 - 2\Delta\theta/3\pi k)$: the perigee drops by $\boxed{2r\Delta\theta/(3\pi k)}$.

Numbers ($v = 7.6686$ km/s, $\Delta\theta = 0.34907$ rad, $k = 2$): $\Delta v \approx 142.0$ m/s against the exact 146.1 m/s, and a drop of 251 km against the exact 252 km.

(c) Show that the phase error at the end is $\sqrt{(\varepsilon\Delta\theta)^2 + (6\pi k\sigma/v)^2}$, and find the k^* at which the absolute error equals the relative one. Evaluate k^* for $\varepsilon = 0.5\%$, $\sigma = 0.1$ m/s, $v = 7669$ m/s, $\Delta\theta = 20^\circ$.

Answer. The error of the phasing burn is $\delta v = \varepsilon dv + \sigma$. After k revolutions (a):

$$\delta\phi = 6\pi k \frac{\delta v}{v}.$$

The **relative** part: $6\pi k \varepsilon \frac{|dv|}{v} = 6\pi k \varepsilon \frac{\Delta\theta}{6\pi k} = \varepsilon \Delta\theta$: the burn gets smaller as $1/k$, the drift per revolution grows as k , and the two cancel. The **absolute** part, $6\pi k \sigma/v$, has no such cancellation and grows with k . Independent errors add in quadrature. They are equal at

$$k^* = \frac{v \varepsilon \Delta\theta}{6\pi \sigma} = \frac{7669 \times 0.005 \times 0.3491}{6\pi \times 0.1} = \boxed{7.1}.$$

(d) At 400 km altitude the phasing perigee must stay above 200 km. Find the smallest k for $\Delta\theta = 20^\circ$. Is a larger k better?

Answer. From (b) the perigee altitude is $h_0 - 2r\Delta\theta/(3\pi k) \geq 200$ km, so

$$k \geq \frac{2r\Delta\theta}{3\pi(h_0 - h_{\min})} = \frac{2(6778)(0.3491)}{3\pi(200)} = 2.51$$

$$\Rightarrow k_{\min} = 3.$$

For $k < k^*$ the relative error dominates and does not depend on k , so nothing is gained by going beyond the minimum except a smaller Δv . For $k > k^* \approx 7$ the absolute error dominates and grows linearly with k . So: take the smallest k that the perigee allows if the Δv can be afforded; take larger k only to save propellant, and then pay for it with correction burns. (This is Q7.)

Q4 — Apse-line rotation: cheap, but fragile

In Example 8 of the deck ($a = 12000$ km, $e = 0.4$, so $p = 10080$ km), the apse line is rotated by $\eta = 30^\circ$ with a single impulse at the crossing $\nu = \eta/2$.

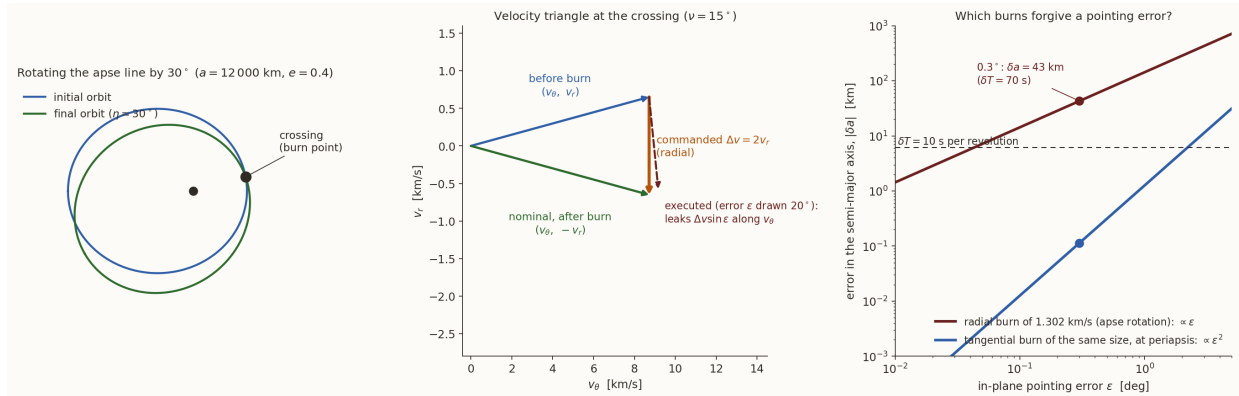


Figure 4: The two orbits, the velocity triangle at the crossing, and the error δa against the in-plane pointing error. [Script](#).

(a) Show that the two orbits (same p , same e , apse lines η apart) cross at $\nu = \eta/2$ and $\eta/2 + \pi$, that the required impulse is **purely radial**, and that $\Delta v = 2\sqrt{\mu/p} e \sin(\eta/2)$, independent of r .

Answer. A point at true anomaly ν on the first orbit is at $\nu - \eta$ on the second. Equal radii need $\cos \nu = \cos(\nu - \eta)$, i.e. $\nu = -(\nu - \eta)$ (the other root is $\eta = 0$): $\nu = \eta/2$, or $\eta/2 + \pi$.

With $v_r = \sqrt{\mu/p} e \sin \nu$ and $v_\theta = \sqrt{\mu/p} (1 + e \cos \nu)$, on the second orbit $\nu' = -\eta/2 = -\nu$. Cosine is even and sine odd, so

$$v'_\theta = v_\theta, \quad v'_r = -v_r$$

$$\Rightarrow \Delta \mathbf{v} = -2v_r \hat{\mathbf{e}}_r, \quad \Delta v = 2\sqrt{\frac{\mu}{p}} e \sin \frac{\eta}{2}$$

purely radial, with no r in it: both crossings cost the same. Here $\sqrt{\mu/p} = 6.28838$ km/s, so $\Delta v = 2(6.28838)(0.4) \sin 15^\circ = 1.30204$ km/s, with $v_r = 0.65102$ and $v_\theta = 8.71802$ km/s at the crossing ($r = 7270.8$ km).

(b) The pointing is off by ε (the thrust is turned from radial towards the transverse direction). Show that the burn now changes the energy at **first** order, $\delta\varepsilon_{\text{orb}} \approx v_\theta \Delta v \varepsilon$. Why does this differ from a tangential burn?

