

Orbital Maneuvers — Analytical Problems

Ten problems with no numbers: derive the cost, then prove something about it

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How to use these problems

In the *Orbital Maneuvers* deck every transfer is costed with numbers: one example, one table, one verdict. These problems take the same transfers and ask for the cost **as a formula**, and then for a **property** of that formula that no single example can show: which burn is always larger, which route always wins, at what angle the verdict changes. Question 6 of the [Midsem 2026 paper](#) is the model.

No problem needs a calculator. Where a numerical value is quoted, it is only there so you can check a root or a limit against the deck. Answers are folded under **Answer**. Every identity, inequality and threshold below is checked symbolically and on a grid by [codes/maneuvers-analytic-verify.py](#).

Conventions. All burns are impulsive. The cost of a manoeuvre is the sum of the magnitudes of its impulses. “Circle r ” is the circular orbit of radius r about a centre of gravitational parameter μ . Orbits are coplanar unless a problem says otherwise. For the Hohmann transfer between circles $r_1 < r_2$ we write

symbol	meaning
$R = r_2/r_1$	radius ratio
$v_{c1} = \sqrt{\mu/r_1}, v_{c2} = \sqrt{\mu/r_2}$	circular speeds
v_{t1}, v_{t2}	speeds on the transfer ellipse at r_1 and at r_2
$\Delta v_1 = v_{t1} - v_{c1}, \Delta v_2 = v_{c2} - v_{t2}$	the two Hohmann burns

#	transfer	what is proved
1	Hohmann	the burn on the inner circle is always the larger
2	Hohmann	the total cost peaks at the root of a cubic
3	coaxial ellipses	with aligned apsides one route is always cheaper; with anti-aligned apsides neither is
4	one-burn circularisation	apoapsis is the cheapest place, the latus rectum the dearest
5	apse-line rotation	the crossover angle in closed form; it never exceeds 60°
6	circular phasing	$\Delta v \cdot t \approx \frac{2}{3} r \Delta \theta$; catching up costs more than dropping back
7	plane change via a high apoapsis	three thresholds, $\sin \frac{\theta}{2} = \frac{1}{3}, \sqrt{2} - 1, \frac{1}{2}$
8	plane change folded into a Hohmann transfer	the extra cost is second order; the optimal split
9	node change	$\Delta v = 2v \sin i \sin \frac{\Delta \Omega}{2}$, and where to burn
10	escape with an Oberth dive	the dive pays exactly when v_∞ exceeds the local escape speed

Problems

1. The larger burn

A Hohmann transfer joins coplanar circles of radii $r_1 < r_2$. Show that

$$\Delta v_1 - \Delta v_2 = \sqrt{\mu} \frac{\sqrt{2(r_1 + r_2)} - \sqrt{r_1} - \sqrt{r_2}}{\sqrt{r_1 r_2}},$$

and hence that the burn on the inner circle is the larger of the two for every $R > 1$, whichever way the transfer is flown. Find $\Delta v_1 / \Delta v_2$ in the limits $R \rightarrow 1$ and $R \rightarrow \infty$. Show also that the inner burn supplies the fraction $R/(1 + R)$ of the energy.

Answer. We have to compare two burns. Each burn is the difference between a circular speed and a speed on the transfer ellipse. So the plan is:

1. find the two speeds on the transfer ellipse;
2. write Δv_1 and Δv_2 , and subtract them;
3. decide the sign of the difference.

The speeds on the transfer ellipse. The transfer ellipse has its periapsis on circle r_1 and its apoapsis on circle r_2 , so its semi-major axis is $a_t = (r_1 + r_2)/2$. *Vis-viva* at the two apsides gives

$$v_{t1}^2 = \mu \left(\frac{2}{r_1} - \frac{2}{r_1 + r_2} \right) = \frac{2\mu r_2}{r_1(r_1 + r_2)},$$

$$v_{t2}^2 = \mu \left(\frac{2}{r_2} - \frac{2}{r_1 + r_2} \right) = \frac{2\mu r_1}{r_2(r_1 + r_2)}.$$

As a check, $r_1 v_{t1} = r_2 v_{t2}$, which is conservation of angular momentum.

The two burns. Taking out the circular speed from each,

$$\Delta v_1 = v_{t1} - v_{c1} = \sqrt{\frac{\mu}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right), \quad \Delta v_2 = v_{c2} - v_{t2} = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right).$$

The two burns carry different factors in front, $\sqrt{\mu/r_1}$ and $\sqrt{\mu/r_2}$. To subtract them we want a common factor. The natural one is $\sqrt{\mu/(r_1 r_2)}$, because $\sqrt{\mu/r_1} = \sqrt{\mu/(r_1 r_2)} \sqrt{r_2}$ and $\sqrt{\mu/r_2} = \sqrt{\mu/(r_1 r_2)} \sqrt{r_1}$:

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1 r_2}} \left(\frac{\sqrt{2} r_2}{\sqrt{r_1 + r_2}} - \sqrt{r_2} \right),$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_1 r_2}} \left(\sqrt{r_1} - \frac{\sqrt{2} r_1}{\sqrt{r_1 + r_2}} \right).$$

Now subtract. The two fractions have the same denominator, and they add up to $\sqrt{2}(r_1 + r_2)/\sqrt{r_1 + r_2} = \sqrt{2(r_1 + r_2)}$. So

$$\Delta v_1 - \Delta v_2 = \sqrt{\frac{\mu}{r_1 r_2}} \left(\sqrt{2(r_1 + r_2)} - \sqrt{r_1} - \sqrt{r_2} \right),$$

which is the formula asked for.

The sign. The factor in front is positive, so we need to show that

$$\sqrt{r_1} + \sqrt{r_2} < \sqrt{2(r_1 + r_2)} \quad (r_1 \neq r_2).$$

Both sides are positive, so we may compare their squares instead:

$$r_1 + r_2 + 2\sqrt{r_1 r_2} < 2r_1 + 2r_2 \iff 0 < r_1 + r_2 - 2\sqrt{r_1 r_2} = (\sqrt{r_2} - \sqrt{r_1})^2.$$

This is true whenever $r_1 \neq r_2$. Hence $\Delta v_1 > \Delta v_2$ for every $R > 1$. (Divided by 2, the inequality says that the arithmetic mean of $\sqrt{r_1}$ and $\sqrt{r_2}$ is less than their root-mean-square.)

If the transfer is flown inwards, the spacecraft uses the same ellipse in the opposite sense. Each burn has the same magnitude as before and the opposite direction (deck, *Inbound and outbound costs are equal*). So the burn on the inner circle is again the larger one, although it is now the second burn.

Limits. For limits it is easier to have a single variable. So divide each burn by v_{c1} , and use $r_2 = Rr_1$ and $v_{c2} = v_{c1}/\sqrt{R}$:

$$\frac{\Delta v_1}{v_{c1}} = \sqrt{\frac{2R}{1+R}} - 1, \quad \frac{\Delta v_2}{v_{c1}} = \frac{1}{\sqrt{R}} \left(1 - \sqrt{\frac{2}{1+R}} \right).$$

- *Nearby circles.* Put $R = 1 + \epsilon$ with ϵ small. Then $\frac{2R}{1+R} = \frac{2+2\epsilon}{2+\epsilon} = 1 + \frac{\epsilon}{2+\epsilon} \approx 1 + \frac{\epsilon}{2}$, so its square root is $\approx 1 + \frac{\epsilon}{4}$ and $\Delta v_1 \approx v_{c1} \epsilon/4$. In the same way $\frac{2}{1+R} = \frac{2}{2+\epsilon} \approx 1 - \frac{\epsilon}{2}$, its square root is $\approx 1 - \frac{\epsilon}{4}$, and with $1/\sqrt{R} \approx 1$ we get $\Delta v_2 \approx v_{c1} \epsilon/4$. The ratio tends to 1: between nearby circles the transfer ellipse is nearly a circle, and the two burns are nearly equal.
- *Distant circles.* As $R \rightarrow \infty$, $\sqrt{2R/(1+R)} \rightarrow \sqrt{2}$, so $\Delta v_1 \rightarrow (\sqrt{2} - 1) v_{c1}$. That is the escape burn: to reach a very distant circle the first burn must nearly escape. In Δv_2 the root $\sqrt{2/(1+R)} \rightarrow 0$, so $\Delta v_2 \approx v_{c1}/\sqrt{R} = v_{c2}$. The spacecraft arrives almost at rest, and the second burn supplies nearly all of the circular speed there. So $\Delta v_1/\Delta v_2 \approx (\sqrt{2} - 1)\sqrt{R} \rightarrow \infty$.

Energy. The specific energy of circle r is $-\mu/(2r)$. The transfer ellipse has $-\mu/(2a_t) = -\mu/(r_1 + r_2)$. So the two burns add

$$\Delta \varepsilon_1 = \frac{\mu}{2r_1} - \frac{\mu}{r_1 + r_2} = \frac{\mu(r_2 - r_1)}{2r_1(r_1 + r_2)}, \quad \Delta \varepsilon_2 = \frac{\mu}{r_1 + r_2} - \frac{\mu}{2r_2} = \frac{\mu(r_2 - r_1)}{2r_2(r_1 + r_2)}.$$

Their ratio is $\Delta \varepsilon_1/\Delta \varepsilon_2 = r_2/r_1 = R$. So the inner burn supplies R parts out of $1 + R$, the fraction $R/(1 + R)$.

