Orbital Elements

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Table of contents

1 Where we are going **1**

2 Why exactly six numbers? **3**

3 A frame to measure angles in **3**
3.1 Directions on the sky . . . 3
3.2 A year of noons . . . 4
3.3 The geocentric equatorial frame . . . 5
3.4 The axes do not hold still . . . 5
3.5 Every direction needs an epoch . . . 6

4 The three orientation angles **6**
4.1 Three vectors do all the work . . . 6
4.2 The definitions . . . 7
4.3 Orbit classes by inclination . . . 8
4.4 Sun-synchronous orbits . . . 8
4.5 When the elements break down . . . 9
4.6 The elements at a glance . . . 10

5 From $(\mathbf{r}, \mathbf{v})$ to the elements **10**
5.1 The algorithm . . . 10
5.2 Why arccos is not enough . . . 11
5.3 The remaining elements . . . 13

6 From the elements to $(\mathbf{r}, \mathbf{v})$ **13**
6.1 Build in the orbital plane, then rotate . . . 13

7 Worked examples **15**
7.1 Example 1: from a state to the elements . . . 15
7.2 Example 2: back again . . . 16
7.3 Example 3: the cost of one dropped test . . . 16
7.4 Example 4: all three tests fire . . . 17
7.5 Example 5: a hyperbola, forward . . . 17

8 Summary **18**

9 Exercises **18**

1 Where we are going

The two-body note gave us the orbit *in its own plane*: a conic

$$r = \frac{p}{1 + e\cos f}, \qquad p = \frac{h^2}{\mu} = a(1 - e^2),$$

with the attracting body at a focus and the true anomaly f measured from periapsis. That is enough to say how far away a satellite is and how fast it moves. It is not enough to point an antenna at it, to know which part of the Earth it flies over, or to tell whether two satellites might collide. For that we need to know **how the orbital plane, and the conic inside it, sit in space**.

This note answers that question. The orientation turns out to need three angles, so that an orbit is fixed by six numbers, the **classical orbital elements** (Figure 1): two for the conic (a, or p, and e), three for its orientation (i, Ω, ω), and one for where the body is on it (f).

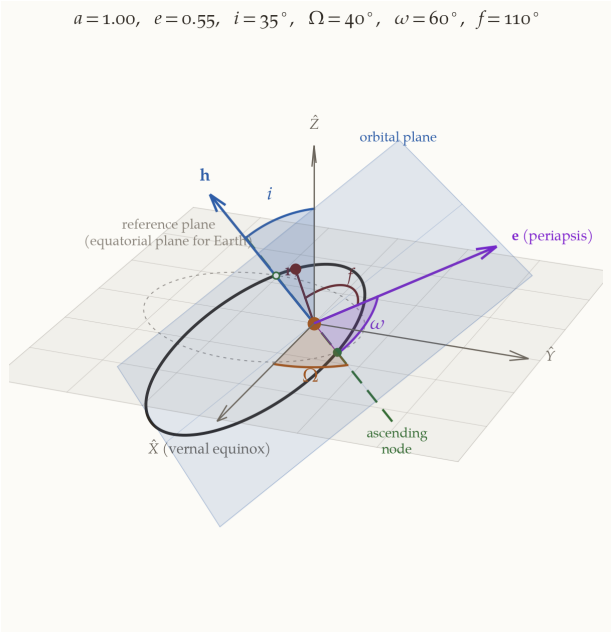

Figure 1: The six classical elements. Two fix the conic (a or p, and e), three fix the orientation of its plane and of the conic within it (i, Ω, ω), and one fixes where the body is (f). The dashed curve is the orbit projected onto the reference plane; the green dashed line is the line of nodes. The axes drawn as $\hat{X}, \hat{Y}, \hat{Z}$ are this note's $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$. *Script: orbital-elements-3d.py* (run it on your computer for sliders on all six).

The route:

1. **Section 2:** why exactly six numbers.
2. **Section 3:** angles need a reference frame; we build the standard one for Earth orbits, and see why even it is not perfectly fixed.
3. **Section 4:** we define i, Ω, ω, and show that three vectors, $\mathbf{h}$, $\mathbf{n}$ and $\mathbf{e}$, measure them.
4. **Section 5:** from a state vector $(\mathbf{r}, \mathbf{v})$ to the elements, where all but one of the angles comes out in the wrong quadrant half the time unless we are careful.
5. **Section 6:** from the elements back to $(\mathbf{r}, \mathbf{v})$.
6. **Section 7:** five worked examples.

As before, a blue **Try it** box asks you to do something, and a green **Check yourself** box asks a short question; each has a hidden answer. Throughout, $\mu = 398\,600$ km^3/s^2 (rounded, so that the examples can be followed by hand; the other notes use 398 600.4418) and $R_\oplus = 6378.137$ km; angles are quoted in degrees and computed in radians.

When you finish, you should be able to answer the following.

1. Why does an orbit need exactly six numbers, and why is one of them different from the other five?
2. What is the vernal equinox, and why must every orbit be given with an epoch?
3. Why is every sun-synchronous orbit retrograde?
4. When you compute Ω, ω and f from a state vector, how do you know which quadrant they are in?
5. Why does a quadrant error leave every *length* correct and still put the satellite on the other side of the Earth?

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

2 Why exactly six numbers?

The two-body equation $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3$ is a second-order vector ODE in three dimensions, i.e. six first-order scalar ODEs. Its general solution therefore carries **six** constants of integration. The state $(\mathbf{r}, \mathbf{v})$ at one instant is one legitimate choice of the six. The classical elements are another, built from the integrals we already have:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \;\; (3), \qquad \mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \hat{\mathbf{r}} \;\; (3), \qquad t_p \;\; (1).$$

That is seven numbers. Which relation removes one of them?

Answer

$\mathbf{h} \cdot \mathbf{e} = 0$: the eccentricity vector lies in the orbital plane. So $\mathbf{h}$ and $\mathbf{e}$ carry five independent numbers, and the time of periapsis passage t_p is the sixth.

5 constants that fix the orbit + 1 constant that fixes the clock = 6.

Five of the six elements describe a curve that does not change; the sixth says where on the curve the body is, and it is the only one that runs with time. Keep this split in mind: it organises everything below.

3 A frame to measure angles in

An angle is measured *from* something. Before we can orient an orbit we need fixed reference directions, and the honest answer to "fixed with respect to what?" turns out to be surprisingly subtle.

3.1 Directions on the sky

Astronomers describe directions as points on the **celestial sphere**: a sphere of arbitrary radius centred on the Earth, on which only direction matters. Four terms are needed.

- The **celestial equator** is the great circle in which the plane of the Earth's equator cuts the celestial sphere: the terrestrial equator projected onto the sky. Its poles are the celestial poles, directly above the Earth's poles.
- An **hour circle** is a great circle through both celestial poles and the object: the sky's version of a meridian of longitude. The **observer's meridian** is the hour circle through the observer's zenith; a body is at its highest (it *culminates*) when it crosses it, and for the Sun that moment is *apparent noon*.

- The **declination** $\delta \in [-90°, 90°]$ of an object is its angle from the celestial equator, measured along its hour circle, positive north: the sky's latitude.
- The **right ascension** $\alpha \in [0°, 360°)$ is the angle, measured eastward along the celestial equator, from a reference direction $\hat{\mathbf{I}}$ to the object's hour circle: the sky's longitude. It is often quoted in hours, $24^{\mathrm{h}} = 360°$.

In a Cartesian frame $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$ with $\hat{\mathbf{K}}$ along the Earth's axis, these are just spherical coordinates:

$$\hat{\mathbf{r}} = (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)\,.$$

So $\delta = 0$ means "in the equatorial plane", and α is measured from a direction in that plane that we have not yet chosen. How do we choose it? Look at the Sun.

3.2 A year of noons

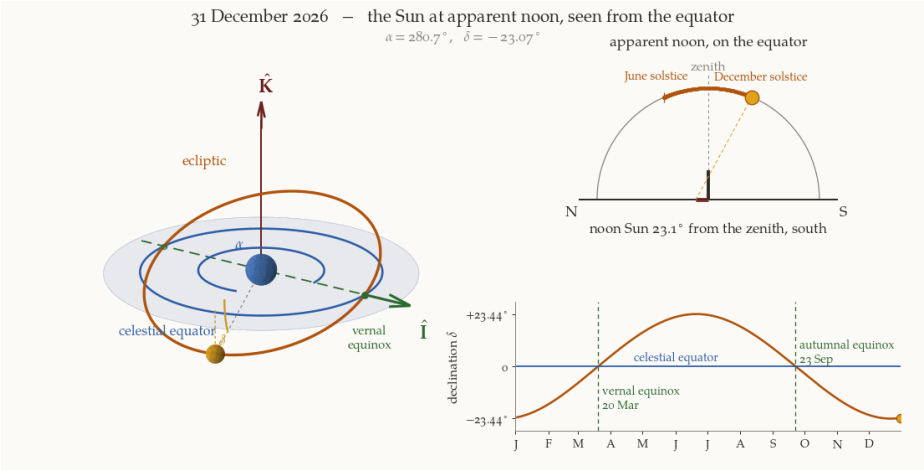

Figure 2: The Sun at apparent noon, seen from a point on the equator, every third day of 2026. *Script: sun-year-animation.py.*

Watch Figure 2. On the equator, at apparent noon, the zenith is on the celestial equator, so the Sun's distance from the zenith equals its declination $|\delta|$, and a vertical stick's noon shadow measures it. (a) Between what limits does δ swing over the year? (b) On how many days is there no noon shadow? (c) What would the noon zenith distance be at latitude φ?

What you should find

(a) Between $\pm 23.44°$: the Sun is **not** in the equatorial plane. It moves round the **ecliptic**, the plane of the Earth's orbit, which is tilted by the *obliquity* $\varepsilon = 23.44°$ to the equator.