Answer. The velocity after the burn is $(v_r - \Delta v \cos \varepsilon, v_\theta + \Delta v \sin \varepsilon)$. Compared with the nominal $(-v_r, v_\theta)$ the energy changes by

$$\begin{aligned} \delta\varepsilon_{\text{orb}} &= \frac{1}{2} [(v_r - \Delta v \cos \varepsilon)^2 - v_r^2] + v_\theta \Delta v \sin \varepsilon + \frac{1}{2} \Delta v^2 \sin^2 \varepsilon \\ &\approx v_\theta \Delta v \varepsilon, \end{aligned}$$

since $\Delta v = 2v_r$ makes the first bracket $O(\varepsilon^2)$. The cause: a nominally radial burn has $\mathbf{v} \cdot \Delta \mathbf{v} = 0$ in the energy, so *any* leak into the direction of $\mathbf{v}_\theta$ enters at first order, weighted by the large speed v_θ . A tangential burn already sits at the maximum of $\cos \beta$ (Oberth slide of the deck): its energy depends on $\cos \varepsilon$ and so changes only at second order.

(c) For Example 8 find δa and the period error δT per revolution for $\varepsilon = 1^\circ$. Compare with a tangential burn of the same size at the periapsis of this orbit. What pointing accuracy keeps $|\delta T| < 10$ s?

Answer. $\delta\varepsilon_{\text{orb}} = 8.71802 \times 1.30204 \times 0.017453 = 0.1981$ km²/s². With $\delta a = 2a^2 \delta\varepsilon_{\text{orb}}/\mu$ and $T = 2\pi\sqrt{a^3/\mu} = 13082$ s:

$$\delta a = \frac{2(12000)^2(0.1981)}{398600} = \boxed{143 \text{ km}},$$

$$\delta T = \frac{3}{2} T \frac{\delta a}{a} = \boxed{234 \text{ s per revolution}}.$$

Tangential burn of the same size at periapsis ($v_p = 8.8037$ km/s): $\delta\varepsilon_{\text{orb}} = -v_p \Delta v (1 - \cos 1^\circ) = -1.75 \times 10^{-3}$, so $\delta a = -1.3$ km: **113 times smaller**.

For $|\delta T| < 10$ s: $\delta a = \frac{2}{3} a (10/T) = 6.1$ km, $\delta\varepsilon_{\text{orb}} = 8.5 \times 10^{-3}$, and $\varepsilon = 8.5 \times 10^{-3} / (8.718 \times 1.302) = 7.4 \times 10^{-4}$ rad = $\boxed{0.043^\circ = 2.6'}$.

(d) The two-burn alternative (circularise at apoapsis, then re-eccentricise) costs 2.196 km/s whatever η is. A 0.3° (1σ) pointing error is realistic. Which would you fly?

Answer. With $\varepsilon = 0.3^\circ$: $\delta\varepsilon_{\text{orb}} = 0.0595$, $\delta a = 43$ km, $\delta T = 70$ s per revolution: too large to leave. A **trim** at periapsis ($v_p = 8.804$ km/s) removes it:

$$dv = \frac{\mu \delta a}{2v_p a^2} = \frac{398600 \times 43}{2(8.804)(1.44 \times 10^8)} = 6.8 \text{ m/s}.$$

Total: $1.302 + 0.007 = 1.309$ km/s, against 2.196 km/s. The single radial burn still wins by nearly a kilometre per second; what changes is that it **must** be followed by an orbit-determination arc and a trim, and that the pointing requirement (0.04° for the burn to be good without a trim) is far tighter than for the tangential burns of Part I.

Part II — Real missions

Q5 — Chandrayaan-3: five burns to the Moon

After launch by LVM3 the Chandrayaan-3 propulsion module (about 3900 kg with lander and rover; one 440 N engine, $I_{sp} \approx 310$ s) was in orbit **A**. Five Earth-bound burns raised the apogee to orbit

B, and the trans-lunar injection (TLI) of 1 August 2023 raised it to C.

orbit	perigee $\times$ apogee [km altitude]	after
A	$170 \times 36\,500$	LVM3 injection
B	$236 \times 127\,603$	fifth Earth-bound burn
C	$288 \times 369\,328$	TLI

Assume: every burn is tangential at perigee; take the perigee radius as $R_\oplus + 200 = 6578.137$ km for all orbits; the five Earth-bound burns are of equal impulsive Δv . (Values rounded from ISRO's releases.)

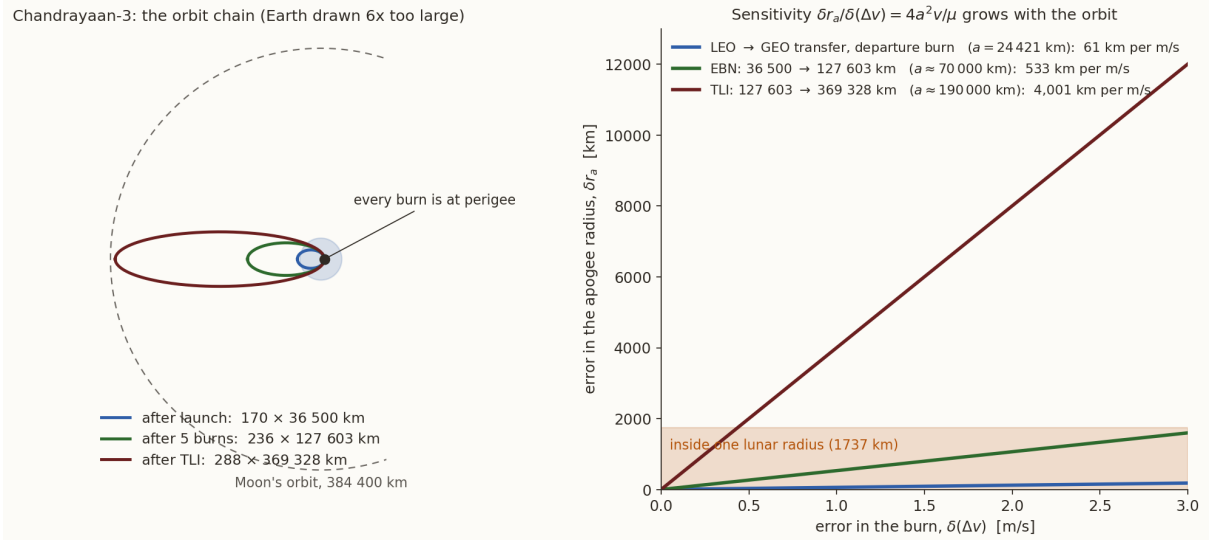


Figure 5: The orbit chain to scale, and how far a 1 m/s burn error moves the apogee for burns at three different semi-major axes. [Script](#).

(a) Find the impulsive Δv for $A \rightarrow B$ and $B \rightarrow C$. Show that doing $A \rightarrow B$ in five burns costs the same as in one. What Δv would a direct 200 km circular $LEO \rightarrow C$ need, and why is each m/s so much more productive at A?