This gives a second way to see why the inner burn is the larger one. A tangential burn from speed v to speed v' changes the energy by

$$\Delta\varepsilon = \frac{1}{2}(v'^2 - v^2) = \Delta v \cdot \frac{v + v'}{2},$$

the size of the burn times the mean of the speeds before and after. So a burn made at high speed buys more energy per unit Δv (the Oberth effect). Two effects now compete. The inner burn must supply R times as much energy, but it is made at a higher speed. Dividing the two energies,

$$\frac{\Delta v_1}{\Delta v_2} = \frac{\Delta\varepsilon_1}{\Delta\varepsilon_2} \cdot \frac{v_{t2} + v_{c2}}{v_{c1} + v_{t1}} = R \frac{v_{t2} + v_{c2}}{v_{c1} + v_{t1}}.$$

Angular momentum gives $v_{t1} = Rv_{t2}$, and the circular speeds give $v_{c1} = \sqrt{R}v_{c2}$. Hence $v_{c1} + v_{t1} = \sqrt{R}v_{c2} + Rv_{t2} < R(v_{c2} + v_{t2})$ for $R > 1$, and the ratio is greater than 1. The speeds at the inner circle are higher, but not by the full factor R , because the circular speed grows only like $\sqrt{R}$. So the energy share wins.

2. The most expensive Hohmann transfer

Write the total cost of Problem 1 as $\Delta v_1 + \Delta v_2 = v_{c1} g(R)$.

- (a) Show that $2R^{3/2}(1+R)^{3/2}g'(R) = \sqrt{2}(1+3R) - (1+R)^{3/2}$. Hence show that the cost is greatest at the only root above 1 of

$$R^3 - 15R^2 - 9R - 1 = 0.$$

- (b) Show that the arrival burn alone is greatest at the root of $R^3 - 5R^2 - 5R - 1 = 0$, and that the departure burn increases with R for all R .
- (c) Explain in one sentence why a more distant circle can be cheaper to reach.

Answer. (a) We want the R at which the total cost is greatest. Once we divide by v_{c1} , the cost depends on R alone, so this is an ordinary one-variable maximum. We differentiate g , find where g' changes sign, and check that the change is from $+$ to $-$. The derivative will contain powers of $\sqrt{R}$ and $\sqrt{1+R}$. The factor $2R^{3/2}(1+R)^{3/2}$ in the question is there to clear them.

From the limits part of Problem 1, with each term written as a product of powers (the easiest form to differentiate),

$$g(R) = \underbrace{\sqrt{2}R^{1/2}(1+R)^{-1/2} - 1}_{\text{departure}} + \underbrace{R^{-1/2} - \sqrt{2}R^{-1/2}(1+R)^{-1/2}}_{\text{arrival}}.$$

Differentiate term by term with the product rule:

$$\begin{aligned} \frac{d}{dR} \left[\sqrt{2}R^{1/2}(1+R)^{-1/2} \right] &= \frac{\sqrt{2}}{2} R^{-1/2}(1+R)^{-3/2} [(1+R) - R] = \frac{1}{\sqrt{2}} R^{-1/2}(1+R)^{-3/2}, \\ \frac{d}{dR} R^{-1/2} &= -\frac{1}{2} R^{-3/2}, \\ \frac{d}{dR} \left[-\sqrt{2}R^{-1/2}(1+R)^{-1/2} \right] &= \frac{\sqrt{2}}{2} R^{-3/2}(1+R)^{-3/2} [(1+R) + R]. \end{aligned}$$

The lowest powers that appear are $R^{-3/2}$ and $(1+R)^{-3/2}$. Multiplying by the positive factor $2R^{3/2}(1+R)^{3/2}$ clears them, and does not change the sign of g' . The three terms become $\sqrt{2}R$, $-(1+R)^{3/2}$ and $\sqrt{2}(1+2R)$, so

$$2R^{3/2}(1+R)^{3/2}g'(R) = \sqrt{2}(1+3R) - (1+R)^{3/2}.$$

When is this positive? Both $\sqrt{2}(1+3R)$ and $(1+R)^{3/2}$ are positive, so we may compare their squares:

$$g' > 0 \iff 2(1+3R)^2 > (1+R)^3 \iff 2+12R+18R^2 > 1+3R+3R^2+R^3 \iff P(R) \equiv R^3-15R^2-9R-1 < 0.$$

So g' has the opposite sign to the cubic P . We cannot solve $P = 0$ by hand, but we do not need to. We only need to know that P changes sign exactly once for $R > 1$, from $-$ to $+$. Divide by R^3 :

$$\frac{P(R)}{R^3} = 1 - \frac{15}{R} - \frac{9}{R^2} - \frac{1}{R^3}.$$

Each of the three subtracted terms decreases as R grows, so P/R^3 increases strictly with R , and it can vanish at most once. At $R = 1$ we have $P = 1 - 15 - 9 - 1 = -24 < 0$, and $P \rightarrow +\infty$ as $R \rightarrow \infty$, so it does vanish, at exactly one $R^\ddagger > 1$. Hence $g' > 0$ on $(1, R^\ddagger)$ and $g' < 0$ beyond it: g rises, peaks at $R^\ddagger$, and then falls. The root is $R^\ddagger = 15.5817$, as in the deck. Beyond it, g decreases towards its limit $\sqrt{2} - 1$ (Problem 1, distant circles).

(b) The arrival burn alone is $h(R) = R^{-1/2} - \sqrt{2}R^{-1/2}(1+R)^{-1/2}$. Its derivative is made of the second and third terms above:

$$2R^{3/2}(1+R)^{3/2}h'(R) = \sqrt{2}(1+2R) - (1+R)^{3/2}.$$

Again we compare squares:

$$h' > 0 \iff 2(1+2R)^2 > (1+R)^3 \iff 2+8R+8R^2 > 1+3R+3R^2+R^3 \iff Q(R) \equiv R^3-5R^2-5R-1 < 0.$$

As before, $Q/R^3 = 1 - 5/R - 5/R^2 - 1/R^3$ increases with R , and $Q(1) = -10 < 0$. So Q has a single root above 1, at $R \approx 5.879$. The arrival burn grows until $R \approx 5.88$ and falls after that. The departure burn has derivative $v_{c1}R^{-1/2}(1+R)^{-3/2}/\sqrt{2} > 0$, so it grows for every R , towards the escape burn $(\sqrt{2}-1)v_{c1}$.

(c) For large R , compare how fast the two burns change. The departure derivative behaves like $R^{-2}/\sqrt{2}$. The arrival burn is $\approx R^{-1/2} - \sqrt{2}R^{-1}$, whose derivative is $\approx -\frac{1}{2}R^{-3/2}$. For large R , $R^{-3/2}$ is much larger than R^{-2} , so the saving at arrival outweighs the extra cost at departure. In one sentence: beyond $R^\ddagger$ the departure burn is already almost the escape burn and can hardly grow, while the arrival burn, which is about the target's circular speed $v_{c1}/\sqrt{R}$, keeps falling as the target moves out.

3. Coaxial ellipses: is one route always cheaper?

Two coplanar ellipses share their apse line, with their periapses on the same side (aligned apsides). Their apsidal radii satisfy $r_{p1} < r_{p2}$ and $r_{a1} < r_{a2}$. Two tangential two-impulse transfers join them: **A1**, from the periapsis of orbit 1 to the apoapsis of orbit 2, and **A2**, from the apoapsis of orbit 1 to the periapsis of orbit 2. Put

$$x_1 = \frac{1}{r_{p1}}, \quad y_1 = \frac{1}{r_{a1}}, \quad x_2 = \frac{1}{r_{p2}}, \quad y_2 = \frac{1}{r_{a2}}, \quad \Phi(s) = \sqrt{\frac{2\mu}{s}}, \quad L(\sigma, \tau) = (\sigma - \tau) \Phi(\sigma + \tau).$$

- Show that a conic with apsidal radii r and r' has $h = \Phi(1/r + 1/r')$, and that a tangential burn at an apsis of radius r costs $|\Delta h|/r$.
- Show that $\Delta v_{A2} - \Delta v_{A1} = L(x_1, y_1) + L(y_1, x_2) + L(x_2, y_2) + L(y_2, x_1)$.
- Prove that A1 is always the cheaper. (*Write the four-term sum as two three-term sums, and differentiate a three-term sum with respect to its first argument.*)
- With anti-aligned apsides, the two routes are T2 (periapsis to periapsis) and T1 (apoapsis to apoapsis). Show that $\Delta v_{T1} - \Delta v_{T2}$ is the cyclic sum taken in the order $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow x_2$. Show that it is negative when orbit 1 is a circle and positive when orbit 2 is a circle, so that here neither route is always the cheaper.

Answer. We have to compare the two routes for *every* pair of aligned orbits, so no numerical example will settle it. We need the difference of the two costs as a formula whose sign we can read off. The plan:

- find a simple formula for the cost of one burn at an apsis (part a);
- use it to write the cost of each route, and subtract (part b);

3. find the sign of the difference (part c).