(b) Two: the days when $\delta = 0$ and the Sun lies *in* the equatorial plane, the **equinoxes** (in 2026, 20 March and 23 September).

(c) $|\varphi - \delta|$.

The two planes, equator and ecliptic, meet in a line, and the Sun lies on that line at the two equinoxes. At the March equinox the Sun crosses the equator going **north**; this is the **vernal equinox**, and the direction from the Earth to the Sun at that moment is the reference direction $\hat{\mathbf{I}}$ (Figure 3). Right ascension is measured from it.

3.3 The geocentric equatorial frame

The standard frame for Earth orbits, often called ECI (Earth-centred inertial), is:

- origin at the centre of the Earth;
- $\hat{\mathbf{I}}$ towards the vernal equinox ♈;
- $\hat{\mathbf{K}}$ along the Earth's rotation axis, towards the north pole;
- $\hat{\mathbf{J}} = \hat{\mathbf{K}} \times \hat{\mathbf{I}}$, completing a right-handed set.

The $\hat{\mathbf{I}}\hat{\mathbf{J}}$ plane, the equator, is the **reference plane**. (For orbits about the Sun the reference plane is the ecliptic instead; every definition below is unchanged.)

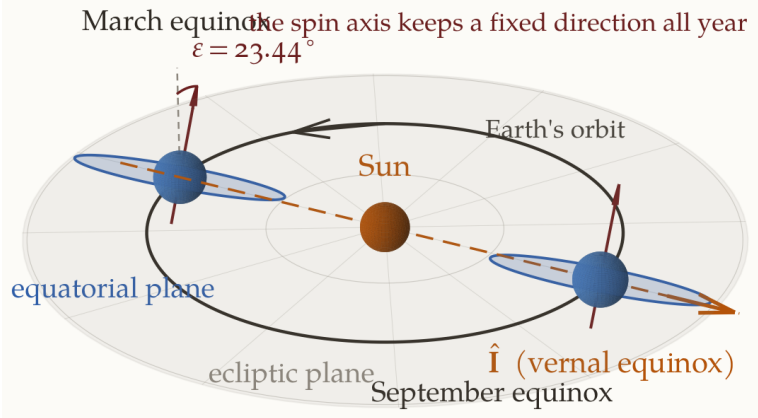

Figure 3: The Earth's equatorial plane is tilted by $\varepsilon = 23.44°$ to the ecliptic. The two planes meet in a line that cuts the orbit at the equinoxes. $\hat{\mathbf{I}}$ is the Earth-to-Sun direction at the March crossing and $\hat{\mathbf{K}}$ is the spin axis. Not to scale. *Script: fig-equinox-frame.py.*

3.4 The axes do not hold still

The frame is built from two planes, the equator and the ecliptic. If either moves, so do the axes. Both move.

The Earth's axis precesses. The Earth is slightly flattened, and the Sun and Moon pull harder on the near side of its equatorial bulge than on the far side. The resulting torque tries to tip the equator into the ecliptic; a spinning body answers a torque at right angles to it, so the axis instead sweeps out a **cone** of half-angle ε, at about 50.3″ per year: one turn in about 25 770 years (the Moon supplies about two thirds of the effect, the Sun one third). On top of the cone the axis **nods** (*nutation*), mainly with the 18.6-year period of the Moon's orbital node, by 9.2″ in obliquity and 17.2″ in longitude. Meanwhile the planets slowly tilt the ecliptic itself, by about 47″ per century, which makes the obliquity drift (it oscillates between about 22.1° and 24.5° over some 41 000 years).

So $\hat{\mathbf{I}}$, the intersection of two moving planes, slides westward along the ecliptic, about 1° every 72 years.

The vernal equinox is called the "First Point of Aries". At 1° per 72 years, how long does it take to cross one 30° sign of the zodiac? Is it still in Aries?

Answer

About 2150 years. It was in Aries when it was named, about two thousand years ago; it crossed into **Pisces** around 68 BC, is there now, and will reach Aquarius around 2600 AD. The name is a fossil.

Nor can a single star stand in for $\hat{\mathbf{I}}$. Stars move across the sky (Barnard's Star by 10.3″ per year), and even a star that does not move *appears* to move, by up to 20.5″ through the aberration of light and by up to about 1″ through parallax.

History: Hipparchus and precession. Hipparchus discovered precession around 130 BC by comparing his own observations of the star Spica with those made by Timocharis some 150 years earlier, and finding that its longitude had changed by about 2°.

There is one more motion, often confused with precession. The rotation axis is not exactly the Earth's axis of figure, so it wanders **through the crust**: the *Chandler wobble* (a free wobble of the non-rigid Earth, period about 433 days, amplitude 3–6 m at the pole) plus an annual term. This *polar motion* moves the ground under $\hat{\mathbf{K}}$, not $\hat{\mathbf{K}}$ among the stars; it enters the conversion from the inertial frame to Earth-fixed coordinates, not the inertial axes.

3.5 Every direction needs an epoch

Precession and nutation are why the frame is frozen at an instant. The standard choice is **J2000.0** (Julian date 2 451 545.0, 1 January 2000 at 12:00 TT): the mean equator and equinox of that moment.

Since J2000 the equinox has moved by about 26 years × 50.3″ per year. How large an angle is that? If an orbit is given in J2000 axes but used as if it were in "axes of date", the whole orbit is rotated about $\hat{\mathbf{K}}$ by that angle. How far does a satellite at $r = 7000$ km move?

What you should find

$26 \times 50.3'' = 1308'' = 0.36°$, and 7000×0.0063 rad ≈ 44 km. The semi-major axis, eccentricity, inclination and radius are all still correct; only Ω is wrong (to this approximation: the real precession is about the pole of the ecliptic, not $\hat{\mathbf{K}}$, so the inclination changes slightly too). A position error of tens of kilometres from a label.

The modern solution is the **International Celestial Reference Frame** (ICRF), whose axes are fixed by the positions of a few hundred distant quasars. Its definition contains no equator and no equinox, so nothing in it precesses; it agrees with J2000 to about 20 milliarcseconds (about 4 m at geostationary distance).

> Important
>
> An orbit needs **two** epochs: the epoch of the **frame** (which equator and equinox the components are measured in) and the epoch of the **state** (when the body was there). Two common traps: two-line element sets (TLEs) are given in a frame called TEME, not J2000, and JPL planetary ephemerides are in ICRF. Adding vectors from different frames produces a number, never an error message.

4 The three orientation angles

4.1 Three vectors do all the work

Given a state $(\mathbf{r}, \mathbf{v})$ in the frame $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$, three vectors carry all the orientation information:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} && \text{normal to the orbital plane,} \\ \mathbf{n} &= \hat{\mathbf{K}} \times \mathbf{h} && \text{along the line of nodes,} \\ \mathbf{e} &= \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \hat{\mathbf{r}} && \text{towards periapsis.} \end{aligned}$$

$\mathbf{h}$ is perpendicular to the orbital plane, so it *is* the plane. $\mathbf{n}$ is perpendicular to both $\hat{\mathbf{K}}$ and $\mathbf{h}$, so it lies in both the equatorial and the orbital plane: along their line of intersection, the **line of nodes**, where the orbit crosses the equator. And $\mathbf{e}$ points from the focus to periapsis, with length e.

The line of nodes is a line; it meets the orbit twice. The **ascending node** is the crossing where the body moves from south to north ($\dot{z} > 0$). Does $\mathbf{n}$ point to it, or to the other one?

At a node, $z = 0$. Show that $\mathbf{r} \cdot \mathbf{n} = r^2 \dot{z}$ there. (Cycle the triple product $\mathbf{r} \cdot (\hat{\mathbf{K}} \times \mathbf{h})$, then expand $\mathbf{h} \times \mathbf{r} = (\mathbf{r} \times \mathbf{v}) \times \mathbf{r}$ with the BAC–CAB rule.) What does this tell you?

What you should find

$\mathbf{r} \cdot \mathbf{n} = \mathbf{r} \cdot (\hat{\mathbf{K}} \times \mathbf{h}) = \hat{\mathbf{K}} \cdot (\mathbf{h} \times \mathbf{r})$, and $\mathbf{h} \times \mathbf{r} = (\mathbf{r} \times \mathbf{v}) \times \mathbf{r} = r^2 \mathbf{v} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{r}$. Taking the $\hat{\mathbf{K}}$ component, $\mathbf{r} \cdot \mathbf{n} = r^2 \dot{z} - (\mathbf{r} \cdot \mathbf{v}) z = r^2 \dot{z}$ at a node. So $\mathbf{r} \cdot \mathbf{n} > 0$ exactly when $\dot{z} > 0$: $\mathbf{n}$ points to the node where the body is climbing, the **ascending node**.

4.2 The definitions

- **Inclination** $i \in [0°, 180°]$: the angle between $\hat{\mathbf{K}}$ and $\mathbf{h}$; equivalently, the angle between the equatorial and orbital planes, measured about the line of nodes.
- **Right ascension of the ascending node** (RAAN) $\Omega \in [0°, 360°)$: the angle from $\hat{\mathbf{I}}$ to $\mathbf{n}$, measured **in the equatorial plane**, anticlockwise about $\hat{\mathbf{K}}$ (eastward).
- **Argument of periapsis** $\omega \in [0°, 360°)$: the angle from $\mathbf{n}$ to $\mathbf{e}$, measured **in the orbital plane**, in the direction of motion (anticlockwise about $\hat{\mathbf{h}}$).
- **True anomaly** $f \in [0°, 360°)$: the angle from $\mathbf{e}$ to $\mathbf{r}$, in the orbital plane, about $\hat{\mathbf{h}}$.