Answer. Perigee speed, $v_p = \sqrt{\mu(2/r_p - 1/a)}$ with $r_p = 6578.137$ km:

orbit	a [km]	e	period	v_p [km/s]
A	24 728	0.7340	10.8 h	10.2504
B	70 280	0.9064	51.5 h	10.7479
C	191 142	0.9656	231 h	10.9135

$$\Delta v_{AB} = 10.7479 - 10.2504 = \boxed{497.5 \text{ m/s}},$$

$$\Delta v_{BC} = 10.9135 - 10.7479 = \boxed{165.6 \text{ m/s}} \text{ (TLI),}$$

$$\Delta v_{AC} = 663.1 \text{ m/s.}$$

Every burn is at the same point and along $\mathbf{v}$, so the Δv 's are differences of the same speed and **telescope:** $\sum \Delta v_i = v_p(B) - v_p(A)$ for any number of burns. Impulsively, five burns cost exactly as much as one. (Q2 and (b) show why they are still better.)

From a 200 km circular orbit ($v_c = 7.7843$ km/s): $\Delta v = 10.9135 - 7.7843 = 3129$ m/s. The launcher has already supplied most of this: at A the craft is moving at 10.25 km/s and each m/s adds energy at the rate v_p ($\Delta \varepsilon = v_p \Delta v$), 32% more than at LEO speed. The remaining 0.66 km/s is cheap.

(b) Find the duration and arc of each of the five equal burns ($\Delta v = 99.5$ m/s each), using $t_b = \frac{m_0 v_e}{F} (1 - e^{-\Delta v/v_e})$ and Kepler's equation. Use Q2 to find the total extra Δv . Compare with doing A $\rightarrow$ B in **one** burn.

Answer. $v_e = 310 \times 9.80665 = 3040.1$ m/s. For burn 1, $m_0 = 3900$ kg: $m_0 v_e / F = 26947$ s and $1 - e^{-0.03273} = 0.03220$, so $t_b = 868$ s. For the arc: $T_A = 38700$ s, $n = 1.6236 \times 10^{-4}$ rad/s, $M = n(t_b/2) = 0.0705$; $E - 0.734 \sin E = 0.0705 \Rightarrow E = 0.2571$; $\tan \frac{f}{2} = \sqrt{1.734/0.266} \tan \frac{E}{2} \Rightarrow f = 36.5^\circ$. Q2(b) then gives $\eta = 1 - 0.734(0.637)^2/(6 \times 1.734^2) = 0.9835$.

burn	apogee before [km]	t_b	θ	η	loss [m/s]
1	36 500	868 s	36.5°	0.9835	1.6
2	43 706	840 s	35.7°	0.9841	1.6
3	53 479	813 s	35.0°	0.9847	1.5
4	67 487	787 s	34.2°	0.9853	1.5
5	89 239	761 s	33.5°	0.9859	1.4

The total extra Δv is ≈ 7.6 m/s (1.5%). (Direct integration of the thrust gives 8.9 m/s, and also shows the perigee creeping from 200 to 224 km: the finite burn lifts the perigee, as the real ones did.)

One burn, A $\rightarrow$ B: $\Delta v_{AB} = 497.5$ m/s needs $t_b = 4068$ s (68 min) and spans $\pm 103^\circ$. The small-arc formula is no longer good there; integrating the orbit with thrust gives 86 min and $\boxed{645 \text{ m/s}}$ to reach the same energy: 147 m/s (30%) **more**, with the perigee thrown up to 694 km.

(c) Find the TLI burn duration (mass after the five burns: $3900 e^{-497.5/3040} = 3311$ kg), its arc, and its extra Δv .

Answer. $t_b = \frac{3311(3040.1)}{440} (1 - e^{-165.6/3040.1}) = 22877 \times 0.05302 = 1213$ s (20.2 min). In orbit B: $T = 185400$ s, $n = 3.388 \times 10^{-5}$ rad/s, $M = n(606.5) = 0.02055$, $E = 0.2053$, $f = 49.8^\circ$. $\eta = 1 - 0.9064(0.869)^2/(6 \times 1.9064^2) = 0.9685$, so the loss is $3.2\% \times 165.6 = \boxed{5.2 \text{ m/s}}$ (integration: 6.4 m/s, 3.9%).

(d) How far does a 1 m/s error in the TLI burn move the apogee? Compare with the LEO $\rightarrow$ GEO departure burn of Q1. How accurate must the TLI be if the apogee must be within 5000 km?

Answer. By Q1(a), $\delta r_a = \frac{4a^2 v}{\mu} \delta(\Delta v)$ for a burn at periapsis. With the TLI orbit ($a = 191142$ km, $v = 10.9135$ km/s):

$$\frac{\delta r_a}{\delta(\Delta v)} = \frac{4(191142)^2(10.9135)}{398600} = 4.00 \times 10^6 \frac{\text{km}}{\text{km/s}}$$

$$= \boxed{4000 \text{ km per m/s}},$$

against 61 km per m/s for LEO $\rightarrow$ GEO, 533 for burn 5 of A $\rightarrow$ B: the gain grows as $a^2 v$. For 5000 km: $\delta(\Delta v) < 1.25$ m/s, i.e. 0.75% of the 165.6 m/s burn; a 0.3% burn error (0.5 m/s) is already 2000 km, more than a lunar radius. A lunar mission therefore cannot be flown open-loop: the TLI is followed by trajectory-correction burns, and the *tiny* Δv of TLI (165 m/s) is deceptive about how accurately it must be delivered.

Q6 — Mangalyaan: a 22-minute burn

Mangalyaan's trans-Mars injection (1 December 2013) was flown from an orbit of 252×192874 km. The 440 N engine burned for 1328.89 s and the reported Δv was 648 m/s (rounded ISRO figures; take $I_{sp} = 310$ s and constant thrust acceleration). The burn was centred in time on periapsis.

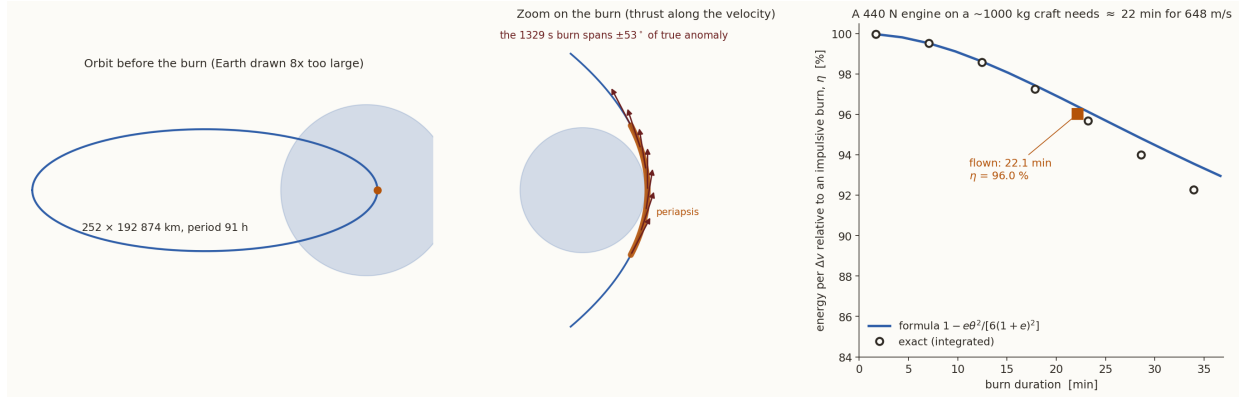


Figure 6: The orbit, the burn arc, and the efficiency of a burn of any duration in this orbit. [Script](#).