(a) Both routes are made of tangential burns at apsides. At an apsis the velocity is purely transverse, so $v = h/r$. A tangential burn there leaves r unchanged and changes only h , so it costs

$$|\Delta v| = \frac{|h_{\text{after}} - h_{\text{before}}|}{r}.$$

So all we need are the angular momenta of the conics involved. A conic with apsidal radii r and r' has $a = (r + r')/2$ and $e = |r' - r|/(r + r')$, so $p = a(1 - e^2) = 2rr'/(r + r')$, as in the midsem Q6 solution. With $h^2 = \mu p$,

$$h = \sqrt{\frac{2\mu rr'}{r + r'}}.$$

Divide the numerator and the denominator inside the root by rr' :

$$h = \sqrt{\frac{2\mu}{1/r + 1/r'}}.$$

This is the step that suggests the variables of the problem. The radii enter h only through the *sum of their reciprocals*, and the cost $|\Delta h|/r$ of a burn is again multiplied by a reciprocal. So we work with reciprocal radii, $x = 1/r_p$ and $y = 1/r_a$, and give the one function that appears a name:

$$\Phi(s) = \sqrt{\frac{2\mu}{s}}, \quad h = \Phi\left(\frac{1}{r} + \frac{1}{r'}\right).$$

A burn at an apsis of reciprocal radius $u = 1/r$ then costs $u|\Delta h|$. We will need two properties of Φ :

$$\Phi'(s) = -\frac{1}{2}\sqrt{2\mu} s^{-3/2} < 0, \quad \Phi''(s) = \frac{3}{4}\sqrt{2\mu} s^{-5/2} > 0.$$

Φ is *decreasing*: bringing the apsides closer to the centre (a larger s) lowers h . Φ is *convex*: the proof in (c) rests on this.

(b) First translate the hypotheses into the new variables. Taking reciprocals reverses an inequality, so:

- each orbit has $r_p \leq r_a$, hence $x_1 \geq y_1$ and $x_2 \geq y_2$;
- $r_{p1} < r_{p2}$ and $r_{a1} < r_{a2}$ give $x_1 > x_2$ and $y_1 > y_2$.

Orbit 1 has $h_1 = \Phi(x_1 + y_1)$, and orbit 2 has $h_2 = \Phi(x_2 + y_2)$.

Route A1. Its transfer conic has apsides r_{p1} and r_{a2} , so $h_{A1} = \Phi(x_1 + y_2)$. The first burn is at r_{p1} (reciprocal x_1) and takes h from h_1 to h_{A1} . The second burn is at r_{a2} (reciprocal y_2) and takes h from h_{A1} to h_2 :

$$\Delta v_{A1} = x_1[\Phi(x_1 + y_2) - \Phi(x_1 + y_1)] + y_2[\Phi(x_2 + y_2) - \Phi(x_1 + y_2)].$$

Can we drop the absolute values? Yes: $y_2 < y_1$ gives $x_1 + y_2 < x_1 + y_1$, and since Φ decreases, the first bracket is positive. In the same way $x_2 < x_1$ makes the second bracket positive. Both burns are prograde.

Route A2. Its transfer conic has apsides r_{a1} and r_{p2} , so $h_{A2} = \Phi(y_1 + x_2)$. The burns are at r_{a1} (reciprocal y_1) and at r_{p2} (reciprocal x_2):

$$\Delta v_{A2} = y_1[\Phi(y_1 + x_2) - \Phi(x_1 + y_1)] + x_2[\Phi(x_2 + y_2) - \Phi(y_1 + x_2)],$$

and both brackets are positive for the same reasons.

The difference. Four values of Φ appear, one for each of the four conics in the problem. Collect the coefficient of each in $\Delta v_{A2} - \Delta v_{A1}$:

value	conic	coefficient
$\Phi(x_1 + y_1)$	orbit 1	$-y_1 + x_1$
$\Phi(y_1 + x_2)$	transfer A2	$y_1 - x_2$
$\Phi(x_2 + y_2)$	orbit 2	$x_2 - y_2$
$\Phi(y_2 + x_1)$	transfer A1	$-x_1 + y_2$

Every term has the same shape, $(\text{first} - \text{second}) \times \Phi(\text{first} + \text{second})$. That is why the question defines $L(\sigma, \tau) = (\sigma - \tau) \Phi(\sigma + \tau)$:

$$\Delta v_{A2} - \Delta v_{A1} = L(x_1, y_1) + L(y_1, x_2) + L(x_2, y_2) + L(y_2, x_1).$$

Read the arguments in order: $x_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow x_1$. Each term starts where the last one ended, so the sum goes once round a cycle. Two facts follow at once from the definition of L : $L(\sigma, \tau) = -L(\tau, \sigma)$, and $L(\sigma, \sigma) = 0$.

(c) We must show that this cyclic sum is positive. Its terms have different signs (for example $L(x_1, y_1) \geq 0$ but $L(y_2, x_1) < 0$), so we cannot argue term by term. The hint suggests a simpler object, a cycle of only three terms,

$$C_3(\sigma, \tau, \kappa) = L(\sigma, \tau) + L(\tau, \kappa) + L(\kappa, \sigma),$$

and then reducing the four-term cycle to two of these.

Step 1: the sign of a three-term cycle. We claim that $C_3(\sigma, \tau, \kappa) > 0$ whenever $\sigma > \tau > \kappa$. The idea is this. When $\sigma = \tau$,

$$C_3(\tau, \tau, \kappa) = L(\tau, \tau) + L(\tau, \kappa) + L(\kappa, \tau) = 0.$$

So if we can show that C_3 *increases* as σ is raised above τ , then C_3 must be positive. Only the first and the last terms contain σ :

$$\begin{aligned} \frac{\partial}{\partial \sigma} L(\sigma, \tau) &= \Phi(\sigma + \tau) + (\sigma - \tau) \Phi'(\sigma + \tau), \\ \frac{\partial}{\partial \sigma} L(\kappa, \sigma) &= \frac{\partial}{\partial \sigma} [(\kappa - \sigma) \Phi(\kappa + \sigma)] = -[\Phi(\sigma + \kappa) + (\sigma - \kappa) \Phi'(\sigma + \kappa)]. \end{aligned}$$

Both have the same form. With $m(t) = \Phi(\sigma + t) + (\sigma - t) \Phi'(\sigma + t)$,

$$\frac{\partial C_3}{\partial \sigma} = m(\tau) - m(\kappa).$$

So C_3 grows with σ if $m(\tau) > m(\kappa)$, that is (since $\tau > \kappa$) if m is an increasing function. Differentiate it:

$$m'(t) = \Phi'(\sigma + t) - \Phi'(\sigma + t) + (\sigma - t) \Phi''(\sigma + t) = (\sigma - t) \Phi''(\sigma + t).$$

This is positive for $t < \sigma$, because Φ is convex. So, as σ rises from τ , $\partial C_3 / \partial \sigma > 0$ all the way, and C_3 rises from 0. That proves the claim. In words: **C_3 is positive when its arguments are in decreasing order, and it increases with its first argument.**

Step 2: split the four-term cycle. Add and subtract $L(y_1, y_2)$, a “diagonal” of the cycle, and use $-L(y_1, y_2) = L(y_2, y_1)$:

$$\begin{aligned} \Delta v_{A2} - \Delta v_{A1} &= [L(x_1, y_1) + L(y_1, y_2) + L(y_2, x_1)] + [L(y_1, x_2) + L(x_2, y_2) + L(y_2, y_1)] \\ &= C_3(x_1, y_1, y_2) + C_3(y_1, x_2, y_2). \end{aligned}$$

Step 3: the signs. In the first term $x_1 \geq y_1 > y_2$, so by Step 1 it is positive. It is zero only if orbit 1 is a circle ($x_1 = y_1$). The second term depends on the order of y_1 and x_2 , that is, on whether r_{a1} is below or above r_{p2} .

- *Nested orbits*, $r_{a1} \leq r_{p2}$: then $y_1 \geq x_2 > y_2$, the arguments are in decreasing order, and the second term is not negative either.
- *Overlapping orbits*, $r_{a1} > r_{p2}$: then $x_2 > y_1$, and the arguments are out of order. Running a cycle backwards changes the sign of every term, so $C_3(y_1, x_2, y_2) = -C_3(x_2, y_1, y_2)$, and

$$\Delta v_{A2} - \Delta v_{A1} = C_3(x_1, y_1, y_2) - C_3(x_2, y_1, y_2) > 0.$$

Both cycles have the same last two arguments, and $x_1 > x_2 > y_1 > y_2$, so the first is larger because C_3 increases with its first argument.

In both cases A1 is the cheaper. The two costs are equal only when both orbits are circles, and then A1 and A2 are the same Hohmann transfer. ■

What the proof says. There are two rules here: leave the inner orbit from its periapsis, and arrive at the outer orbit at its apoapsis. With aligned apsides, route A1 obeys both. Example 2 of the deck (A1 cheaper by 644 m/s) is one instance. The convexity of Φ that drives the proof is the Oberth effect: burning deep in the field, or matching the target where it moves slowly, buys the most.

(d) With anti-aligned apsides, orbit 1 has its apoapsis on the side of orbit 2's periapsis. For the orbits to be nested, $r_{a1} < r_{p2}$, so now $x_1 \geq y_1 > x_2 \geq y_2$. T2 joins the two periapses, so its conic has $h = \Phi(x_1 + x_2)$. T1 joins the two apoapses, so its conic has $h = \Phi(y_1 + y_2)$. As in (b), all four burns are prograde:

$$\begin{aligned}\Delta v_{T2} &= x_1[\Phi(x_1 + x_2) - \Phi(x_1 + y_1)] + x_2[\Phi(x_2 + y_2) - \Phi(x_1 + x_2)], \\ \Delta v_{T1} &= y_1[\Phi(y_1 + y_2) - \Phi(x_1 + y_1)] + y_2[\Phi(x_2 + y_2) - \Phi(y_1 + y_2)].\end{aligned}$$

Collecting the coefficient of each value of Φ , exactly as in (b),

$$\Delta v_{T1} - \Delta v_{T2} = L(x_1, y_1) + L(y_1, y_2) + L(y_2, x_2) + L(x_2, x_1),$$

the cycle $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow x_2 \rightarrow x_1$. To see that its sign is not fixed, look at the two extreme cases, where one term drops out and the cycle becomes a C_3 .