Notice the three different axes: Ω about $\hat{\mathbf{K}}$, i about the line of nodes, and ω and f about $\hat{\mathbf{h}}$. This is a "3–1–3" sequence of rotations, and it will give the rotation matrix of Section 6.

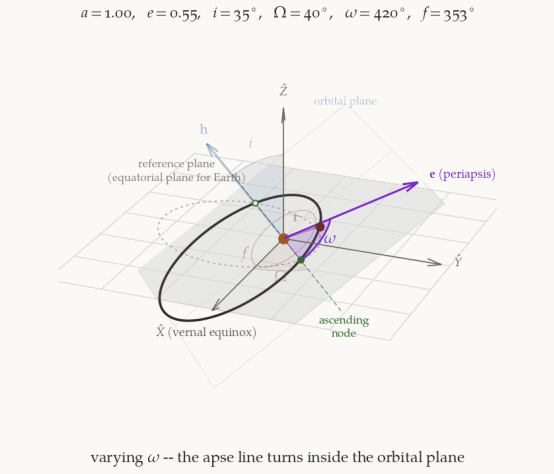

Figure 4: A tour of the three orientation angles, one at a time. Ω swings the line of nodes round the reference plane, i tilts the orbital plane about that line, and ω turns the line of apsides inside the plane. The conic never changes size or shape, only the way it is held. *Script: orbital-elements-3d.py.*

Which of the six elements can change the distance r of the body at a given true anomaly f? Which can change its speed?

Answer

Only a and e (through $r = a(1 - e^2)/(1 + e \cos f)$ and vis-viva). Changing i, Ω or ω moves the orbit **rigidly**: it cannot change any length or speed. Figure 4 is exactly this statement.

4.3 Orbit classes by inclination

i	name	remark
$0°$	equatorial, prograde	no line of nodes
$0° < i < 90°$	**prograde** (direct)	$h_z > 0$, moves eastward
$90°$	**polar**	passes over both poles
$90° < i < 180°$	**retrograde**	$h_z < 0$, moves westward
$180°$	equatorial, retrograde	no line of nodes

Two named cases are worth knowing. The Earth's flattening (measured by the coefficient $J_2 = 1.08263 \times 10^{-3}$; see the gravity note) makes the argument of periapsis drift at a rate proportional to $5\cos^2 i - 1$. At the **critical inclination** $i = 63.4°$ this vanishes, so the apogee stays put: Molniya orbits use it to keep their apogee over the northern hemisphere. The second case is the sun-synchronous orbit.

4.4 Sun-synchronous orbits

The same flattening also turns the orbital plane: the node drifts at

$$\dot{\Omega} = -\frac{3}{2}\,\frac{\sqrt{\mu}\,J_2 R_\oplus^2}{(1-e^2)^2 a^{7/2}}\,\cos i. \tag{1}$$

A satellite whose plane turns at exactly the rate at which the Sun moves round the sky,

$$\dot{\alpha}_\odot = \frac{360°}{365.2422\ \text{days}} = 0.9856° \text{ per day, eastward},$$

keeps the same orientation relative to the Sun all year (Figure 5). Such an orbit is called **sun-synchronous**: it crosses each latitude at the same *local solar time* on every pass.

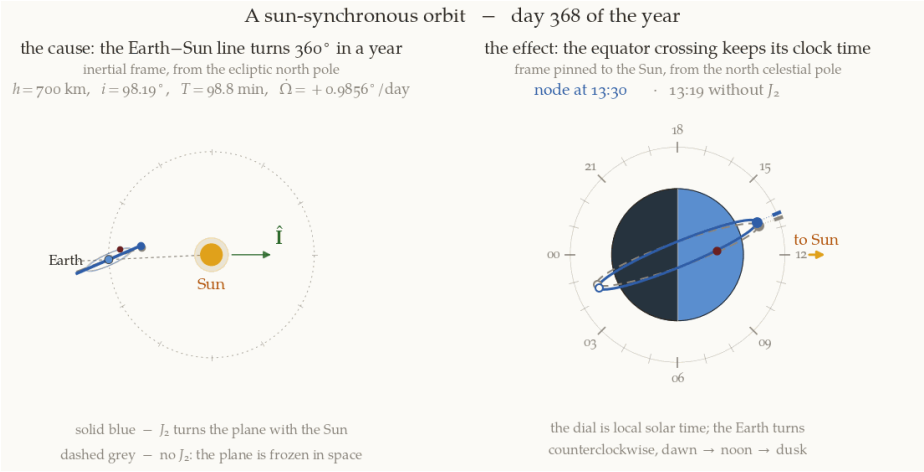

Figure 5: A sun-synchronous orbit does not follow the Sun; its *plane* does. **Left,** inertial: the Earth–Sun line turns once a year. **Right,** in a frame that turns with the Sun: the blue plane, turned by J_2 at $+0.9856°$/day, keeps a 13:30 local time all year; the grey plane, on a spherical Earth, drifts through all 24 hours. *Script: sso-animation.py.*

In Equation 1, everything except $\cos i$ is positive. What sign must $\cos i$ have for a sun-synchronous orbit? What does that say about the inclination?

Answer

The node must move **eastward**, $\dot{\Omega} > 0$, so $\cos i < 0$, i.e. $i > 90°$. **Every sun-synchronous orbit is retrograde.** That is where the familiar $i \approx 98°$ comes from; it is not a convention.

Since $\dot{\Omega}$ is linear in $\cos i$, setting $\dot{\Omega} = \dot{\alpha}_\odot$ gives the inclination directly:

$$\boxed{\cos i = -\frac{2\,\dot{\alpha}_\odot\,(1-e^2)^2 a^{7/2}}{3\sqrt{\mu}\,J_2 R_\oplus^2}.}$$

Compute the sun-synchronous inclination and the period of circular orbits at 400, 600, 800, 1000 and 1200 km altitude. (Convert $\dot{\alpha}_\odot$ to rad/s.) Then find the altitude at which the right-hand side reaches -1.

What you should find

altitude	a [km]	i	T [min]
400 km	6778.1	97.03°	92.6
600 km	6978.1	97.79°	96.7
800 km	7178.1	98.60°	100.9
1000 km	7378.1	99.48°	105.1
1200 km	7578.1	100.42°	109.4

The right-hand side grows as $a^{7/2}$ and reaches -1 ($i = 180°$) at $a = 12\,352$ km, i.e. 5974 km altitude. Above that, J_2 cannot turn a plane fast enough and no sun-synchronous orbit exists. Note that a and i are no longer independent: altitude alone fixes the inclination.

What it buys. The same sunlight on every pass: the same Sun elevation in every image, so pictures taken years apart are comparable (Landsat and Sentinel-2 fly such orbits); a thermal and solar-array geometry that never drifts; and a free choice of the local time itself, about 10:30 for optical imaging, or a dawn–dusk orbit (06:00/18:00) for radar satellites that want their solar panels always lit. **What it costs.** The inclination is spent: it cannot also be chosen for coverage, and a retrograde launch gives up the eastward boost from the Earth's rotation (about 0.46 km/s at the equator) that a due-east launch gets for free.

Is a satellite in a 13:30 sun-synchronous orbit always in sunlight?

Answer

No. "Sun-synchronous" fixes the angle between the orbital *plane* and the Sun, not the satellite's illumination. A 13:30 orbit passes through the Earth's shadow on every revolution. Only the dawn–dusk orbit, whose plane is nearly perpendicular to the Earth–Sun line, avoids eclipses, and even it does so only for part of the year.

4.5 When the elements break down

Two of the orientation angles are measured from *directions* that can cease to exist.

- **Circular orbit,** $e = 0$**.** Then $\mathbf{e} = \mathbf{0}$: there is no periapsis, so ω and f are both undefined. Their **sum** is not: the *argument of latitude* $u = \omega + f$, the angle from the ascending node to the body, with $\cos u = \mathbf{n}\cdot\mathbf{r}/(nr)$ and $u > 180°$ if $z < 0$.

- **Equatorial orbit,** $i = 0°$ **or** $180°$**.** Then $\mathbf{n} = \mathbf{0}$: there is no node, so Ω and ω are undefined, but the *longitude of periapsis* ϖ, the angle of $\mathbf{e}$ from $\hat{\mathbf{I}}$ measured about $+\hat{\mathbf{K}}$, is not: $\cos\varpi = e_I/e$, with $\varpi > 180°$ if $e_J < 0$. For a prograde orbit ($i = 0°$) it equals $\Omega + \omega$; for a retrograde one ($i = 180°$) the in-plane angles run the other way round $\hat{\mathbf{K}}$, and it equals $\Omega - \omega$.
- **Both**: use the *true longitude* λ, the angle of $\mathbf{r}$ from $\hat{\mathbf{I}}$ about $+\hat{\mathbf{K}}$: $\cos\lambda = r_I/r$, with $\lambda > 180°$ if $r_J < 0$. It equals $\Omega + \omega + f$ for $i = 0°$ and $\Omega - \omega - f$ for $i = 180°$.

These are legitimate only because the angle between the two planes in which their pieces are measured has collapsed to zero. Numerically the trouble starts *before* the exact degeneracy: for small e the direction of $\mathbf{e}$ is poorly determined, and for small i the direction of $\mathbf{n}$ is. Software either tests for it ($e < 10^{-8}$, $\sin i < 10^{-8}$) or uses a non-singular set such as the **equinoctial elements** $\left(a,\ e\cos\varpi,\ e\sin\varpi,\ \tan\frac{i}{2}\cos\Omega,\ \tan\frac{i}{2}\sin\Omega,\ \text{a longitude}\right)$, singular only for $i = 180°$.