(a) Find e , a , the period, v_p , and the true-anomaly arc spanned by the burn.

Answer. $r_p = 6630.137$ km, $r_a = 199\,252.137$ km, so $a = 102\,941$ km and $e = 0.93559$. The period is $T = 2\pi\sqrt{a^3/\mu} = 328\,700$ s (91.3 h), $v_p = 10.7873$ km/s, $v_{\text{esc}} = \sqrt{2\mu/r_p} = 10.9654$ km/s and $\varepsilon_0 = -\mu/2a = -1.9361$ km²/s².

Arc: half the burn is 664.4 s; $n = 1.9116 \times 10^{-5}$ rad/s, so $M = 0.01270$. Solving $E - 0.93559 \sin E = M$ gives $E = 0.1825$, and $\tan \frac{f}{2} = \sqrt{1.93559/0.06441} \tan \frac{E}{2} = 0.5016 \Rightarrow \boxed{\theta = 53.3^\circ}$.

(b) Use Q2 to find η and the energy delivered. Hence find the v_∞ and C_3 the burn gave, and the impulsive Δv that would have done the same.

Answer. $\eta = 1 - \frac{e\theta^2}{6(1+e)^2} = 1 - \frac{0.93559(0.9301)^2}{6(1.93559)^2} = \boxed{0.964}$ (integrating the orbit: 0.960). From Q2(a), $\Delta\varepsilon = \eta v_p \Delta v + \frac{1}{2} \Delta v^2$:

$$\Delta\varepsilon = 0.964(10.7873)(0.648) + \frac{1}{2}(0.648)^2 = 6.739 + 0.210 = 6.949 \text{ km}^2/\text{s}^2,$$

$$\varepsilon_f = -1.936 + 6.949 = 5.013 \Rightarrow$$

$$\boxed{v_\infty = \sqrt{2\varepsilon_f} = 3.17 \text{ km/s}, \quad C_3 = 10.0 \text{ km}^2/\text{s}^2}$$

(Numerical integration of the burn: $v_\infty = 3.14$ km/s, $C_3 = 9.9$.) An impulsive burn at periapsis giving the same v_∞ needs $v = \sqrt{v_{\text{esc}}^2 + v_\infty^2} = 11.413$ km/s, i.e. $\Delta v_{\text{imp}} = \boxed{626 \text{ m/s}}$. The finite burn “wastes” about 22 m/s (3.4%; 28 m/s by integration). The $C_3 \approx 10$ is above the Hohmann minimum ($v_\infty = 2.94$ km/s, $C_3 = 8.6$), as it must be for a real launch window.

(c) What was the spacecraft mass at the start of the burn, and the propellant used?

Answer. $\dot{m} = F/(I_{sp}g_0) = 440/3040.1 = 0.14473$ kg/s, so $m_{\text{prop}} = 0.14473 \times 1328.89 = \boxed{192 \text{ kg}}$, and from $\Delta v = v_e \ln(m_0/m_1)$:

$$m_0 = \frac{m_{\text{prop}}}{1 - e^{-648/3040.1}} = \frac{192.3}{0.19196} = \boxed{1002 \text{ kg}}, \quad m_1 = 810 \text{ kg}.$$

(d) The engine cuts off 1 s late. By how much does v_∞ change, and what miss does it cause after a 300-day cruise?

Answer. A second more of thrust is $\delta(\Delta v) = a_T \cdot 1 \text{ s} = 648/1329 = 0.49 \text{ m/s}$. Near cutoff the craft is at $v \approx 11.4 \text{ km/s}$ and $d\varepsilon = v d(\Delta v)$, so $v_\infty dv_\infty = v d(\Delta v)$:

$$\frac{dv_\infty}{d(\Delta v)} = \frac{11.4}{3.17} = 3.6 \Rightarrow \delta v_\infty = 1.8 \text{ m/s}.$$

Cruising for $300 \text{ d} = 2.6 \times 10^7 \text{ s}$, a velocity error of 1.8 m/s becomes $\approx 4.7 \times 10^4 \text{ km}$ if the path were straight. In the Sun's field the error is amplified rather than just carried: $4a^2v/\mu_\odot \approx 3.5 \times 10^4 \text{ km}$ per m/s for a Mars Hohmann arc. Either way the miss is 10^4 to 10^5 km , against the few hundred km needed to hit the Mars capture window.

(e) A correction made t days after TMI removes a miss d for $\Delta v \approx d/(T_{\text{flight}} - t)$. Evaluate for $d = 4.7 \times 10^4 \text{ km}$, $T = 300 \text{ d}$, at $t = 10, 100, 250 \text{ d}$. When would you do TCM-1?

Answer.

$$\Delta v = \frac{4.7 \times 10^7 \text{ m}}{(300 - t) \times 86400 \text{ s}} :$$

$$t = 10 \text{ d} : 1.9 \text{ m/s}, \quad 100 \text{ d} : 2.7 \text{ m/s}, \quad 250 \text{ d} : \boxed{10.9 \text{ m/s}}.$$

The same miss costs 6 times more at 250 d than at 10 d: **correct early**. But the orbit must first be *determined* well enough to know the error (a few days of tracking), so TCM-1 is flown ~ 10 days after injection, with later TCMs trimming what navigation then refines. Compare the 1.9 m/s with the 0.49 m/s one second of cutoff error: the correction is about the size of the error it removes.

Q7 — ISS rendezvous: three hours or two days?

A visiting vehicle trails the ISS (circular, altitude 400 km , $r = 6778.137 \text{ km}$) by $\Delta\theta = 20^\circ$ and closes the gap in k revolutions with the two-burn phasing of Q3. The Soyuz/Progress “fast” profile takes about two orbits (about 3 h); the standard profile about 34 orbits (about 2 days). The phasing perigee must stay above 200 km . Vehicle: 7000 kg , $I_{sp} = 300 \text{ s}$.