- *Orbit 1 a circle* ($x_1 = y_1$, so $L(x_1, y_1) = 0$). The sum is $L(x_1, y_2) + L(y_2, x_2) + L(x_2, x_1)$, which is the cycle $x_1 \rightarrow x_2 \rightarrow y_2$ run backwards, $-C_3(x_1, x_2, y_2)$. Since $x_1 > x_2 > y_2$, this is negative, and **T1 is cheaper**.
- *Orbit 2 a circle* ($x_2 = y_2$, so $L(y_2, x_2) = 0$). The sum is $L(x_1, y_1) + L(y_1, x_2) + L(x_2, x_1) = C_3(x_1, y_1, x_2)$. Since $x_1 > y_1 > x_2$, this is positive, and **T2 is cheaper**.

The difference is a continuous function of the four radii, so somewhere between these two cases it changes sign. With anti-aligned apsides the cheaper route has to be computed. The deck's Example 3 (T1 cheaper) and its variant with $e_1 = 0.45$ (T2 cheaper) lie on either side of the change of sign.

The two rules of (c) explain why. With anti-aligned apsides, the periapsis of orbit 1 and the apoapsis of orbit 2 lie on the *same* side of the centre. No tangential transfer can join them, because the apsides of a transfer conic are 180° apart. T2 obeys the first rule, and T1 obeys the second, and which rule matters more depends on the orbits.

4. Circularising with one burn

A spacecraft on an ellipse of semi-latus rectum p and eccentricity e is to be put on a circular orbit with a single impulse: the circle of whatever radius it has when it burns.

- (a) Show that a burn at true anomaly f costs

$$\Delta v^2 = \frac{\mu}{p} \left[e^2 - (x - 1)^2(2x + 1) \right], \quad x^2 = 1 + e \cos f.$$

- (b) Show that the burn is dearest at the ends of the latus rectum ($f = \pm 90^\circ$), where it is purely radial and costs $e\sqrt{\mu/p}$. Show that it is cheapest at one of the apsides.

- (c) Show that

$$\Delta v_{\text{apo}} = \sqrt{\frac{\mu}{p}} s_- (1 - s_-), \quad \Delta v_{\text{peri}} = \sqrt{\frac{\mu}{p}} s_+ (s_+ - 1), \quad s_{\pm} = \sqrt{1 \pm e},$$

and prove that the apoapsis burn is the cheaper for every $0 < e < 1$. What does (b) say about Question 6 of the midsem?

Answer. We are asked how the cost of a one-burn circularisation depends on *where* on the ellipse the burn is made. So first we need Δv as a function of the position on the orbit, and then we look for its largest and smallest values.

- (a) The burn must turn the velocity on the ellipse into the velocity on the circle through the same point.

On that circle the velocity is purely transverse, with magnitude $\sqrt{\mu/r}$. So the burn must

- remove the whole radial velocity v_r , and
- change the transverse velocity from v_θ to $\sqrt{\mu/r}$.

These two changes are perpendicular, so

$$\Delta v^2 = v_r^2 + \left(v_\theta - \sqrt{\mu/r}\right)^2.$$

On the ellipse, $v_r = \sqrt{\mu/p} e \sin f$ and $v_\theta = \sqrt{\mu/p} (1 + e \cos f)$. Since $r = p/(1 + e \cos f)$, the circular speed is $\sqrt{\mu/r} = \sqrt{\mu/p} \sqrt{1 + e \cos f}$.

That square root $\sqrt{1 + e \cos f}$ is the awkward part, and it is the reason for the variable $x = \sqrt{1 + e \cos f}$. With it, $v_\theta = \sqrt{\mu/p} x^2$ and $\sqrt{\mu/r} = \sqrt{\mu/p} x$, which are both polynomials in x :

$$\frac{p}{\mu} \Delta v^2 = e^2 \sin^2 f + (x^2 - x)^2.$$

We also want $e^2 \sin^2 f$ in terms of x . Since $e \cos f = x^2 - 1$, $e^2 \sin^2 f = e^2 - (e \cos f)^2 = e^2 - (x^2 - 1)^2$. So

$$\frac{p}{\mu} \Delta v^2 = e^2 - (x^2 - 1)^2 + x^2(x - 1)^2.$$

The last two terms share the factor $(x - 1)^2$, because $(x^2 - 1)^2 = (x - 1)^2(x + 1)^2$. Take it out:

$$\frac{p}{\mu} \Delta v^2 = e^2 - (x - 1)^2[(x + 1)^2 - x^2] = e^2 - (x - 1)^2(2x + 1).$$

(b) Now the cost depends on the position only through x . As f goes round the orbit, $1 + e \cos f$ runs from $1 - e$ (apoapsis) to $1 + e$ (periapsis), so x runs over $[s_-, s_+]$. Write $B(x) = (x - 1)^2(2x + 1)$. The cost is largest where B is smallest, and smallest where B is largest.

The dearest point. On the orbit $x > 0$, so $B \geq 0$, and $B = 0$ only at $x = 1$. Hence $\Delta v^2 \leq (\mu/p) e^2$, with equality at $x = 1$, that is at $\cos f = 0$, $f = \pm 90^\circ$: the ends of the latus rectum. There $r = p$, so $v_\theta = \sqrt{\mu/p}$ is already the circular speed. Only $v_r = e\sqrt{\mu/p}$ has to be removed, and the burn is purely radial.

The cheapest point. Differentiate B :

$$B'(x) = 2(x - 1)(2x + 1) + 2(x - 1)^2 = 2(x - 1)[(2x + 1) + (x - 1)] = 6x(x - 1).$$

This is negative for $x < 1$ and positive for $x > 1$. So B falls from $x = s_-$ to $x = 1$ and rises from $x = 1$ to $x = s_+$. Its largest value on the interval is therefore at one of the two ends: the cost is least at apoapsis or at periapsis. Part (c) decides which.

(c) At an apsis $v_r = 0$, so we can read the cost directly from the two speeds.

- *Apoapsis*, $x = s_-$: the speed is $v_\theta = \sqrt{\mu/p} s_-^2$ and the circular speed is $\sqrt{\mu/p} s_-$. Since $s_- < 1$, the burn speeds the spacecraft up, by $\Delta v_{\text{apo}} = \sqrt{\mu/p} (s_- - s_-^2) = \sqrt{\mu/p} s_- (1 - s_-)$.
- *Periapsis*, $x = s_+$: in the same way the burn slows it down, by $\Delta v_{\text{peri}} = \sqrt{\mu/p} (s_+^2 - s_+) = \sqrt{\mu/p} s_+ (s_+ - 1)$.

(As a check against (a): at $x = s_-$ we have $e = 1 - s_-^2 = (1 - s_-)(1 + s_-)$, so $e^2 - B = (1 - s_-)^2[(1 + s_-)^2 - (2s_- + 1)] = (1 - s_-)^2 s_-^2$.)

The apoapsis burn is the cheaper if

$$s_- - s_-^2 < s_+^2 - s_+ \iff s_- + s_+ < s_-^2 + s_+^2.$$

Here we see why $s_\pm$ are convenient: their squares add up to $(1 - e) + (1 + e) = 2$. So we need $s_- + s_+ < 2$. Both sides are positive, so square: $(s_- + s_+)^2 = s_-^2 + s_+^2 + 2s_-s_+ = 2 + 2\sqrt{1 - e^2}$, which is less than 4 for every $e > 0$. Hence **the apoapsis burn is always the cheaper**. (Divided by 2, this is again “the arithmetic mean is less than the root-mean-square”, now for s_- and s_+ , as in Problem 1.)

The midsem. In Question 6 the circle had radius $r_1 = p$ and crossed the ellipse at $f = \pm 90^\circ$, which is exactly where (b) puts the dearest one-burn change between that ellipse and a circle. Running the burn backwards (circle to ellipse) costs the same, $e\sqrt{\mu/p} = e v_c$, so the single impulse of Q6 was made at the worst possible point. The two-impulse routes avoid that point, and every burn on them is tangential: it changes the size of $\mathbf{v}$, never its direction.

5. Apse-line rotation: when to go round by a circle

The apse line of an orbit (p, e) is to be turned through η , keeping p and e . Route S is the single radial impulse at a crossing point, $\Delta v_S = 2\sqrt{\mu/p} e \sin(\eta/2)$ (deck, Part IV). Route C circularises at apoapsis, coasts, and leaves the circle at the new apoapsis.

- (a) Show that $\Delta v_C = 2\sqrt{\mu/p} s_-(1 - s_-)$, with $s_- = \sqrt{1 - e}$, whatever η .
(b) Prove that S is the cheaper exactly when

$$\sin \frac{\eta}{2} < \frac{\sqrt{1 - e}}{1 + \sqrt{1 - e}}.$$

- (c) Show that the crossover angle falls monotonically from 60° to 0 as e goes from 0 to 1 . Explain the 60° limit by comparing how efficiently radial and tangential impulses change e .
(d) Show that going round by any other intermediate circle, joined by single burns at crossing points, costs more than Route C.

Answer. We have two routes. The cost of S is given. The cost of C can be built from Problem 4. Then we find the angle at which the two are equal.