4.6 The elements at a glance

	element	what it fixes	constant in the two-body problem?
size	a	the energy, $\varepsilon = -\mu/2a$	yes
shape	e	the type and flattening of the conic	yes
	i	tilt of the plane	yes
orientation	Ω	swing of the plane about $\hat{\mathbf{K}}$	yes
	ω	turn of the apse line in the plane	yes
position	f (or M, or t_p)	where the body is	**no**

Only the last runs with time. Under perturbations (J_2, drag, the Sun and Moon) the first five drift *slowly*, which is why this set is the natural language of perturbation theory: the fast variable is isolated in one coordinate.

5 From $(\mathbf{r}, \mathbf{v})$ to the elements

5.1 The algorithm

Tracking gives a state vector; catalogues want elements. Given $\mathbf{r}$, $\mathbf{v}$ in $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$ and μ:

#	quantity	formula	quadrant test
1	radius, speed	$r = \lvert\mathbf{r}\rvert$, $v = \lvert\mathbf{v}\rvert$	
2	radial speed	$v_r = \mathbf{r}\cdot\mathbf{v}/r$	(used in 9)
3	angular momentum	$\mathbf{h} = \mathbf{r}\times\mathbf{v}$, $h = \lvert\mathbf{h}\rvert$	
4	**inclination**	$i = \arccos(h_K/h)$	**none needed**
5	node vector	$\mathbf{n} = \hat{\mathbf{K}}\times\mathbf{h} = (-h_J,\ h_I,\ 0)$	

#	quantity	formula	quadrant test
6	**RAAN**	$\Omega = \arccos(n_I/n)$	if $n_J < 0$: $\Omega \to 360^\circ - \Omega$
7	eccentricity	$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}\right]$	
8	**argument of periapsis**	$\omega = \arccos\frac{\mathbf{n}\cdot\mathbf{e}}{ne}$	if $e_K < 0$: $\omega \to 360^\circ - \omega$
9	**true anomaly**	$f = \arccos\frac{\mathbf{e}\cdot\mathbf{r}}{er}$	if $v_r < 0$: $f \to 360^\circ - f$
10	size	$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1}$, $p = \frac{h^2}{\mu}$	

Guards. Steps 6, 8 and 9 divide by n and e, so the table fails for the special orbits of Section 4.5. If $n \approx 0$ (equatorial orbit), set $\Omega = 0$ and report the longitude of periapsis ϖ instead of ω; if $e \approx 0$ (circular orbit), set $\omega = 0$ and report the argument of latitude u instead of f; if both, report the true longitude λ. Near $e = 1$, step 10 gives a huge a with few correct digits: carry p, which is always well defined. Figure 6 shows the steps on a worked example.

Show that the formula in step 7 is the same as $\mathbf{e} = \mathbf{v}\times\mathbf{h}/\mu - \hat{\mathbf{r}}$. (Expand $\mathbf{v}\times(\mathbf{r}\times\mathbf{v})$.)

What you should find

$\mathbf{v}\times(\mathbf{r}\times\mathbf{v}) = v^2\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}$, so $\mathbf{e} = \frac{1}{\mu}\left[v^2\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}\right] - \frac{\mathbf{r}}{r}$, and $\frac{\mathbf{r}}{r} = \frac{1}{\mu}\frac{\mu}{r}\mathbf{r}$. The form in the table needs no cross product, which is convenient by hand.

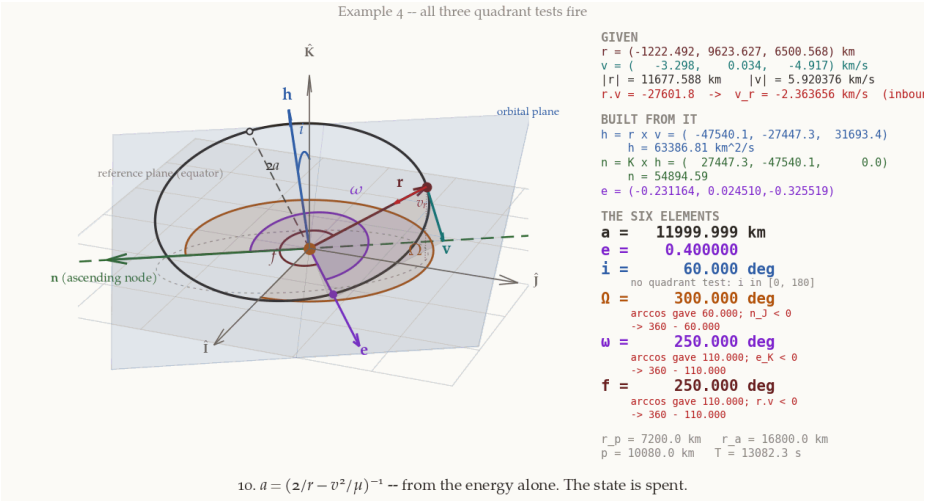

Figure 6: The algorithm, step by step. **h** fixes the plane, **n** the line of nodes, **e** the line of apsides, and only then, with $p = h^2/\mu$, is the conic itself determined. This is Example 4 of Section 7, where all three quadrant tests fire: watch the figure contradict the raw arccos three times. *Script: rv-to-elements.py.*

Steps 4, 6, 8 and 9 produce the four angles, and three of the four carry a quadrant test. Where do the tests come from, and why does i escape?

5.2 Why arccos is not enough

Every angle in the table comes from a dot product, i.e. from its **cosine**. But $\cos\theta = \cos(360^\circ - \theta)$, and arccos always returns the answer in $[0^\circ, 180^\circ]$. It cannot tell θ from $360^\circ - \theta$.

To decide, we need one more bit of information: the **sign of the sine**. For an angle measured from **a** to **b** about a unit axis $\hat{\mathbf{k}}$ (with both vectors perpendicular to $\hat{\mathbf{k}}$),

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}, \qquad \sin\theta = \frac{(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{k}}}{ab}.$$

So for each angle, compute that triple product and keep its sign. We now show that in every case it reduces to a single number we already have.

i **needs no test**, for a definitional reason. The inclination is *defined* on $[0°, 180°]$, and on that interval $\sin i \geq 0$, so the sign of the sine carries no information. (The price is that i itself must carry the distinction between eastward and westward orbits, and Ω must run over the full circle.) Note also that $n = |\hat{\mathbf{K}}\times\mathbf{h}| = h\sin i \geq 0$.

Ω**: the sign of** n_J**.** $\mathbf{n}$ lies in the equatorial plane, at angle Ω from $\hat{\mathbf{I}}$, so $\mathbf{n} = n(\cos\Omega, \sin\Omega, 0)$ and $\sin\Omega = n_J/n$. Since $n > 0$,

$$\boxed{\operatorname{sign}(\sin\Omega) = \operatorname{sign}(n_J), \qquad \Omega = \operatorname{atan2}(n_J,\ n_I).}$$

Find the corresponding tests for ω and f. For ω, measured from $\mathbf{n}$ to $\mathbf{e}$ about $\hat{\mathbf{h}}$, compute $(\mathbf{n}\times\mathbf{e})\cdot\hat{\mathbf{h}}$ using $\mathbf{n} = \hat{\mathbf{K}}\times\mathbf{h}$ and $\mathbf{h}\cdot\mathbf{e} = 0$. For f, measured from $\mathbf{e}$ to $\mathbf{r}$, compute $(\mathbf{e}\times\mathbf{r})\cdot\hat{\mathbf{h}}$ using the definition of $\mathbf{e}$ and $\mathbf{h}\cdot\mathbf{r} = 0$.

What you should find

With $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$,

$$\mathbf{n}\times\mathbf{e} = (\hat{\mathbf{K}}\times\mathbf{h})\times\mathbf{e} = \mathbf{h}\,(\hat{\mathbf{K}}\cdot\mathbf{e}) - \hat{\mathbf{K}}\,(\mathbf{h}\cdot\mathbf{e}) = e_K\,\mathbf{h},$$

so $\sin\omega = he_K/(ne) = e_K/(e\sin i)$: **the sign of** e_K. Periapsis is north of the equator exactly when $\omega < 180°$.

The $\hat{\mathbf{r}}$ part of $\mathbf{e}$ contributes nothing to $\mathbf{e}\times\mathbf{r}$, so

$$\mathbf{e}\times\mathbf{r} = \frac{(\mathbf{v}\times\mathbf{h})\times\mathbf{r}}{\mu} = \frac{\mathbf{h}(\mathbf{v}\cdot\mathbf{r}) - \mathbf{v}(\mathbf{h}\cdot\mathbf{r})}{\mu} = \frac{\mathbf{r}\cdot\mathbf{v}}{\mu}\,\mathbf{h},$$

so $\sin f = h\,v_r/(\mu e)$: **the sign of** v_r, i.e. of $\mathbf{r}\cdot\mathbf{v}$. The body is past periapsis and climbing exactly when $f < 180°$, which is the $e\sin f$ in $v_r = (\mu/h)e\sin f$.

Collecting the three, with both arguments of each atan2 scaled consistently (sin and cos multiplied by the same positive number):

$$\begin{aligned}
\Omega &= \operatorname{atan2}\left(n_J,\ n_I\right),\\
\omega &= \operatorname{atan2}\left(\frac{e_K}{\sin i},\ \frac{\mathbf{n}\cdot\mathbf{e}}{n}\right),\\
f &= \operatorname{atan2}\left(\frac{h\,v_r}{\mu},\ \frac{\mathbf{e}\cdot\mathbf{r}}{r}\right).
\end{aligned} \tag{2}$$

The ω line still divides by $\sin i$, which is small exactly where Section 4.5 warns of trouble. Multiplying both of its arguments by $h\sin i = n$, and both arguments of the f line by μr, gives forms with no division at all:

$$\omega = \operatorname{atan2}(h\,e_K,\ \mathbf{n}\cdot\mathbf{e}), \qquad f = \operatorname{atan2}(h\,(\mathbf{r}\cdot\mathbf{v}),\ h^2 - \mu r),$$

and the second does not even need $\mathbf{e}$ (use $h^2 - \mu r = \mu r\,e\cos f$, which is the orbit equation).

angle	flip when	meaning
Ω	$n_J < 0$	ascending node in the half of the equator with $J < 0$
ω	$e_K < 0$	periapsis south of the equator
f	$\mathbf{r} \cdot \mathbf{v} < 0$	body inbound, falling towards periapsis

In the f formula, why must the second argument be $\mathbf{e} \cdot \mathbf{r}/r$ and not $\mathbf{e} \cdot \mathbf{r}/e$?