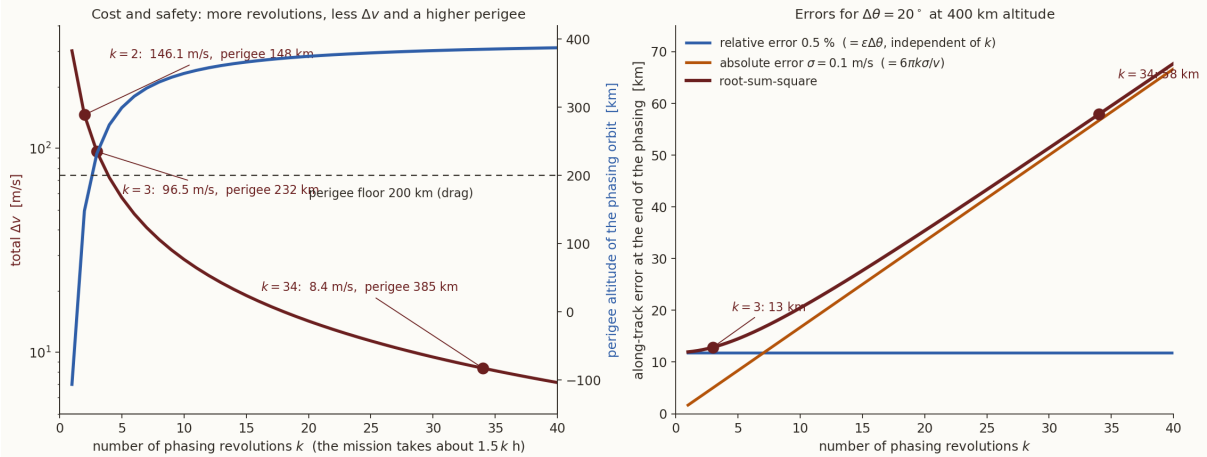


Figure 7: Cost and perigee against k , and the along-track error for a 0.5% relative and 0.1 m/s absolute burn error. [Script](#).

(a) For $k = 2, 3$ and 34 find T_{ph} , a_{ph} , the perigee altitude, Δv_{total} and the mission duration. Which k are admissible?

Answer. $T = 2\pi\sqrt{r^3/\mu} = 5553.6 \text{ s}$, $v = 7.6686 \text{ km/s}$. $T_{ph} = T(1 - \Delta\theta/2\pi k)$, $a_{ph} = (\mu(T_{ph}/2\pi)^2)^{1/3}$, $r_p = 2a_{ph} - r$, $\Delta v_{\text{total}} = 2(v - v_{ph})$ with $v_{ph} = \sqrt{\mu(2/r - 1/a_{ph})}$:

k	T_{ph} [s]	a_{ph} [km]	perigee alt.	Δv_{total}	mission
2	5399.4	6652.0	148 km	146.1 m/s	3.0 h
3	5450.8	6694.2	232 km	96.5 m/s	4.5 h
34	5544.5	6770.8	385 km	8.4 m/s	52.4 h

$k = 2$ puts the perigee at 148 km, inside the drag layer: **not admissible**. $k_{min} = 3$ (Q3(d)). The fast profile is therefore not a pure phasing manoeuvre; real fast rendezvous gets its phasing from launch timing and a lower insertion orbit, and the burns above are the *equivalent* cost.

(b) Propellant for $k = 3$ and $k = 34$. What does the fast profile cost?

Answer. $m_{prop} = m_0 (1 - e^{-\Delta v / (I_{sp} g_0)})$ with $I_{sp} g_0 = 2941.995$ m/s:

$$k = 3 : 7000 (1 - e^{-96.46/2942}) = \boxed{226 \text{ kg}}, \quad k = 34 : 7000 (1 - e^{-8.37/2942}) = \boxed{20 \text{ kg}}.$$

About 206 kg of propellant buys 48 hours. On a cargo vehicle 206 kg is mass that cannot be cargo.

(c) The phasing burn has a relative error 0.5% and an absolute error $\sigma = 0.1$ m/s (1σ). Find the along-track error at the end for $k = 3$ and $k = 34$.

Answer. By Q3(c): relative part $\varepsilon \Delta \theta = 0.005 \times 20^\circ = 0.100^\circ$ (11.8 km), the same for both. Absolute part $6\pi k \sigma / v$:

$$k = 3 : 6\pi(3)(0.1)/7668.6 = 7.4 \times 10^{-4} \text{ rad} = 0.042^\circ \text{ (5.0 km),}$$

$$\text{RSS} = 0.109^\circ = \boxed{12.8 \text{ km}}$$

$$k = 34 : 6\pi(34)(0.1)/7668.6 = 8.4 \times 10^{-3} \text{ rad} = 0.479^\circ \text{ (56.6 km),}$$

$$\text{RSS} = 0.489^\circ = \boxed{57.9 \text{ km}}$$

Here $k^* = 7.1$: the two-day profile is dominated by the *absolute* error. Tens of km are still inside a radar acquisition range, but this is why the long profile carries scheduled correction burns, and why it trades propellant for operations and time.

Q8 — GTO → GEO: Kourou, Sriharikota or the Cape?

A 3000 kg satellite ($I_{sp} = 315$ s, $F = 440$ N) is in a GTO of $185 \times 35\,786$ km. At apogee it makes **one** burn that circularises at GEO and removes the inclination. Compare the three launch sites (rounded GTO inclinations): Kourou 7° , Sriharikota 19.2° , Cape Canaveral 28.5° .

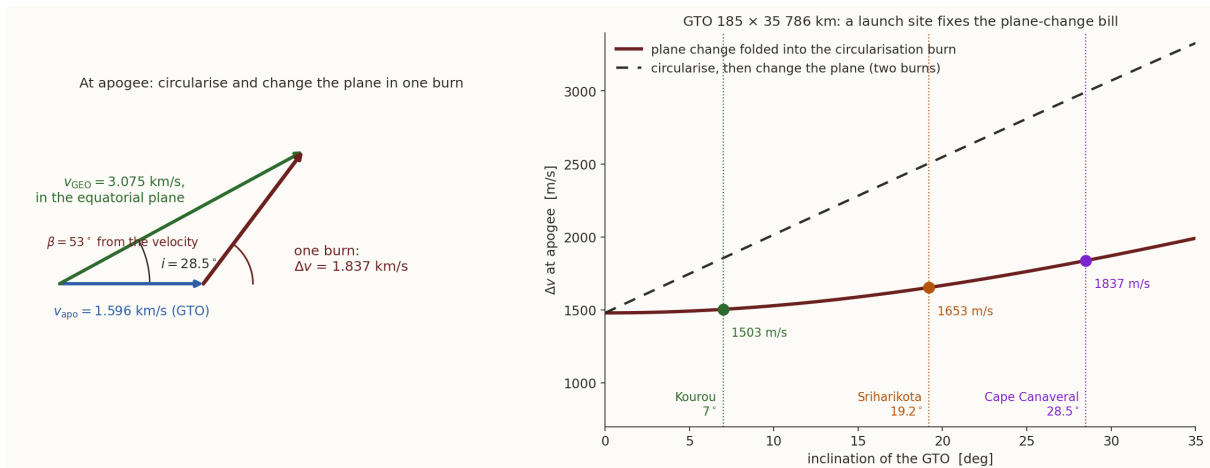


Figure 8: The velocity triangle at apogee for 28.5° , and the cost against inclination. [Script](#).