(a) Route C has two burns. The first circularises at the apoapsis of the original orbit, and by Problem 4(c) it costs $\sqrt{\mu/p} s_-(1 - s_-)$. The coast on the circle costs nothing. The second burn leaves the circle, at the new apoapsis, onto the rotated ellipse. Run backwards in time, it is the apoapsis circularisation of the rotated ellipse. That ellipse has the same p and e , so the burn costs the same. The total is $2\sqrt{\mu/p} s_-(1 - s_-)$. Nothing here depends on η : a circle has no apse line, so it does not matter where we leave it.

(b) Both costs carry the factor $2\sqrt{\mu/p}$, which cancels. S is the cheaper when

$$e \sin \frac{\eta}{2} < s_-(1 - s_-).$$

To compare the two sides we want e in terms of s_- as well. Since $s_-^2 = 1 - e$,

$$e = 1 - s_-^2 = (1 - s_-)(1 + s_-).$$

Now the factor $1 - s_-$ appears on both sides, and it is positive for $e > 0$, so it cancels:

$$(1 + s_-) \sin \frac{\eta}{2} < s_- \iff \sin \frac{\eta}{2} < \frac{s_-}{1 + s_-}.$$

Check: for $e = 0.4$, $s_- = 0.7746$ and $s_-/(1 + s_-) = 0.4365$, so the crossover is at $\eta_c = 51.8^\circ$, as in the deck.

(c) Write the threshold as $s_-/(1 + s_-) = 1 - 1/(1 + s_-)$. This increases with s_- , and $s_- = \sqrt{1 - e}$ decreases as e increases. So η_c decreases with e . At the two ends: as $e \rightarrow 0$, $s_- \rightarrow 1$ and $\sin(\eta_c/2) \rightarrow \frac{1}{2}$, so $\eta_c \rightarrow 60^\circ$; as $e \rightarrow 1$, $s_- \rightarrow 0$ and $\eta_c \rightarrow 0$.

Why 60° ? Think of the eccentricity vector $\mathbf{e}$. Turning the apse line through η at fixed e means turning $\mathbf{e}$ through η . Its tip moves along a chord of length $|\Delta \mathbf{e}| = 2e \sin(\eta/2)$. Route S moves the tip straight along that chord, with a radial impulse. Route C goes round the other two sides of a triangle: it shrinks $\mathbf{e}$ to zero and then grows it again in the new direction, a total path of $2e$, with tangential impulses at apsides. How much does each kind of impulse change $\mathbf{e}$? For small e the speed is close to the circular speed v , and $h/\mu \approx 1/v$.

- *Radial impulse.* From $e \sin f = h v_r / \mu$, a radial Δv moves the tip of $\mathbf{e}$ by $h \Delta v / \mu = \Delta v / v$.
- *Tangential impulse at an apsis.* There $e = h^2 / (\mu r) - 1 = r v^2 / \mu - 1$, with r fixed. So $\Delta e = 2 r v \Delta v / \mu = 2 \Delta v / v$.

A tangential burn is twice as effective. So

$$\Delta v_S \approx v \cdot 2e \sin \frac{\eta}{2}, \quad \Delta v_C \approx v \cdot \frac{2e}{2} = e v,$$

and these are equal at $\sin(\eta/2) = \frac{1}{2}$. In words: going the long way round pays once the direct chord is longer than half the path round the other two sides.

(d) Take any circle of radius r between r_p and r_a . The route through it costs the one-burn circularisation at r on the old ellipse, plus the reverse burn at r onto the new ellipse. The two ellipses have the same p and e , so the two burns are equal, and the route costs $2 \Delta v(r)$, with $\Delta v(r)$ from Problem 4(a). By Problem 4(b, c), that is least at $r = r_a$, which is Route C.

6. Phasing: an invariant, and an asymmetry

A chaser on a circle of radius r trails its target by a small angle $\Delta\theta$. It closes the gap with a circular phasing manoeuvre of k revolutions (deck, Part III). Let $x = \Delta\theta/(2\pi k)$ and $v_c = \sqrt{\mu/r}$.

- (a) Show that each of the two burns is $v_c(\frac{1}{3}x + \frac{1}{3}x^2 + O(x^3))$. Show that the total cost and the total time t satisfy

$$\Delta v_{\text{total}} t = \frac{2}{3} r \Delta\theta [1 + O(x^2)],$$

whatever μ , k and r .

- (b) If instead the chaser is *ahead* by $\Delta\theta$ and drops back, show that each burn is $v_c(\frac{1}{3}x - \frac{1}{3}x^2 + O(x^3))$. Which situation costs more, and by what factor?
- (c) The phasing perigee must stay above $r_{\min}$. Show that the trailing chaser needs

$$k \geq \frac{\Delta\theta}{2\pi \left[1 - \left(\frac{r+r_{\min}}{2r}\right)^{3/2}\right]}.$$

Answer. The claim in (a) is that cost times time depends only on the size of the gap. Before calculating, let us see why that should be so. Then we do the calculation properly, because the question also asks how big the correction is.

Why the product is nearly constant. A small tangential burn Δv on a circle changes the energy by $v_c \Delta v$. Since $\varepsilon = -\mu/(2a)$, that is $\mu \Delta a/(2a^2) = v_c \Delta v$, or $\Delta a/a = 2 \Delta v/v_c$. By Kepler's third law, $\Delta T/T = \frac{3}{2} \Delta a/a = 3 \Delta v/v_c$. On each revolution the chaser gains the angle $2\pi |\Delta T|/T = 6\pi \Delta v/v_c$ on the target, so in k revolutions it gains $\Delta\theta = 6\pi k \Delta v/v_c$. The manoeuvre takes $t \approx kT = 2\pi k r/v_c$. Multiply: $\Delta v \cdot t = \frac{1}{3} r \Delta\theta$ for one burn. There are two burns, one to enter the phasing orbit and one to leave it, so $\Delta v_{\text{total}} t \approx \frac{2}{3} r \Delta\theta$. A cheaper manoeuvre must use a smaller Δv . Its phasing orbit is then closer to the circle and gains angle more slowly, so it takes longer in exactly the same proportion.

(a) Now the calculation. The phasing orbit is fixed by its period. In k phasing periods the chaser must return to its burn point just as the target, which started $\Delta\theta$ ahead, arrives there. The target covers $2\pi k - \Delta\theta$ in that time, so

$$T_{ph} = T \left(1 - \frac{\Delta\theta}{2\pi k}\right) = T(1 - x).$$

This is why x is the natural small quantity: it is the fractional change in the period. By Kepler's third law $a_{ph} = r(1 - x)^{2/3}$. The phasing ellipse is smaller than the circle, so the burn point is its apoapsis, and there *vis-viva* gives

$$v = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a_{ph}}\right)} = v_c \sqrt{2 - (1 - x)^{-2/3}}.$$

Since x is small, and we want the leading term and the first correction, we expand in powers of x . The binomial series gives $(1 - x)^{-2/3} = 1 + \frac{2}{3}x + \frac{5}{9}x^2 + O(x^3)$, so the quantity under the root is $1 + u$ with

$u = -\frac{2}{3}x - \frac{5}{9}x^2$. With $\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + O(u^3)$,

$$\sqrt{1 - \frac{2}{3}x - \frac{5}{9}x^2} = 1 - \frac{1}{3}x - \frac{5}{18}x^2 - \frac{1}{8} \cdot \frac{4}{9}x^2 + O(x^3) = 1 - \frac{1}{3}x - \frac{1}{3}x^2 + O(x^3).$$

So each burn is $v_c - v = v_c(\frac{1}{3}x + \frac{1}{3}x^2 + O(x^3))$. Its leading term $\frac{1}{3}x$ is the estimate above: $3\Delta v/v_c = x$. The manoeuvre lasts k phasing periods,

$$t = kT_{ph} = k \frac{2\pi r}{v_c} (1 - x).$$

Multiply the total cost, two burns, by the time:

$$\Delta v_{\text{total}} t = 2v_c \left(\frac{1}{3}x + \frac{1}{3}x^2 \right) \cdot \frac{2\pi k r}{v_c} (1 - x) + O(x^3) = \frac{2}{3} r (2\pi k x) (1 + x)(1 - x) + O(x^3).$$

Now $2\pi k x = \Delta\theta$, and $(1+x)(1-x) = 1 - x^2$. The first-order corrections cancel: the cost carries a factor $1+x$ and the time a factor $1-x$. So

$$\Delta v_{\text{total}} t = \frac{2}{3} r \Delta\theta [1 + O(x^2)].$$

That is why the estimate is so good: its error is second order. (The script carries the expansion one order further, and the factor is $1 + \frac{2}{27}x^2$.) For the deck's Example 6 the product is 788.70 km, against $\frac{2}{3}r\Delta\theta = 788.66$ km.