Answer

atan2(y, x) returns the angle of the point (x, y), so y and x must be $\sin f$ and $\cos f$ times the *same* positive factor. $hv_r/\mu = e \sin f$, and $\mathbf{e} \cdot \mathbf{r}/r = e \cos f$: both are scaled by e. With $\mathbf{e} \cdot \mathbf{r}/e = r \cos f$ instead, the point would be $(r \cos f,\ e \sin f)$, whose angle is not f. The quadrant would still be right, but the angle would be wrong. (The same care applies to ω: $e_K / \sin i = e \sin \omega$ and $\mathbf{n} \cdot \mathbf{e}/n = e \cos \omega$.)

There is a second reason to prefer atan2: arccos is **ill-conditioned** near $0°$ and $180°$. From $\delta\theta = -\delta(\cos\theta)/\sin\theta$, an error ϵ in the cosine becomes an error of order $\sqrt{2\epsilon}$ in the angle as $\theta \to 0$. With $\epsilon = 10^{-16}$ (double precision) that is 10^{-8} rad: half the significant digits lost. atan2 has no such loss, and it also survives a cosine that rounding has pushed just outside $[-1, 1]$, where arccos returns "not a number".

5.3 The remaining elements

The scalars come straight from the integrals:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}, \qquad a = -\frac{\mu}{2\varepsilon}, \qquad p = \frac{h^2}{\mu}, \qquad e = \sqrt{1 - \frac{p}{a}},$$

and, for an ellipse, $r_p = a(1 - e)$, $r_a = a(1 + e)$ and $T = 2\pi\sqrt{a^3/\mu}$. Some books (Curtis, for example) use h instead of a as the size element. That is the better choice near the parabola, where $a \to \infty$ while h and p stay finite.

6 From the elements to $(\mathbf{r}, \mathbf{v})$

6.1 Build in the orbital plane, then rotate

The forward map is easier, and comes in two steps.

Step 1: the conic in its own plane. In the *perifocal* frame $\{\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}\}$ ($\hat{\mathbf{p}}$ towards periapsis, $\hat{\mathbf{w}}$ along $\mathbf{h}$; see the perifocal note, which derives them)

$$\mathbf{r}_{PQW} = \frac{p}{1 + e\cos f} \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix}, \qquad \mathbf{v}_{PQW} = \frac{\mu}{h} \begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix}.$$

Step 2: rotate into the inertial frame. Apply the 3–1–3 sequence of rotations to the perifocal triad (derived step by step in the perifocal note):

$$\mathbf{Q} = \mathbf{R}_3(\Omega)\, \mathbf{R}_1(i)\, \mathbf{R}_3(\omega), \qquad \mathbf{r} = \mathbf{Q}\, \mathbf{r}_{PQW}, \qquad \mathbf{v} = \mathbf{Q}\, \mathbf{v}_{PQW},$$

where $\mathbf{R}_1(\theta)$ and $\mathbf{R}_3(\theta)$ rotate a vector by θ about the first and third axes. Read the product from right to left: ω turns $\hat{\mathbf{p}}$ from the line of nodes to periapsis, i tilts the plane up from the equator about the line of nodes, and Ω swings the line of nodes round from $\hat{\mathbf{I}}$. Figure 7 shows the other reading, left to right.

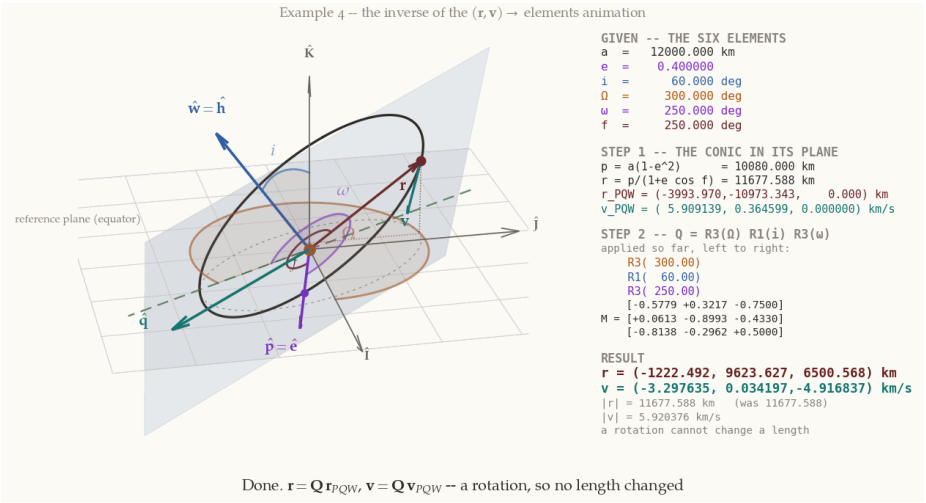

Figure 7: The same product read left to right, as three rotations of the triad, each about an axis the previous one has just put in place: $\hat{\mathbf{K}}$, then the frame's own first axis (now the line of nodes), then its own third axis (now $\mathbf{h}$). The matrix in the panel fills in as the triad moves; at the end it is $\mathbf{Q}$. *Script: elements-to-rv.py.*

Multiplying out (the perifocal note does it line by line), with $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$,

$$\mathbf{Q} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega s_\omega c_i & -c_\Omega s_\omega - s_\Omega c_\omega c_i & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega s_\omega c_i & -s_\Omega s_\omega + c_\Omega c_\omega c_i & -c_\Omega s_i \\ s_\omega s_i & c_\omega s_i & c_i \end{bmatrix}. \tag{3}$$

Check Equation 3 in three ways. (a) The third column should be $\hat{\mathbf{h}}$ in inertial components; why should it not contain ω? (b) The third row holds the out-of-plane components of $\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}$; what should happen to it when $i = 0$? (c) What should $\mathbf{Q}$ be when $i = \Omega = \omega = 0$?

What you should find

(a) $(\sin\Omega\sin i,\ -\cos\Omega\sin i,\ \cos i)$, with no ω: turning the apse line inside the plane cannot move the plane's normal. (b) Its first two entries carry $\sin i$ and vanish, and the third is 1: for an equatorial orbit $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ lie in the equator. (c) The identity matrix. $\mathbf{Q}$ is orthogonal, $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathsf{T}}$, with $\det\mathbf{Q} = +1$.

Convention trap

This $\mathbf{Q}$ maps perifocal components to inertial ones. Many books tabulate the matrix that maps the other way, which is its transpose (Curtis writes out the geocentric-to-perifocal matrix, for example), built from rotations of the *frame* rather than of the vector. The transpose of a product reverses its order: $\mathbf{Q}^{\mathsf{T}} = \mathbf{R}_3(-\omega)\mathbf{R}_1(-i)\mathbf{R}_3(-\Omega)$. Flipping the signs of a formula from such a book but keeping its order swaps the roles of Ω and ω. Always test a matrix against cases you know: $i = \Omega = \omega = 0 \Rightarrow \mathbf{Q} = \mathbf{1}$, and, with $i = 90°, \Omega = 90°, \omega = 0 \Rightarrow \hat{\mathbf{p}} = \hat{\mathbf{J}}$ but $\Omega = 0, \omega = 90° \Rightarrow \hat{\mathbf{p}} = \hat{\mathbf{K}}$.

The forward map has **no quadrant problem**: it uses only sines and cosines of known angles, with no inverse trigonometry anywhere. All the difficulty of Section 5 came from inverting it. Together with Kepler's equation, the two maps close the propagation loop:

$$(\mathbf{r},\mathbf{v})_{t_0} \to (a,e,i,\Omega,\omega,f_0) \to M_0 \to M(t) \to E \to f \to (\mathbf{r},\mathbf{v})_t,$$

in which only the last element changed. Five elements are the orbit; the sixth is the clock.

7 Worked examples

7.1 Example 1: from a state to the elements

Given $\mathbf{r} = (-6045,\ -3490,\ 2500)$ km and $\mathbf{v} = (-3.457,\ 6.618,\ 2.533)$ km/s.

Run the algorithm yourself before reading the solution below. Keep six significant figures.

Steps 1–3. $r = 7414.319$ km, $v = 7.884470$ km/s, $\mathbf{r}\cdot\mathbf{v} = 4133.245$ km^2/s, so $v_r = +0.557468$ km/s (outbound). Then

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = (-25\,385.17,\ 6669.485,\ -52\,070.74)\ \text{km}^2/\text{s},$$

with $h = 58\,311.67$ km^2/s.

Step 4. $i = \arccos(-52\,070.74/58\,311.67) = \arccos(-0.892973) = 153.249°$. More than 90°: the orbit is **retrograde**.

Steps 5–6. $\mathbf{n} = (-h_J,\ h_I,\ 0) = (-6669.485,\ -25\,385.17,\ 0)$, $n = 26\,246.69$. $\arccos(n_I/n) = \arccos(-0.254108) = 104.721°$. But $n_J < 0$, so the ascending node is in the half of the equator with $J < 0$, and $\Omega = 360° - 104.721° = 255.279°$. (Directly: atan2$(-25\,385.17,\ -6669.485) = -104.721° \equiv 255.279°$.)