(a) Find Δv , the thrust direction β from the velocity, and the propellant mass for each site. How much does folding the plane change into the circularisation save?

Answer. GTO: $a = 24\,363.6$ km, $v_{\text{apo}} = 1.5958$ km/s; GEO: $v_{\text{GEO}} = 3.0747$ km/s. The law of cosines and the triangle of the figure:

$$\Delta v = \sqrt{v_{\text{apo}}^2 + v_{\text{GEO}}^2 - 2v_{\text{apo}}v_{\text{GEO}}\cos i}, \quad \tan \beta = \frac{v_{\text{GEO}} \sin i}{v_{\text{GEO}} \cos i - v_{\text{apo}}},$$

$$m_p = m_0 (1 - e^{-\Delta v/(I_{sp}g_0)})$$

site	i	Δv [m/s]	β	m_p	two burns [m/s]
Kourou	7°	1503	14°	1156 kg (38.5%)	1854
Sriharikota	19.2°	1653	38°	1243 kg (41.4%)	2504
Cape Canaveral	28.5°	1837	53°	1345 kg (44.8%)	2993

The “two burns” column is $(v_{\text{GEO}} - v_{\text{apo}}) + 2v_{\text{GEO}} \sin(i/2)$: 1479 m/s to circularise plus the plane change at GEO speed. Combining saves 351, 851 and 1156 m/s (19, 34, 39%): the plane change is done at 1.6 km/s, not 3.1. The Cape-to-Kourou difference (334 m/s) is 189 kg of propellant: about 6% of the satellite.

(b) The burn lasts $t_b = \frac{m_0 v_e}{F} (1 - e^{-\Delta v/v_e})$. For the three sites find t_b and the arc at apogee ($\dot{f}_a = v_{\text{apo}}/r_a$), and the loss $\frac{\theta^2}{6} \Delta v$ (θ = half the arc) if the thrust direction is held fixed in inertial space, so that the misalignment runs from $-\theta$ to $+\theta$, in **one** burn against **three** equal burns.

Answer. $v_e = 3089.1$ m/s, $\dot{f}_a = 1.5958/42164 = 3.785 \times 10^{-5}$ rad/s.

site	t_b	arc $\dot{f}_a t_b$	loss, 1 burn	loss, 3 burns
Kourou	8116 s (2.25 h)	17.6°	5.9 m/s (0.39%)	0.7 m/s
Sriharikota	8728 s (2.42 h)	18.9°	7.5 m/s (0.45%)	0.8 m/s
Cape	9443 s (2.62 h)	20.5°	9.8 m/s (0.53%)	1.1 m/s

The misalignment is spread uniformly over $\pm\theta$ (θ = half the arc), so the mean loss per unit Δv is $\langle 1 - \cos \delta \rangle \approx \langle \delta^2 \rangle / 2 = \theta^2 / 6$, and it is 1/9 of that when each burn is 1/3 as long: the same $1/n^2$ as in Q2. A 2.5-hour burn spreads over an arc of $\sim 20^\circ$.

(c) A yaw error of 1° in the thrust direction: how much inclination is left over, and what does it cost to remove at GEO?

Answer. The final inclination follows $\tan \Delta i = \frac{\Delta v \sin \beta}{v_{\text{apo}} + \Delta v \cos \beta}$. Differentiating, $\frac{d \tan \Delta i}{d \beta} = \frac{\Delta v (v_{\text{apo}} \cos \beta + \Delta v)}{(v_{\text{apo}} + \Delta v \cos \beta)^2}$, then $d(\Delta i)/d\beta = \frac{1}{1 + \tan^2 \Delta i} \frac{d \tan \Delta i}{d \beta}$. For the Cape ($\Delta v = 1.837$, $\beta = 53^\circ$, $\Delta i = 28.5^\circ$): 0.544. So 1° of yaw leaves 0.54° of inclination at GEO. Removing it later costs

$$2v_{\text{GEO}} \sin \frac{0.544^\circ}{2} = \boxed{29 \text{ m/s}} \text{ (Kourou: 26, Sriharikota: 27),}$$

more than half a year of north-south station-keeping (≈ 50 m/s/yr, Q10). A yaw error is paid for in propellant that was supposed to last for years.

(d) The burn has a 1% (1σ) magnitude error. Compare the final orbit after a **single** Cape burn with the last of **three** equal burns (whose error the earlier ones, tracked from the ground, have

not left to correct). Use $\delta a = (2a^2 v / \mu) \delta(\Delta v)$ at GEO and $\delta T / T = 1.5 \delta a / a$.

Answer. At GEO radius $2a^2 v / \mu = 27.4$ km per m/s.

	1% error	δa	$\delta T / T$	drift
single burn, 1837 m/s	18.4 m/s	504 km	1.8%	6.4°/day
last of 3, 612 m/s	6.1 m/s	168 km	0.60%	2.2°/day

The drift is $360^\circ \times \delta T / T$ per day. A drift of 6.4°/day sweeps through the whole GEO belt in two months; the cost of stopping it ($\approx v \delta T / (3T)$, here 18 m/s) would be over a third of a year's north-south budget. Splitting the burn, with an orbit determination between burns to size the last one, cuts the final error by n ; a short trim burn at the end cuts it by a further factor ~ 10 .

Q9 — Bi-elliptic to lunar distance

A relay spacecraft is to be placed into a **circular orbit at the Moon's distance** ($r_2 = 384\,400$ km) from a 300 km LEO ($r_1 = 6678.137$ km), so $R = 57.56$. Compare the Hohmann transfer with a bi-elliptic transfer whose intermediate apoapsis is $r_b = 10^6$ km.