(b) A chaser that is ahead must let the target catch up, so it needs a *longer* period, $T_{ph} = T(1+x)$, and $a_{ph} = r(1+x)^{2/3}$. The phasing ellipse is now larger than the circle, the burn point is its perigee, and each burn is

$$v - v_c = v_c \left[\sqrt{2 - (1+x)^{-2/3}} - 1 \right].$$

The same two expansions, with x replaced by $-x$ in the binomial series, $(1+x)^{-2/3} = 1 - \frac{2}{3}x + \frac{5}{9}x^2 + O(x^3)$, give

$$\sqrt{1 + \frac{2}{3}x - \frac{5}{9}x^2} = 1 + \frac{1}{3}x - \frac{5}{18}x^2 - \frac{1}{18}x^2 + O(x^3) = 1 + \frac{1}{3}x - \frac{1}{3}x^2 + O(x^3).$$

So each burn is $v_c(\frac{1}{3}x - \frac{1}{3}x^2 + O(x^3))$. Comparing with (a),

$$\frac{\Delta v_{\text{behind}}}{\Delta v_{\text{ahead}}} = \frac{1+x}{1-x} + O(x^2) = 1 + 2x + O(x^2).$$

The trailing chaser pays more, by the factor $1 + 2x$. To first order the two cases are mirror images: both burns are $\frac{1}{3}x v_c$. The difference comes at second order. The cost of a burn, as a function of the change y in the period ($T_{ph} = T(1+y)$), is built from two increasing and *concave* functions: $a \propto (1+y)^{2/3}$, and the *vis-viva* speed $\sqrt{2 - r/a}$. A concave function changes more on the way down than on the way up. So cutting the period by x costs more than raising it by x .

(c) The phasing ellipse of the trailing chaser has its apoapsis at r , so its perigee is $2a_{ph} - r$. The perigee must stay above $r_{\min}$:

$$2r(1-x)^{2/3} - r \geq r_{\min} \iff (1-x)^{2/3} \geq \frac{r+r_{\min}}{2r} \iff x \leq 1 - \left(\frac{r+r_{\min}}{2r} \right)^{3/2}.$$

Putting in $x = \Delta\theta/(2\pi k)$ and solving for k gives the stated bound. A chaser that is ahead goes *higher*, and has no such constraint.

7. Turning a plane from far away

The plane of a circle of radius r_1 is to be turned through θ . Write $v_c = \sqrt{\mu/r_1}$ and $\sigma = \sin(\theta/2)$. Route D turns the velocity at r_1 and costs $2v_c\sigma$. Route F raises the apoapsis tangentially to $r_b = Kr_1$, turns the plane at apoapsis, and circularises when it is back at r_1 .

(a) Show that

$$\Delta v_F(K) = 2v_c \left[\sqrt{\frac{2K}{1+K}} - 1 + \sigma \sqrt{\frac{2}{K(1+K)}} \right].$$

- (b) Show that $d\Delta v_F/dK$ has the sign of $K - \sigma(1 + 2K)$. Hence show that the best K is 1 (Route D) for $\sigma \leq \frac{1}{3}$, is $K^* = \sigma/(1 - 2\sigma)$ for $\frac{1}{3} < \sigma < \frac{1}{2}$, and is $K \rightarrow \infty$ for $\sigma \geq \frac{1}{2}$.
- (c) Show that the optimal cost in the middle range is $\Delta v^* = 2v_c[2\sqrt{2\sigma(1-\sigma)} - 1]$. Show that it joins the other two branches continuously.
- (d) At what angle does the limit $K \rightarrow \infty$ first beat Route D? Compare the three thresholds $\theta = 38.9^\circ, 48.9^\circ, 60^\circ$ with the two bi-elliptic thresholds of the deck.

Answer. For each angle θ we must choose the apoapsis K that makes Route F cheapest, and compare the result with Route D. The plan: write the cost as a function of K (a), find where its derivative vanishes (b), evaluate the cost there (c), and compare the cases (d).

(a) Route F has three burns.

1. *Raise the apoapsis.* A tangential burn at r_1 takes the circle to the ellipse with $r_p = r_1$ and $r_a = Kr_1$. By the transfer-ellipse speeds of Problem 1, with R replaced by K , the periapsis speed is $v_p = v_c \sqrt{2K/(1+K)}$. The burn costs $v_p - v_c$.
2. *Turn the plane at apoapsis.* Angular momentum gives the apoapsis speed, $v_b = v_p/K = v_c \sqrt{2/(K(1+K))}$. The velocity is turned through θ with its size kept, so the old and new velocities form an isosceles triangle with two sides v_b and apex angle θ . Its base is the burn, $2v_b \sin(\theta/2) = 2v_b\sigma$.
3. *Circularise.* The spacecraft falls back along the turned copy of the same ellipse, and reaches r_1 at speed v_p . Circularising costs $v_p - v_c$ again.

The total is $2(v_p - v_c) + 2\sigma v_b$, which is the stated expression. (The first burn must be placed so that the apoapsis falls on the line where the two planes meet. Only on that line is there a point common to both orbits at which the plane can be turned.)

The structure of this cost is the whole story. The in-plane part $2(v_p - v_c)$ grows with K . The plane-change part $2\sigma v_b$ falls with K , because the spacecraft moves slowly far out. The best K balances the two.

(b) We need two derivatives. As in Problem 2,

$$\frac{d}{dK} \sqrt{\frac{2K}{1+K}} = \frac{1}{\sqrt{2K}(1+K)^{3/2}}, \quad \frac{d}{dK} \sqrt{\frac{2}{K(1+K)}} = -\frac{1+2K}{\sqrt{2}K^{3/2}(1+K)^{3/2}}.$$

To put them over a common denominator, write $1/\sqrt{2K} = K/(\sqrt{2}K^{3/2})$. Then

$$\frac{d\Delta v_F}{dK} = \frac{\sqrt{2}v_c}{K^{3/2}(1+K)^{3/2}} [K - \sigma(1+2K)].$$

The factor in front is positive, so the sign is that of the bracket, which we write as $K(1 - 2\sigma) - \sigma$. Now there are cases.

- $\sigma \geq \frac{1}{2}$: then $1 - 2\sigma \leq 0$, the bracket is at most $-\sigma < 0$ for every K , and the cost falls all the way as $K \rightarrow \infty$.
- $\sigma < \frac{1}{2}$: the bracket is negative for $K < K^* = \sigma/(1 - 2\sigma)$ and positive beyond, so the cost falls until K^* and rises after it. But K must be at least 1. If $K^* \leq 1$, which means $\sigma \leq 1 - 2\sigma$, i.e. $\sigma \leq \frac{1}{3}$, the cost rises over the whole allowed range, and the best choice is $K = 1$: Route D. If $\sigma > \frac{1}{3}$, the best choice is K^* .

(c) To evaluate the cost at K^* we need the two square roots there. First the pieces:

$$1 + K^* = \frac{1 - \sigma}{1 - 2\sigma}, \quad \frac{2K^*}{1 + K^*} = \frac{2\sigma}{1 - \sigma}, \quad K^*(1 + K^*) = \frac{\sigma(1 - \sigma)}{(1 - 2\sigma)^2}.$$

So the plane-change term is

$$\sigma \sqrt{\frac{2}{K^*(1 + K^*)}} = \sigma(1 - 2\sigma) \sqrt{\frac{2}{\sigma(1 - \sigma)}} = (1 - 2\sigma) \sqrt{\frac{2\sigma}{1 - \sigma}},$$

and both terms now carry the same root:

$$\frac{\Delta v^*}{2v_c} = \sqrt{\frac{2\sigma}{1 - \sigma}} [1 + (1 - 2\sigma)] - 1 = 2(1 - \sigma) \sqrt{\frac{2\sigma}{1 - \sigma}} - 1 = 2\sqrt{2\sigma(1 - \sigma)} - 1.$$

Continuity. At $\sigma = \frac{1}{3}$: $2\sqrt{2 \cdot \frac{1}{3} \cdot \frac{2}{3}} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$, so $\Delta v^* = \frac{2}{3}v_c = 2v_c\sigma$, the cost of Route D. At $\sigma = \frac{1}{2}$: $2\sqrt{\frac{1}{2}} - 1 = \sqrt{2} - 1$, so $\Delta v^* = 2(\sqrt{2} - 1)v_c$, which is the limit of Δv_F as $K \rightarrow \infty$ (the plane-change term vanishes there).

(d) As $K \rightarrow \infty$, $\Delta v_F \rightarrow 2(\sqrt{2} - 1)v_c$: the plane change itself becomes free, and we pay only to go out and come back. This beats $2v_c\sigma$ when $\sigma > \sqrt{2} - 1$, i.e. $\theta > 48.9^\circ$. So there are three thresholds:

σ	θ	what happens
$\frac{1}{3}$	38.9°	a small raise of the apoapsis starts to pay
$\sqrt{2} - 1$	48.9°	going out to infinity beats the direct turn
$\frac{1}{2}$	60°	infinity is the best choice

The bi-elliptic transfer of the deck also has two thresholds: infinity beats Hohmann at $R = 11.94$, but the smallest detour pays only from $R = 15.58$. Here the order is reversed: the local threshold (38.9°) comes *before* the one at infinity (48.9°). The reason is the shape of the cost. The cost of Route F has an interior **minimum** in K , which beats both ends. The bi-elliptic cost, in the window $11.94 < R < 15.58$, has an interior **maximum** in r_b .

8. Folding a small plane change into a Hohmann transfer

A Hohmann transfer $r_1 \rightarrow r_2 = Rr_1$ must also turn the plane by a small angle Δi .

- Show that a burn which changes the speed from v to v' and turns the velocity through δ costs $\sqrt{(v' - v)^2 + 4vv' \sin^2(\delta/2)}$. Hence show that folding the whole turn into the second burn raises its cost by only $v_{t2}v_{c2} \Delta i^2 / (2\Delta v_2)$ to lowest order, whereas a separate turn at r_2 costs $v_{c2} \Delta i$.
- Give δ of the turn to the first burn and $\Delta i - \delta$ to the second. Show that, to lowest order, the best split is

$$\frac{\delta}{\Delta i} = \frac{1}{1 + R^{3/2} \Delta v_2 / \Delta v_1}.$$

- Show that this fraction tends to $\frac{1}{2}$ as $R \rightarrow 1$ and to 0 as $R \rightarrow \infty$, and that it is below $\frac{1}{2}$ for every $R > 1$. State what “small” means here.