Step 7. With $v^2 - \mu/r = 62.16486 - 53.76084 = 8.40402$ km^2/s^2,

$$\mathbf{e} = \frac{1}{\mu}\left[8.40402\,\mathbf{r} - 4133.245\,\mathbf{v}\right] = (-0.091605,\ -0.142207,\ 0.026444),$$

so $e = 0.171212$.

Step 8. $\cos\omega = \mathbf{n}\cdot\mathbf{e}/(ne) = 4220.92/(26\,246.69\times 0.171212) = 0.939284$, so arccos $= 20.068°$. Test: $e_K > 0$, periapsis is north of the equator, and $\omega = 20.068°$ stands.

Step 9. $\cos f = \mathbf{e}\cdot\mathbf{r}/(er) = 1116.16/(0.171212\times 7414.319) = 0.879270$, so arccos $= 28.446°$; $v_r > 0$, so $f = 28.446°$ stands.

Step 10. $\varepsilon = 31.08243 - 53.76084 = -22.67841$ km^2/s^2, so $a = 8788.10$ km.

$a = 8788.10$ km, $\quad e = 0.171212$, $\quad i = 153.249°$,
$\Omega = 255.279°$, $\quad \omega = 20.068°$, $\quad f = 28.446°$

Sanity-check the answer: is the perigee above the atmosphere, and is $r_p < r < r_a$?

Answer

$r_p = a(1-e) = 7283.46$ km (905 km altitude), $r_a = a(1+e) = 10\,292.73$ km, and $r = 7414$ km lies between them, just past perigee as $f = 28°$ says. Also $p = h^2/\mu = 8530.48$ km and $T = 8198.9$ s $= 2.28$ h.

7.2 Example 2: back again

Take the six numbers just found and run the forward map. With $p = 8530.48$ km and $f = 28.446°$,

$$\mathbf{r}_{PQW} = \begin{bmatrix} 6519.19 \\ 3531.62 \\ 0 \end{bmatrix} \text{ km}, \qquad \mathbf{v}_{PQW} = \begin{bmatrix} -3.25600 \\ 7.18076 \\ 0 \end{bmatrix} \text{ km/s}.$$

Before rotating: what should $|\mathbf{r}_{PQW}|$ and $|\mathbf{v}_{PQW}|$ be?

Answer

7414.32 km and 7.88447 km/s, the r and v of Example 1. A rotation cannot change a length.

With $c_\Omega = -0.254107$, $s_\Omega = -0.967176$, $c_i = -0.892973$, $s_i = 0.450111$, $c_\omega = 0.939284$, $s_\omega = 0.343140$, Equation 3 gives

$$\mathbf{Q} = \begin{bmatrix} -0.535036 & -0.724029 & -0.435336 \\ -0.830591 & 0.545011 & 0.114377 \\ 0.154451 & 0.422782 & -0.892973 \end{bmatrix},$$

and then $\mathbf{r} = \mathbf{Q}\mathbf{r}_{PQW} = (-6045.00,\ -3490.01,\ 2500.00)$ km and $\mathbf{v} = \mathbf{Q}\mathbf{v}_{PQW} = (-3.45700,\ 6.61800,\ 2.53300)$ km/s. The round trip closes to about 10 m in position, which is entirely the six-figure rounding: carried in double precision, the residual is below 10^{-8} km. A round trip is the cheapest test that exercises all three quadrant rules at once.

7.3 Example 3: the cost of one dropped test

Suppose that in Example 1 we had trusted arccos and kept $\Omega = 104.721°$, with everything else right.

Run the forward map with $\Omega = 104.721°$ and the other five elements of Example 1. How far is the result from the true $\mathbf{r}$? What is $|\mathbf{r}|$?

What you should find

$\mathbf{r} = (3548.89,\ 6010.62,\ 2500.00)$ km instead of $(-6045,\ -3490,\ 2500)$ km: 13 502 km away, nearly twice the orbital radius, on the other side of the Earth. Yet $|\mathbf{r}| = 7414.32$ km, and a, e, i, f are all still exactly right.

This is the signature of a quadrant error: **every scalar checks out and the vector is nonsense.** A change of Ω rotates the whole orbit about $\hat{\mathbf{K}}$, preserving every length and every angle within the orbit. No magnitude check can catch it; only a component-by-component comparison can.

Common mistakes with the elements

- **Quadrant flips.** arccos returns $[0, 180°]$. Each of Ω, ω and f needs its sign test (n_Y, e_Z, v_r), or the division-free atan2 forms. Example 3 shows what one dropped test costs.
- **The epoch of the frame.** "Equatorial coordinates" means nothing until the equinox and equator are dated: J2000, the mean equator of date, or the true equator of date. In 2026 precession alone has moved the equinox about 0.37° from J2000, some 46 km at a LEO radius.
- **TEME.** Two-line element sets and SGP4 work in the *true equator, mean equinox* frame, which is none of the above. Convert before comparing with a J2000 state.
- **Active against passive matrices.** $\mathbf{Q} = \mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega)$ here turns

vectors. Books that turn the *frame* write the transpose, with the order of the three factors reversed. Check with $i = 90°$, comparing $\Omega = 90°$, $\omega = 0$ against $\Omega = 0$, $\omega = 90°$, as the perifocal note does.

7.4 Example 4: all three tests fire

Given $\mathbf{r} = (-1222.492,\ 9623.627,\ 6500.568)$ km and $\mathbf{v} = (-3.297635,\ 0.034197,\ -4.916837)$ km/s. Then $r = 11\,677.59$ km, $v = 5.920376$ km/s, $\mathbf{r} \cdot \mathbf{v} = -27\,601.8$, so $v_r = -2.363656$ km/s (**inbound**), and

$$\begin{aligned} \mathbf{h} &= (-47\,540.11,\ -27\,447.29,\ 31\,693.40), & h &= 63\,386.81, \\ \mathbf{n} &= (27\,447.29,\ -47\,540.11,\ 0), & n &= 54\,894.59, \\ \mathbf{e} &= (-0.231164,\ 0.024510,\ -0.325519), & e &= 0.4, \end{aligned}$$

and $i = \arccos(h_K/h) = \arccos(0.5) = 60.000°$.

Compute Ω, ω and f, applying the quadrant tests. Then find a, r_p, r_a and T.

What you should find

angle	cos	arccos	test	value	corrected
Ω	$+0.500000$	$60.000°$	n_J	$-47\,540 < 0$	**300.000°**
ω	-0.342020	$110.000°$	e_K	$-0.3255 < 0$	**250.000°**
f	-0.342020	$110.000°$	$\mathbf{r} \cdot \mathbf{v}$	$-27\,602 < 0$	**250.000°**

Every one of the three raw answers was wrong. $a = (2/r - v^2/\mu)^{-1} = 12\,000$ km, $r_p = 7200$ km, $r_a = 16\,800$ km and $T = 13\,082.3$ s $= 3.634$ h.

The body is inbound at $f = 250°$. How long until perigee?

Answer

Convert f to E with the quadrant-safe form $E = 2\,\mathrm{atan2}\left(\sqrt{1-e}\sin\frac{f}{2},\ \sqrt{1+e}\cos\frac{f}{2}\right) = 273.851°$. (The plain arctan of the half-angle ratio gives $E = -86.15°$, which is the same angle as $273.85°$: halving and then doubling turns the $180°$ ambiguity of arctan into $360°$, so the half-angle formula is never in the wrong quadrant. The real hazard is not wrapping E, and then M, into $[0°, 360°)$: with $M = -63.28°$ the time-to-perigee formula below would be off by a whole period.) Then $\sin E = -0.997742$ and $M = E - e\sin E = 273.851° + 22.867° = 296.718°$ (the term $e\sin E$ is in radians, 0.399 rad $= 22.867°$). Perigee is at $M = 360°$, so $\Delta t = (360 - 296.718)/360 \times 13\,082.3 = 2299.6$ s $= 38.3$ min.

7.5 Example 5: a hyperbola, forward

The forward map does not care what conic it draws. **Given** $h = 80\,000$ km^2/s, $e = 1.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$, $f = 30°$.

Find p and a, then $\mathbf{r}_{PQW}$, $\mathbf{v}_{PQW}$, and the inertial $\mathbf{r}$, $\mathbf{v}$. Check the energy.

What you should find

$p = h^2/\mu = 16\,056.20$ km and $a = p/(1 - e^2) = -16\,725.20$ km (negative, as it must be). $\mathbf{r}_{PQW} = (6284.96,\ 3628.62,\ 0)$ km, $\mathbf{v}_{PQW} = (-2.49125,\ 11.29047,\ 0)$ km/s, and

$$\mathbf{r} = (-4039.90,\ 4814.56,\ 3628.62)\ \text{km},$$
$$\mathbf{v} = (-10.3860,\ -4.7719,\ 1.7439)\ \text{km/s}.$$

Then $r = 7257.25$ km, $v = 11.5621$ km/s, $\varepsilon = v^2/2 - \mu/r = +11.916\ \text{km}^2/\text{s}^2 > 0$ and $v_\infty = \sqrt{2\varepsilon} = 4.882$ km/s. Nothing in Section 6 had to change.