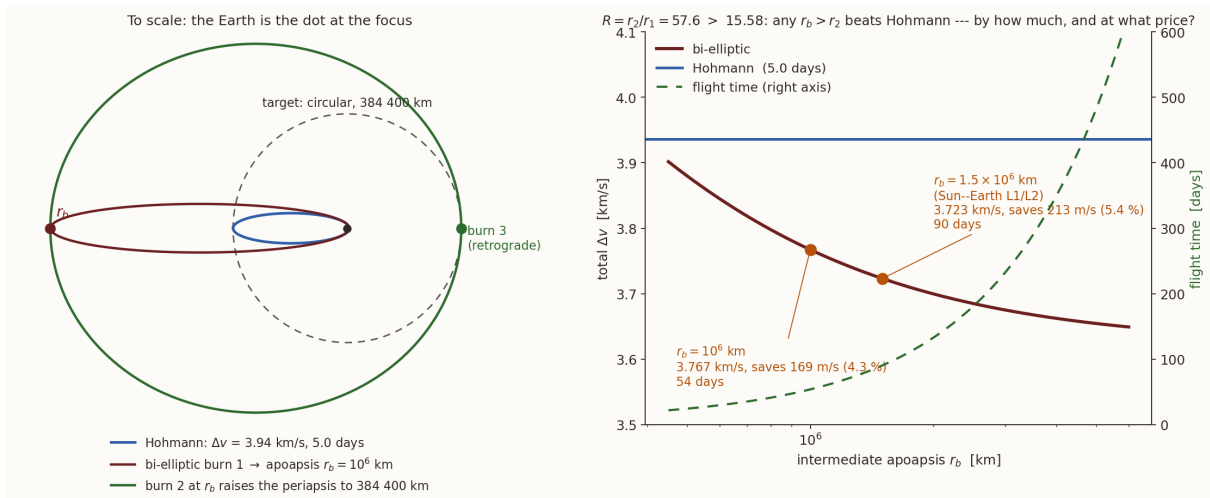


Figure 9: The orbits to scale (the Earth is the dot at the focus), and the total Δv and flight time against r_b . [Script](#).

(a) Hohmann: Δv_1 , Δv_2 , total and flight time. (b) Bi-elliptic: the three burns, total, saving, flight time.

Answer. (a) $a_t = 195\,539$ km: $v_{t1} = 10.8322$, $v_{c1} = 7.7258$, $v_{t2} = 0.18819$, $v_{c2} = 1.01830$ km/s,

$$\Delta v_1 = 3.1064, \quad \Delta v_2 = 0.8301, \quad \boxed{\Delta v_H = 3.9365 \text{ km/s}}, \quad t = \pi \sqrt{a_t^3 / \mu} = \boxed{4.98 \text{ d}}.$$

(b) $a_{t1} = 503\,339$ km, $a_{t2} = 692\,200$ km.

burn	where	from $\rightarrow$ to [km/s]	Δv
1	r_1	7.7258 $\rightarrow$ 10.8896	3.1638
2	r_b	0.0727 $\rightarrow$ 0.4705	0.3978
3	r_2	1.2239 $\rightarrow$ 1.0183 (retrograde)	0.2056

$$\Delta v_{bi} = \boxed{3.7672 \text{ km/s}}, \quad \text{saving } \boxed{169 \text{ m/s (4.3\%)}, \quad t = 20.6 + 33.2 = \boxed{53.7 \text{ d}}.$$

At r_b the craft moves at only 73 m/s, so 0.40 km/s lifts the periapsis by 378 000 km. For $r_b = 1.5 \times 10^6$ km the saving is 213 m/s (5.4%) and the time 90 d.

(c) $R = 57.6 > 15.58$. What does the deck's theorem say? Why does the bi-elliptic transfer *keep* saving more as r_b grows, and where does it stop being worth it?

Answer. For $R > 15.58$ every $r_b > r_2$ beats Hohmann, by a margin that grows with r_b : the saving is 169 m/s at 10^6 km, 213 at 1.5×10^6 , 262 (6.7%) at 3×10^6 km (the figure). The saving grows because the far burn is made where the speed is smallest ($d\varepsilon = v\Delta v$, small v) and *changes the periapsis*, which costs almost nothing. It stops being worth it because the flight time grows as $r_b^{3/2}$ ($5 \rightarrow 54 \rightarrow 233$ d), and because the two-body model fails at large r_b (part (e)).

(d) Compute the error gain $\delta r/\delta(\Delta v) = 4a^2v/\mu$ of Hohmann burn 1 (on r_a), bi-elliptic burn 1 (on r_b) and burn 2 (on the periapsis). Is the first bi-elliptic burn's huge gain a problem?

Answer.

$$\text{Hohmann burn 1: } \frac{4(195\,539)^2(10.832)}{398600} = 4160 \frac{\text{km}}{\text{m/s}},$$

$$\text{bi-elliptic burn 1: } \frac{4(503\,339)^2(10.890)}{398600} = \boxed{27\,700} \frac{\text{km}}{\text{m/s}},$$

$$\text{bi-elliptic burn 2: } \frac{4(692\,200)^2(0.4705)}{398600} = 2260 \frac{\text{km}}{\text{m/s}}.$$

A 1 m/s error in burn 1 moves r_b by 27 700 km (2.8%) but that is **harmless**: the total cost is flat in r_b ($d\Delta v/dr_b = -12.7$ m/s per 10^5 km), so a 2.8% error in r_b changes the total by only ≈ 3.5 m/s, and burn 2 is simply *computed from the orbit actually achieved* (it is made at the measured apoapsis). The burn that *must* be right is burn 2 (the periapsis it sets, 2260 km per m/s) and, for the Hohmann route, burn 1 (which sets the arrival radius at 4160 km per m/s). The bi-elliptic route trades a precision requirement at the start for a real-time design at the apoapsis.

(e) The solar tide on the spacecraft at distance r from the Earth is about $2\mu_\odot r/d^3$ ($d = 1$ AU). Compare it with the Earth's pull at $r_b = 10^6$ and 1.5×10^6 km. Is the two-body analysis valid? Is the bi-elliptic transfer worth flying?

Answer.

$$r_b = 10^6 \text{ km : } \frac{2\mu_\odot r}{d^3} = 7.9 \times 10^{-5} \text{ m/s}^2, \quad \frac{\mu_\oplus}{r^2} = 3.99 \times 10^{-4}$$

$$\text{ratio } \boxed{0.20},$$

$$r_b = 1.5 \times 10^6 \text{ km : } 1.19 \times 10^{-4} \text{ vs } 1.77 \times 10^{-4} \text{ m/s}^2 : \text{ ratio } \boxed{0.67}.$$

The Earth's Hill radius is 1.50×10^6 km, so $r_b = 1.5 \times 10^6$ km (the Sun–Earth L1/L2 distance, where JWST and Aditya-L1 operate) is *at the edge of the sphere where the Earth controls the motion*. At 10^6 km the Sun's tide is 20% of the Earth's pull, and it acts for weeks of the climb and fall: the two-body Δv is an estimate, not a guarantee. Verdict: a 169 m/s saving (4.3%) for 49 extra days, three burns and a poorly predictable apoapsis. Not worth it for this target. The theorem is right; its assumptions run out. A real design of this kind would use the Sun's perturbation deliberately (weak-stability transfers, Part VII of the deck).