Answer. (a) First we need the cost of a burn that changes the speed *and* turns the velocity. The vectors $\mathbf{v}$ and $\mathbf{v}'$ make the angle δ , so by the law of cosines

$$|\mathbf{v}' - \mathbf{v}|^2 = v^2 + v'^2 - 2vv' \cos \delta.$$

We want to separate the in-plane part $(v' - v)^2$, so write $v^2 + v'^2 = (v' - v)^2 + 2vv'$:

$$|\mathbf{v}' - \mathbf{v}|^2 = (v' - v)^2 + 2vv'(1 - \cos \delta) = (v' - v)^2 + 4vv' \sin^2 \frac{\delta}{2}.$$

At the second Hohmann burn, $v = v_{t2}$, $v' = v_{c2}$, and $v' - v = \Delta v_2$ is the in-plane cost. Folding the whole turn Δi into it gives

$$\Delta v_2(\Delta i) = \sqrt{\Delta v_2^2 + 4v_{t2}v_{c2} \sin^2 \frac{\Delta i}{2}}.$$

To see what the turn adds when Δi is small, take out the in-plane part,

$$\Delta v_2(\Delta i) = \Delta v_2 \sqrt{1 + \epsilon}, \quad \epsilon = \frac{4v_{t2}v_{c2} \sin^2(\Delta i/2)}{\Delta v_2^2} \approx \frac{v_{t2}v_{c2} \Delta i^2}{\Delta v_2^2},$$

and use $\sqrt{1 + \epsilon} \approx 1 + \epsilon/2$:

$$\Delta v_2(\Delta i) = \Delta v_2 + \frac{v_{t2}v_{c2}}{2\Delta v_2} \Delta i^2 + O(\Delta i^4).$$

The extra cost is **second order** in Δi . A separate turn at r_2 costs $2v_{c2} \sin(\Delta i/2) = v_{c2} \Delta i + O(\Delta i^3)$, which is first order. The reason is Pythagoras. The plane-change part of the burn is perpendicular to the in-plane part, so the two add in quadrature, and a small perpendicular addition lengthens a vector only at second order. A small plane change folded into a burn that has to be made anyway is almost free.

(b) Now share the turn between the two burns. By (a), each burn's extra cost is a parabola in the angle it takes. For the first burn ($v = v_{c1}$, $v' = v_{t1}$, in-plane cost Δv_1) and the second, the total extra cost is

$$E(\delta) = \frac{1}{2}A\delta^2 + \frac{1}{2}B(\Delta i - \delta)^2, \quad A = \frac{v_{c1}v_{t1}}{\Delta v_1}, \quad B = \frac{v_{t2}v_{c2}}{\Delta v_2}.$$

A and B measure how “stiff” each burn is against a plane change. To minimise E , set its derivative to zero:

$$E'(\delta) = A\delta - B(\Delta i - \delta) = 0 \implies \frac{\delta}{\Delta i} = \frac{B}{A+B} = \frac{1}{1+A/B}.$$

It remains to write A/B in terms of R . The circular speeds give $v_{c1}/v_{c2} = \sqrt{r_2/r_1} = \sqrt{R}$. Angular momentum on the transfer ellipse, $r_1v_{t1} = r_2v_{t2}$, gives $v_{t1}/v_{t2} = R$. So

$$\frac{A}{B} = \frac{v_{c1}v_{t1}}{v_{c2}v_{t2}} \cdot \frac{\Delta v_2}{\Delta v_1} = R^{3/2} \frac{\Delta v_2}{\Delta v_1},$$

which gives the stated split.

(c) *Limits.* As $R \rightarrow 1$, $\Delta v_2/\Delta v_1 \rightarrow 1$ (Problem 1), so $A/B \rightarrow 1$ and the fraction tends to $\frac{1}{2}$: nearby circles share the turn equally. As $R \rightarrow \infty$, $\Delta v_1 \rightarrow (\sqrt{2} - 1)v_{c1}$ and $\Delta v_2 \approx v_{c1}R^{-1/2}$ (Problem 1 again), so $A/B \approx R/(\sqrt{2} - 1)$ and $\delta/\Delta i \approx (\sqrt{2} - 1)/R \rightarrow 0$.

Always below $\frac{1}{2}$. We need $A/B > 1$, i.e. $R^{3/2}\Delta v_2 > \Delta v_1$. Two effects compete. The speeds at the inner circle are higher, which makes turning there dearer: this is the factor $R^{3/2}$. But the inner in-plane burn is the larger one (Problem 1), which makes turning there cheaper. We must show that the speeds win.

Both burns contain the same square root, so give it a name, $q = \sqrt{2/(1+R)}$. Then $\sqrt{2R/(1+R)} = \sqrt{R}q$, and from the limits part of Problem 1,

$$\Delta v_1 = v_{c1}(\sqrt{R}q - 1), \quad R^{3/2}\Delta v_2 = R^{3/2} \cdot \frac{v_{c1}}{\sqrt{R}}(1 - q) = v_{c1}R(1 - q).$$

So

$$R^{3/2}\Delta v_2 > \Delta v_1 \iff R - Rq > \sqrt{R}q - 1 \iff R + 1 > q(R + \sqrt{R}) \iff (1 + R)^{3/2} > \sqrt{2}(R + \sqrt{R}),$$

where the last step puts in q and multiplies by $\sqrt{1+R}$. To remove the roots, put $R = t^2$ ($t > 1$) and square: we need

$$(1+t^2)^3 > 2t^2(1+t)^2.$$

Split the left side as $(1+t^2)^2 \cdot (1+t^2)$ and bound each factor:

- $(1+t^2)^2 \geq (2t)^2 = 4t^2$, because $1+t^2 \geq 2t$ is $(t-1)^2 \geq 0$;
- $1+t^2 \geq \frac{1}{2}(1+t)^2$, because $2+2t^2 - (1+2t+t^2) = (t-1)^2 \geq 0$.

Multiplying, $(1+t^2)^3 \geq 4t^2 \cdot \frac{1}{2}(1+t)^2 = 2t^2(1+t)^2$, and both inequalities are strict for $t \neq 1$. So $A/B > 1$ and $\delta/\Delta i < \frac{1}{2}$: **most of the turn always goes to the outer burn**, where the spacecraft moves slowly.

What “small” means. The expansion in (a) needs the plane-change part of each burn to be small compared with its in-plane part, $4vv' \sin^2(\delta/2) \ll (v' - v)^2$. For LEO $\rightarrow$ GEO the formula gives 9.44% of the turn to the first burn, i.e. 2.69° of 28.5° . The deck’s exact optimum is 2.20° . The difference is because 28.5° is *not* small at the second burn: there the ϵ of (a) is about 0.5, not much less than 1.

9. Moving the node

The node of a circular orbit of radius r and inclination i is to be moved by $\Delta\Omega$, keeping i , with a single impulse. Let $v = \sqrt{\mu/r}$.

- Starting from $\cos \alpha = \cos^2 i + \sin^2 i \cos \Delta\Omega$ (deck, Part VI), show that the plane turns through α with $\sin \frac{\alpha}{2} = \sin i \sin \frac{\Delta\Omega}{2}$. Hence show that $\Delta v = 2v \sin i \sin \frac{\Delta\Omega}{2}$.
- Show that the burn is made at argument of latitude

$$u_1 = 90^\circ + \arctan\left(\cos i \tan \frac{\Delta\Omega}{2}\right)$$

on the initial orbit, or 180° further on, and at $u_2 = 90^\circ - \arctan(\cos i \tan \frac{\Delta\Omega}{2})$ on the final orbit. Where is this for a polar orbit, and for small $\Delta\Omega$?

- Show that moving the node through an angle never costs more than changing the inclination through the same angle. Why is moving the node of a near-equatorial orbit almost free?

Answer. (a) The plane turns through the angle α between the two orbit normals, and the deck gives $\cos \alpha$. A plane-change burn costs $2v \sin(\alpha/2)$, so we want the *half-angle*. Use $1 - \cos \phi = 2 \sin^2(\phi/2)$ on both sides:

$$1 - \cos \alpha = 1 - \cos^2 i - \sin^2 i \cos \Delta\Omega = \sin^2 i (1 - \cos \Delta\Omega) \implies 2 \sin^2 \frac{\alpha}{2} = 2 \sin^2 i \sin^2 \frac{\Delta\Omega}{2}.$$

All the half-angles lie between 0° and 90° , so we take the positive root, $\sin \frac{\alpha}{2} = \sin i \sin \frac{\Delta\Omega}{2}$.

The burn is made at a point common to both orbits, so on the line where the two planes meet. On a circle the velocity is perpendicular to $\mathbf{r}$, so at that point it is perpendicular to the line of intersection. Turning the plane through α about that line therefore turns the velocity through the full α , at constant speed. The isosceles triangle gives $\Delta v = 2v \sin \frac{\alpha}{2} = 2v \sin i \sin \frac{\Delta\Omega}{2}$.

- To find where to burn we must (1) find the direction of the line where the two planes meet, and (2) find the argument of latitude of that direction on each orbit.