8 Summary

1. **Six numbers, one of which moves**: (a, e) size and shape, (i, Ω, ω) orientation, and f (or M, or t_p) position.
2. Angles are measured in the geocentric equatorial frame: $\hat{\mathbf{I}}$ to the vernal equinox, $\hat{\mathbf{K}}$ along the spin axis. Both move (precession, nutation), so every orbit needs a frame epoch (J2000, or the quasar-based ICRF) as well as a state epoch.
3. **Three vectors generate everything**: $\mathbf{h}$ the plane, $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h}$ the ascending node, $\mathbf{e}$ the line of apsides.
4. J_2 turns the node at a rate $\propto -\cos i$; sun-synchronous orbits match it to the Sun and are therefore retrograde.
5. i **never needs a quadrant test**; Ω, ω, f always do, with the tests $\text{sign}(n_J)$, $\text{sign}(e_K)$, $\text{sign}(\mathbf{r} \cdot \mathbf{v})$. Use atan2 with consistently scaled arguments (Equation 2).
6. $e \to 0$ removes ω; $i \to 0$ removes Ω. Use u, ϖ, λ or equinoctial elements.
7. The forward map, $\mathbf{r} = \mathbf{Q}\mathbf{r}_{PQW}$ with $\mathbf{Q} = \mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega)$, is easy; all the difficulty is in the inverse. Test with a component-wise round trip.

Where this goes next. Ground tracks need Ω against the rotating Earth. The Earth's oblateness makes three elements drift steadily: Ω and ω, and the rate at which the mean anomaly advances, while a, e and i only oscillate (the gravity note). Orbit determination produces $(\mathbf{r}, \mathbf{v})$ from observations, and Section 5 turns it into a catalogue entry. Manoeuvres (impulsive-model note) are naturally described as changes of elements: a Hohmann transfer changes a and e, a plane change i and Ω, an apse rotation ω alone. Finally, a caution: TLEs quote the same six numbers (i, Ω, e, ω, M and the mean motion), but they are *mean* elements tied to the SGP4 propagator, not the osculating elements of this note, and feeding them to a two-body propagator is a common and expensive mistake.

9 Exercises

1. **Geostationary.** What are the elements of a geostationary satellite above 75°E at a given instant? Which of them are undefined, and what would you quote instead?

Solution. We start from what "geostationary" demands, and then go through the six elements one at a time.

A geostationary satellite stays above one point of the equator. So it must move in the equatorial plane, on a circle, with the period of the Earth's rotation relative to the stars, one sidereal day, $T = 86\,164.09$ s. Each of these three requirements fixes one element.

- **Size.** Kepler's third law gives $a = (\mu T^2/4\pi^2)^{1/3} = 42\,164$ km. The speed is $\sqrt{\mu/a} = 3.0747$ km/s.
- **Shape.** A circle, so $e = 0$.
- **Tilt.** The equatorial plane, moving east with the Earth, so $i = 0°$.

The other three are where the trouble lies. With $i = 0°$ the orbit plane coincides with the equator, the node vector $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h}$ is zero, and there is no line of nodes.

So Ω is undefined. With $e = 0$ the eccentricity vector is zero, and there is no periapsis. So ω is undefined, and so is f, which is measured from periapsis. These are the two breakdowns of Section 4.5 happening together.

What remains well defined is the direction of the satellite itself. We quote the **true longitude** λ, the angle of $\mathbf{r}$ from $\hat{\mathbf{I}}$ measured eastward about $\hat{\mathbf{K}}$. The satellite is above 75°E, so its angle from the Greenwich meridian is 75°. The Greenwich meridian itself stands at the angle θ_G (the Greenwich sidereal time of the time note) east of $\hat{\mathbf{I}}$ at that instant. Hence

$$\lambda = \theta_G + 75^\circ.$$

So the set we quote is $a = 42\,164$ km, $e = 0$, $i = 0^\circ$ and $\lambda = \theta_G + 75^\circ$. That is four numbers, not six, because a circular equatorial orbit is a special orbit. Only one number changes with time, λ, which grows by 360° every sidereal day, exactly as θ_G does.

In practice. A real geostationary satellite never has exactly $e = 0$ and $i = 0^\circ$. The Sun, the Moon and the Earth's flattening keep pushing e to around 10^{-4} and i to a fraction of a degree. Then Ω and ω do exist, but they swing wildly with tiny changes of the state, as Section 4.5 warns. This is why operators quote these satellites with the non-singular equinoctial elements, or simply by their longitude and the small drifts in longitude and latitude.

2. **The ISS.** The space station has $i = 51.6^\circ$. What is the highest latitude it flies over? Why can it not be reached from a launch site at latitude 60° without a plane change?

Solution. The highest latitude. Let the satellite be at the argument of latitude u, the angle from the ascending node to the satellite, measured in the orbit plane. Its height above the equatorial plane is the component of $\mathbf{r}$ along $\hat{\mathbf{K}}$. The orbit plane is tilted by i about the line of nodes, so a point at the angle u from the node is at the height $r \sin u \sin i$. Its latitude φ therefore satisfies

$$\sin\varphi = \frac{z}{r} = \sin i \, \sin u.$$

This is largest when $\sin u = 1$, that is, 90° after the ascending node, where $\varphi = i$. So the station reaches latitude 51.6°N, and by symmetry 51.6°S half an orbit later. For a prograde orbit the highest latitude is the inclination itself.

Why not from 60°. At the instant of launch the rocket is at the launch site. A short burn cannot carry the rocket far from the plane that contains the centre of the Earth, the site and the direction in which it is heading. So the orbit it reaches passes over the site. By the result above, every point of an orbit with inclination i is at a latitude between $-i$ and $+i$. A site at 60° is outside that band when $i = 51.6^\circ$, so no direct launch can put the vehicle into the station's orbit. The lowest inclination reachable from 60° is 60° itself, with a launch due east.

The difference of 8.4° must then be removed by a plane change. At the station's altitude of about 400 km, $r = 6778$ km and the circular speed is $\sqrt{\mu/r} = 7.669$ km/s. A plane change by Δi at the speed v costs $2v\sin(\Delta i/2)$ (derived in the plane-change note), here

$$2 \times 7.669 \times \sin 4.2^\circ = 1.12 \text{ km/s}.$$

That is about a seventh of the speed of the orbit itself, a very heavy price. This is why the station's inclination was chosen to suit the launch sites that serve it, not the other way round. It can be reached directly from any site at a latitude

below 51.6°, such as Baikonur at 45.9°N, Cape Canaveral at 28.5°N or Sriharikota at 13.7°N, by launching in the right direction.

3. **State to elements.** Find the elements of $\mathbf{r} = (7000,\ 0,\ 0)$ km, $\mathbf{v} = (0,\ 5,\ 5)$ km/s. Which angles are delicate, and why?

Solution. We follow the algorithm of Section 5.1 step by step and look at every quadrant test as we go.

Steps 1 to 4. We have $r = 7000$ km and $v = \sqrt{50} = 7.0711$ km/s. The dot product $\mathbf{r} \cdot \mathbf{v} = 0$, so $v_r = 0$. Then

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (0,\ -35\,000,\ 35\,000)\ \text{km}^2/\text{s}, \qquad h = 49\,497.5\ \text{km}^2/\text{s},$$

and $\cos i = h_K/h = 1/\sqrt{2}$, so $i = 45°$.

Steps 5 and 6. The node vector is $\mathbf{n} = (-h_J,\ h_I,\ 0) = (35\,000,\ 0,\ 0)$. So $\cos\Omega = n_I/n = 1$ and $\Omega = 0°$. The test asks for the sign of n_J, which is exactly zero.

Step 7. Since $\mathbf{r} \cdot \mathbf{v} = 0$, only the first term survives,

$$\mathbf{e} = \frac{1}{\mu}\Big(v^2 - \frac{\mu}{r}\Big)\mathbf{r} = \frac{50 - 56.9429}{398\,600}\,(7000,\ 0,\ 0) = (-0.12193,\ 0,\ 0), \qquad e = 0.12193.$$

The bracket is negative because $v^2 < \mu/r$. The satellite is slower than the circular speed at its radius, 7.546 km/s, so the eccentricity vector points away from it.

Steps 8 and 9. $\cos\omega = \mathbf{n} \cdot \mathbf{e}/(ne) = -1$, so $\omega = 180°$, and the test quantity e_K is zero. $\cos f = \mathbf{e} \cdot \mathbf{r}/(er) = -1$, so $f = 180°$, and the test quantity v_r is zero. The satellite is at apoapsis, which agrees with $v_r = 0$ and $v^2 < \mu/r$.

Step 10. $a = (2/r - v^2/\mu)^{-1} = 6239.3$ km and $p = h^2/\mu = 6146.5$ km. Then $r_a = a(1 + e) = 7000$ km, as it must be, and $r_p = a(1 - e) = 5478.5$ km.

So the elements are $a = 6239.3$ km, $e = 0.12193$, $i = 45°$, $\Omega = 0°$, $\omega = 180°$ and $f = 180°$.

Which angles are delicate. All three angles that carry a quadrant test, Ω, ω and f, are delicate here, for two reasons.

- Each test quantity, n_J, e_K and v_r, is exactly zero. The slightest error in the state, such as a rounding error, decides its sign at random. Fortunately the two branches agree here. For Ω they give 0° and 360°, which are the same direction, and for ω and f they both give 180°. So the wrong branch does no harm in this case.
- Each cosine is exactly ± 1, where arccos is at its worst. Near $x = 1$ we have $\arccos(1 - \delta) \approx \sqrt{2\delta}$, so an error of 10^{-10} in the cosine gives an error of about 1.4×10^{-5} rad in the angle. Half the significant digits are lost. The quadrant-safe atan2 form of Equation 2 does not have this trouble.

Reading the result. The perigee radius, 5478.5 km, is less than the Earth's radius. This is not an orbit at all. The satellite is at the top of a trajectory that would enter the atmosphere less than half an orbit later. The elements tell us at once what the state alone hides.

4. **The sign tests.** Construct a state with $n_J < 0$, $e_K > 0$ and $v_r < 0$, compute its elements, and confirm with a round trip.

Solution. The easy way to construct such a state is backwards. We choose elements for which each sign comes out as required, build the state with the forward map of Section 6, and then run the algorithm of Section 5.1 on it.

Choosing the elements. Each sign condition picks out a range of one angle.