Q10 — The Δv budget of a GEO satellite, and its uncertainty

A 3000 kg GEO satellite is to operate for 15 years. From the deck: north–south station-keeping 50 m/s/yr, east–west 3 m/s/yr, graveyard disposal 11 m/s. Chemical propulsion has $I_{sp} = 320$ s; electric propulsion 1800 s.

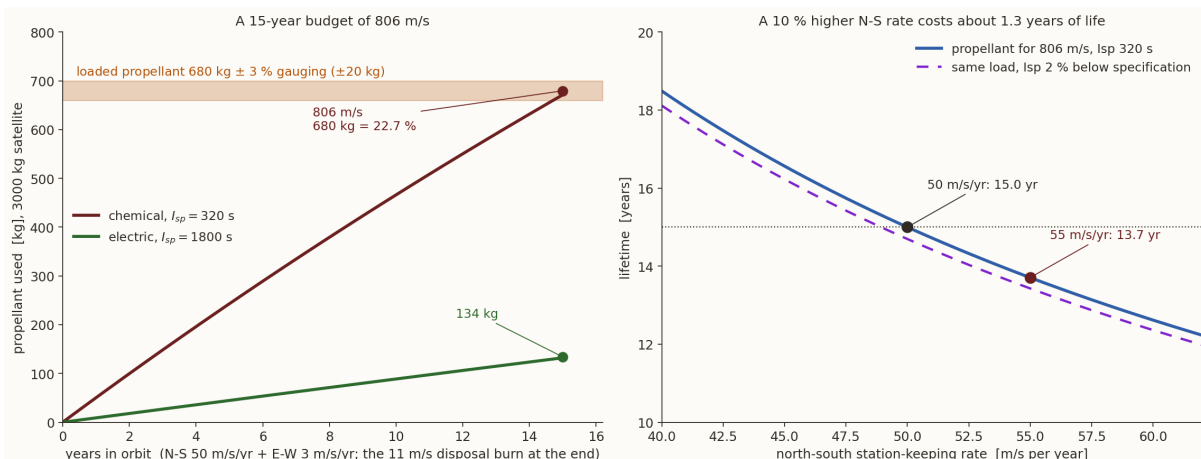


Figure 10: Propellant used against time for chemical and electric propulsion, and the lifetime against the north-south rate. [Script](#).

(a) Total Δv , and the propellant mass for chemical and for electric propulsion. How much heavier is the chemical satellite at launch?

Answer.

$$\Delta v = 15(50 + 3) + 11 = \boxed{806 \text{ m/s}} \quad (750 + 45 + 11).$$

$v_e = 320 \times 9.80665 = 3138.1 \text{ m/s}$; $m_p = 3000(1 - e^{-806/3138.1}) = \boxed{680 \text{ kg}}$ (22.7%). Electric ($v_e = 17652 \text{ m/s}$): $3000(1 - e^{-806/17652}) = \boxed{134 \text{ kg}}$ (4.5%). The chemical satellite carries 546 kg more propellant, 18% of the launch mass, which electric propulsion converts to payload or launch cost.

(b) The propellant gauge is good to $\pm 3\%$ of the loaded mass. Find the uncertainty in the remaining Δv at the end of life, and in years.

Answer. End-of-life mass $3000 - 680 = 2320 \text{ kg}$. A 3% error on 680 kg is $\pm 20.4 \text{ kg}$, so

$$\delta(\Delta v) = v_e \frac{\delta m}{m_f} = 3138 \times \frac{20.4}{2320} = \boxed{\pm 27.7 \text{ m/s}} = \pm 0.52 \text{ yr}.$$

The operator cannot know the end of life to better than ± 6 months and must reserve the 11 m/s disposal burn *with margin*: a spacecraft that runs out one month early cannot be moved out of the GEO belt.

(c) The north-south rate is not known in advance: 50 m/s/yr is the *mean*, with $\sigma = 3 \text{ m/s/yr}$. (i) With the propellant of (a), find the lifetime if the rate turns out to be 55 m/s/yr. (ii) What propellant guarantees 15 years at 3σ ?

Answer.

(i) $\text{life} = \frac{806 - 11}{55 + 3} = \boxed{13.7 \text{ yr}}$: a 10% higher rate costs 1.3 years (9% of the life). (At 45: 16.6 yr.)

(ii) 3σ : $50 + 9 = 59 \text{ m/s/yr}$, so $\Delta v = 15(59 + 3) + 11 = 941 \text{ m/s}$ and $m_p = 3000(1 - e^{-941/3138.1}) = \boxed{777 \text{ kg}}$: 98 kg (14%) more than the nominal 680. The margin costs a lot: that is the price of an uncertain Δv .

(d) The delivered I_{sp} is 2% below specification. What is the effect on the propellant for 806 m/s, or on the life with the propellant of (a)?

Answer. With $v_e \rightarrow 0.98 \times 3138.1 = 3075.3 \text{ m/s}$: $m_p = 3000(1 - e^{-806/3075.3}) = 692 \text{ kg}$: 12 kg more for the same Δv . Or, with 680 kg loaded, the available Δv is $3075.3 \ln(3000/2320) = 790 \text{ m/s}$ and the life $(790 - 11)/53 = \boxed{14.7 \text{ yr}}$: three months lost. (Compare: a 10% N-S excess costs 1.3 years; an I_{sp} shortfall of 2% only 0.3.)

Sources and caveats

- **Chandrayaan-3** (Q5): orbits A, B, C are ISRO's reported values for the orbit after launch, after the fifth Earth-bound burn, and after TLI (rounded); the spacecraft mass, thrust and I_{sp} are round values; the five burns are taken as equal (the real ones were not) and the perigee is held at 200 km (the real one rose from 170 to 236 km).
- **Mangalyaan** (Q6): the orbit $252 \times 192\,874$ km, the burn time 1328.89 s and the 648 m/s are ISRO's reported values; I_{sp} and a constant thrust acceleration are assumed. The v_∞ and C_3 that follow are a consistency check, not a quoted mission figure.
- **ISS, GTO and GEO** (Q7, Q8, Q10): representative values: ISS altitude 400 km; GTO inclinations 7° (Kourou), 19.2° (Sriharikota) and 28.5° (Cape); the station-keeping rates are those of the *Orbital Maneuvers* deck.
- **Q9** is a design exercise, not a flown mission: JWST and Aditya-L1 are cited only for the distance of the Sun–Earth L1/L2 points.
- All scripts: `codes/fig-mq-*.py`, run from this directory, write `img/fig-mq-*.png`.