Choose $\hat{\mathbf{I}}$ midway between the two nodes, so that $\Omega_1 = -\frac{1}{2}\Delta\Omega$ and $\Omega_2 = +\frac{1}{2}\Delta\Omega$. This makes the two orbits mirror images of each other, which halves the work. With $\hat{\mathbf{h}} = (\sin i \sin \Omega, -\sin i \cos \Omega, \cos i)$, the line of intersection is perpendicular to both normals, so it lies along their cross product:

$$\hat{\mathbf{h}}_1 \times \hat{\mathbf{h}}_2 = 2 \sin i \sin \frac{\Delta\Omega}{2} (0, \cos i, \sin i \cos \frac{\Delta\Omega}{2}).$$

Call the normalised direction $\hat{\mathbf{x}} = (0, \cos i, \sin i \cos \frac{\Delta\Omega}{2})/N$, with $N = \sqrt{\cos^2 i + \sin^2 i \cos^2 \frac{\Delta\Omega}{2}}$. Its K component is positive, so it points north.

Now locate $\hat{\mathbf{x}}$ on orbit 1. A point at argument of latitude u is at angle u from the ascending node $\hat{\mathbf{n}}_1 = (\cos \frac{\Delta\Omega}{2}, -\sin \frac{\Delta\Omega}{2}, 0)$, and it has $z/r = \sin i \sin u$. So we can read off both $\cos u_1$ and $\sin u_1$:

$$\cos u_1 = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{x}} = -\frac{\cos i \sin \frac{\Delta\Omega}{2}}{N}, \quad \sin u_1 = \frac{\hat{x}_K}{\sin i} = \frac{\cos \frac{\Delta\Omega}{2}}{N}.$$

Hence $\tan(u_1 - 90^\circ) = -\cos u_1 / \sin u_1 = \cos i \tan \frac{\Delta\Omega}{2}$, which is the stated u_1 . On orbit 2 the node is $\hat{\mathbf{n}}_2 = (\cos \frac{\Delta\Omega}{2}, +\sin \frac{\Delta\Omega}{2}, 0)$. That flips the sign of $\cos u$ and gives u_2 . The planes also meet at $-\hat{\mathbf{x}}$, 180° further on.

For a **polar** orbit, $\cos i = 0$ and $u_1 = u_2 = 90^\circ$. Every polar plane passes through the pole, so that is where they meet. For **small** $\Delta\Omega$, the burn is near $u = 90^\circ$, the most northerly point of the orbit: just past it on a prograde orbit, and just before it on a retrograde one.

(c) Changing the inclination through $\Delta\Omega$ turns the plane by $\Delta\Omega$ and costs $2v \sin \frac{\Delta\Omega}{2}$. The node change costs $\sin i$ times as much: never more, and the same only for a polar orbit. As $i \rightarrow 0$ the orbit plane is nearly the equator, and moving the node moves the plane hardly at all: $\alpha \rightarrow 0$. The element Ω swings round while the orbit barely changes. This is the degeneracy of the elements at $i = 0$ (*Orbital Elements*, “When the elements break down”).

10. An Oberth dive before escape

A spacecraft on a circle of radius r (speed $v_c = \sqrt{\mu/r}$) is to leave on a hyperbola of excess speed v_∞ . Route E burns prograde at r . Route O first burns retrograde to drop the periapsis to ρr ($0 < \rho < 1$), then burns prograde at that periapsis.

(a) Show that $\Delta v_E = \sqrt{v_\infty^2 + 2v_c^2} - v_c$ and

$$\Delta v_O(\rho) = v_c \left[1 - \sqrt{\frac{2\rho}{1+\rho}} \right] + \sqrt{v_\infty^2 + \frac{2v_c^2}{\rho}} - v_c \sqrt{\frac{2}{\rho(1+\rho)}}.$$

(b) Show that $d\Delta v_O/d\rho|_{\rho=1} = v_c \left[\frac{1}{2} - v_c/\sqrt{v_\infty^2 + 2v_c^2} \right]$, and that $\Delta v_O \rightarrow v_c$ as $\rho \rightarrow 0$.

(c) Hence show that the shallowest dive and the deepest dive both start to pay at the same excess speed: $v_\infty = \sqrt{2} v_c$, the local escape speed.

(d) Why can a dive never help a parabolic escape?

Answer. We compare a direct escape burn with “dive first, then burn”. The plan: write both costs (a); ask whether a slight dive helps, from the slope of the cost at $\rho = 1$, and whether the deepest dive helps, from the limit $\rho \rightarrow 0$ (b); then compare (c).

(a) *Route E.* On the hyperbola, energy conservation between r and infinity gives $\frac{1}{2}v^2 - \mu/r = \frac{1}{2}v_\infty^2$, so the speed needed at r is $\sqrt{v_\infty^2 + 2\mu/r} = \sqrt{v_\infty^2 + 2v_c^2}$. The burn is tangential, so $\Delta v_E = \sqrt{v_\infty^2 + 2v_c^2} - v_c$.

Route O. The first burn, at r , puts the spacecraft on the ellipse with apoapsis r and periapsis ρr . Its two apsidal speeds follow from *vis-viva* with $a = (1 + \rho)r/2$:

$$v_a = v_c \sqrt{\frac{2\rho}{1+\rho}}, \quad v_p = v_c \sqrt{\frac{2}{\rho(1+\rho)}}.$$

(As a check, $r v_a = \rho r v_p$.) The first burn slows the spacecraft from v_c to v_a and costs $v_c - v_a$. At periapsis the hyperbola needs the speed $\sqrt{v_\infty^2 + 2\mu/(\rho r)} = \sqrt{v_\infty^2 + 2v_c^2/\rho}$, so the second burn costs that minus v_p . Adding the two gives $\Delta v_O(\rho)$. At $\rho = 1$ the first burn is zero, and $\Delta v_O(1) = \Delta v_E$.

The dive is a trade. The first burn throws away speed. In return, the second burn is made deep in the field, at high speed, where energy is cheap.

(b) *A shallow dive.* Is $\Delta v_O(\rho)$ smaller than Δv_E for ρ a little below 1? It is if the slope of Δv_O at $\rho = 1$ is positive. The two root derivatives are those of Problem 7(b), with K replaced by ρ . At $\rho = 1$ they are

$$\frac{d}{d\rho} \sqrt{\frac{2\rho}{1+\rho}} = \frac{1}{\sqrt{2} \cdot 2^{3/2}} = \frac{1}{4}, \quad \frac{d}{d\rho} \sqrt{\frac{2}{\rho(1+\rho)}} = -\frac{3}{\sqrt{2} \cdot 2^{3/2}} = -\frac{3}{4}.$$

The hyperbola term gives $\frac{d}{d\rho}\sqrt{v_\infty^2 + 2v_c^2/\rho} = -\frac{v_c^2/\rho^2}{\sqrt{v_\infty^2 + 2v_c^2/\rho}}$. Putting the three together at $\rho = 1$,

$$\left.\frac{d\Delta v_O}{d\rho}\right|_{\rho=1} = v_c\left(-\frac{1}{4} + \frac{3}{4}\right) - \frac{v_c^2}{\sqrt{v_\infty^2 + 2v_c^2}} = v_c\left[\frac{1}{2} - \frac{v_c}{\sqrt{v_\infty^2 + 2v_c^2}}\right].$$

The deepest dive. As $\rho \rightarrow 0$, $v_a \rightarrow 0$, so the first burn tends to v_c : it removes all the speed. The second burn is the difference of two speeds that both grow without bound. To find its limit, multiply and divide by their sum (the usual trick for a difference of square roots):

$$\sqrt{v_\infty^2 + \frac{2v_c^2}{\rho}} - \sqrt{\frac{2v_c^2}{\rho(1+\rho)}} = \frac{v_\infty^2 + \frac{2v_c^2}{\rho}\left(1 - \frac{1}{1+\rho}\right)}{\sqrt{v_\infty^2 + \frac{2v_c^2}{\rho}} + \sqrt{\frac{2v_c^2}{\rho(1+\rho)}}} = \frac{v_\infty^2 + \frac{2v_c^2}{1+\rho}}{\sqrt{v_\infty^2 + \frac{2v_c^2}{\rho}} + \sqrt{\frac{2v_c^2}{\rho(1+\rho)}}}.$$

The numerator stays finite and the denominator grows like $\rho^{-1/2}$, so the second burn tends to 0, and $\Delta v_O \rightarrow v_c$. Deep down the periapsis speed is enormous, so any finite energy can be bought there for almost no Δv . The whole cost is in stopping first.

(c) *The shallow dive* helps if the slope in (b) is positive:

$$\frac{1}{2} > \frac{v_c}{\sqrt{v_\infty^2 + 2v_c^2}} \iff v_\infty^2 + 2v_c^2 > 4v_c^2 \iff v_\infty > \sqrt{2}v_c.$$

The deepest dive beats Route E if its cost v_c is less than Δv_E :

$$v_c < \sqrt{v_\infty^2 + 2v_c^2} - v_c \iff 2v_c < \sqrt{v_\infty^2 + 2v_c^2},$$

which is the same inequality. So both switch at $v_\infty = \sqrt{2}v_c = \sqrt{2\mu/r}$, the escape speed at the starting radius. (The script checks on a grid that no intermediate depth breaks this: below the threshold every dive loses, and above it a deeper dive is always cheaper.) In practice ρr is bounded below by the planet's radius or its atmosphere.

(d) A parabolic escape has $v_\infty = 0 < \sqrt{2}v_c$. The dive first destroys speed, which costs Δv , and gets it back by buying energy cheaply at the deep periapsis. That trade pays only if there is enough energy still to buy beyond escape, $\frac{1}{2}v_\infty^2$. For $v_\infty = 0$ there is none.