- The node vector points along $(\cos\Omega,\ \sin\Omega,\ 0)$, so $n_J < 0$ needs $\sin\Omega < 0$, that is, $180^\circ < \Omega < 360^\circ$. We take $\Omega = 250^\circ$.
- The periapsis direction is the first column of Equation 3, whose third entry is $\sin\omega\sin i$. So $e_K = e\sin\omega\sin i$, and for a prograde orbit $e_K > 0$ needs $0^\circ < \omega < 180^\circ$. We take $\omega = 60^\circ$ and $i = 40^\circ$.
- The radial speed is $v_r = (\mu/h)\, e\sin f$, so $v_r < 0$ needs $180^\circ < f < 360^\circ$. We take $f = 250^\circ$.

The size and shape do not affect the signs, so we take $a = 10\,000$ km and $e = 0.3$. Then $p = a(1-e^2) = 9100$ km and $h = \sqrt{\mu p} = 60\,226.7$ km^2/s.

The forward map. In the perifocal frame,

$$r = \frac{p}{1+e\cos f} = 10\,140.47\text{ km}, \qquad v_r = \frac{\mu}{h}\, e\sin f = -1.86576\text{ km/s}, \qquad v_\theta = \frac{\mu}{h}\,(1+e\cos f) = 5.93924\text{ km/s}.$$

Rotating with $\mathbf{Q}$ of Equation 3 gives the state

$$\mathbf{r} = (-7821.151,\ -4089.827,\ -4993.209)\text{ km}, \qquad \mathbf{v} = (2.631062,\ -4.523091,\ 3.372658)\text{ km/s}.$$

Back to the elements. Now we forget where the state came from. The algorithm gives $\mathbf{h} = (-36\,378.32,\ 13\,240.63,\ 46\,136.36)$, so $i = \arccos(h_K/h) = 40.000^\circ$. Then $\mathbf{n} = (-13\,240.63,\ -36\,378.32,\ 0)$ and $\mathbf{e} = (0.135719,\ -0.209024,\ 0.167001)$, with $e = 0.30000$, and $\mathbf{r}\cdot\mathbf{v} = -18\,920$ km^2/s. The three angles are

angle	cos	arccos	test	corrected
Ω	-0.34202	110.000°	$n_J = -36\,378 < 0$	250.000°
ω	$+0.50000$	60.000°	$e_K = +0.167 > 0$	60.000°
f	-0.34202	110.000°	$v_r = -1.866 < 0$	250.000°

Finally $a = (2/r - v^2/\mu)^{-1} = 10\,000.00$ km. Every element comes back as chosen, to the digits shown, so the round trip closes.

Reading the result. Two of the three tests fire and one does not, as the signs demand. The test for ω stays silent because the perigee is north of the equator, where the principal value of arccos is already right. Had we forgotten the tests, the raw arccos would have put the node at 110° and the satellite on its way out instead of on its way in. Constructing a state backwards like this is the standard way to test a program. Every branch of the program can be exercised on purpose, and the true answer is known in advance.

5. **Sun-synchronous design.** For a 700 km circular sun-synchronous orbit, find i. How many revolutions does it make per day? Is the ground track repeated after one day?

Solution. The inclination. For a circle, $e = 0$, and $a = R_\oplus + 700 = 7078.137$ km. The boxed formula after Equation 1, with $\dot\alpha_\odot = 2\pi/(365.2422\times 86\,400) = 1.991\times 10^{-7}$ rad/s, gives

$$\cos i = -\frac{2\,\dot\alpha_\odot\, a^{7/2}}{3\sqrt{\mu}\, J_2 R_\oplus^2} = -0.14242, \qquad i = 98.19^\circ.$$

It falls between the 600 and 800 km rows of the table in Section 4, as it should.

Revolutions per day. The period is $T = 2\pi\sqrt{a^3/\mu} = 5926.4$ s $= 98.77$ min. In one solar day of 86 400 s the satellite makes $86\,400/5926.4 = 14.579$ revolutions.

Is the track repeated after one day? The ground track repeats when the satellite is back at the same point of its orbit and the Earth has also turned back to the same position under the orbit plane. The plane of a sun-synchronous orbit turns with the Sun, so the Earth turns once relative to this plane in exactly one mean solar day. A daily repeat would therefore need a whole number of revolutions in 86 400 s. With 14.579 revolutions it is not repeated.

We can see how far it misses. During one revolution the Earth turns under the plane by $360° \times 5926.4/86\,400 = 24.69°$, so each equator crossing is 24.69° west of the one before. Fifteen revolutions take a little more than a day, and the fifteenth crossing is $15 \times 24.69° - 360° = 10.4°$ west of the first one. The next day's tracks fall between those of the first day, 10.4° to the west of them.

Reading the result. This is what an imaging mission wants. If the track repeated daily, the satellite would see only fifteen strips of the Earth and never what lies between them. Instead the tracks interleave day after day, and the pattern closes only after many days. The orbit is then tuned so that it closes after a chosen number of days. Landsat, at about 705 km, makes 233 revolutions in 16 days. Its design is the subject of the ground-tracks note, which also takes account of the small change that J_2 makes to the period.

6. **Molniya.** A Molniya orbit has a period of half a sidereal day and $e = 0.74$. Find a, r_p, r_a, and explain why $i = 63.4°$ is chosen.

Solution. Size and shape. Half a sidereal day is $T = 43\,082.05$ s. Kepler's third law gives

$$a = \left(\frac{\mu T^2}{4\pi^2}\right)^{1/3} = 26\,561.8 \text{ km}.$$

Then $r_p = a(1-e) = 6906.1$ km and $r_a = a(1+e) = 46\,217.4$ km, at altitudes of 528 km and 39 839 km.

Why 63.4°. A Molniya satellite serves high northern latitudes, such as Russia, which a geostationary satellite sees only very low on the horizon or not at all. By Kepler's second law the satellite moves slowly near apogee. With the apogee over the northern hemisphere, at $\omega = 270°$, it spends about eight of its twelve hours high over the north, and it rushes through the south near perigee.

This only works if the apogee stays where it is put. The Earth's flattening turns the apse line at a rate proportional to $5\cos^2 i - 1$ (Section 4). For an orbit with a low perigee and this large eccentricity the rate is large. Using the formula of the gravity note, $\dot{\omega} = \frac{3}{4} n J_2 (R_\oplus/p)^2 (5\cos^2 i - 1)$, it would be 0.66° per day in the equatorial plane, and 0.18° per day at $i = 50°$. At the second rate the apogee would leave the north in a few months. The rate vanishes when

$$5\cos^2 i - 1 = 0, \qquad \cos i = \frac{1}{\sqrt{5}}, \qquad i = 63.43°.$$

At this critical inclination the apogee stays fixed, and no propellant is spent to hold it. A retrograde orbit at 116.6° would also do, but it would lose the help of the Earth's rotation at launch. There is a bonus. An inclination of 63.4° also suits the high-latitude launch sites from which these satellites are launched.

7. **Rotation matrices.** Show that $\mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega)$, applied to $(0,0,1)$, gives $\hat{\mathbf{h}}$, and applied to $(1,0,0)$ gives the unit vector towards periapsis.

Solution. The plan has two parts for each vector. First we multiply out, applying the three rotations from right to left. Then we work out $\hat{\mathbf{h}}$ and the periapsis direction directly from the definitions of i, Ω and ω, and compare.

The normal. $\mathbf{R}_3(\omega)$ turns about the third axis, so it leaves $(0, 0, 1)$ unchanged. $\mathbf{R}_1(i)$ then gives $(0,\ -\sin i,\ \cos i)$, and $\mathbf{R}_3(\Omega)$ gives

$$\mathbf{Q}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sin\Omega\sin i\\ -\cos\Omega\sin i\\ \cos i\end{bmatrix}.$$

Now from the definitions. The inclination is the angle between $\hat{\mathbf{h}}$ and $\hat{\mathbf{K}}$, so $h_K/h = \cos i$, and the part of $\hat{\mathbf{h}}$ in the equatorial plane has length $\sin i$. The node vector $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h} = (-h_J,\ h_I,\ 0)$ points along $(\cos\Omega,\ \sin\Omega,\ 0)$, by the definition of Ω. So $-h_J$ is proportional to $\cos\Omega$ and h_I to $\sin\Omega$, which gives $\hat{\mathbf{h}} = (\sin i\sin\Omega,\ -\sin i\cos\Omega,\ \cos i)$. This is the vector found above.

The periapsis direction. Multiplying out again, $\mathbf{R}_3(\omega)$ turns $(1, 0, 0)$ into $(\cos\omega,\ \sin\omega,\ 0)$, then $\mathbf{R}_1(i)$ gives $(\cos\omega,\ \sin\omega\cos i,\ \sin\omega\sin i)$, and $\mathbf{R}_3(\Omega)$ gives

$$\mathbf{Q}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i\\ \sin\omega\sin i\end{bmatrix}.$$

Now from the definitions. The argument of periapsis ω is the angle from the ascending node to periapsis, measured in the orbit plane in the direction of motion. Two unit vectors in the plane make this easy to use. The first is $\hat{\mathbf{n}} = (\cos\Omega,\ \sin\Omega,\ 0)$. The second, $\hat{\mathbf{m}} = \hat{\mathbf{h}} \times \hat{\mathbf{n}}$, is $\hat{\mathbf{n}}$ turned by $90°$ about $\hat{\mathbf{h}}$, which is the direction of motion. Using the $\hat{\mathbf{h}}$ found above,

$$\hat{\mathbf{m}} = \hat{\mathbf{h}} \times \hat{\mathbf{n}} = (-\sin\Omega\cos i,\ \cos\Omega\cos i,\ \sin i).$$

The unit vector at the angle ω from $\hat{\mathbf{n}}$ towards $\hat{\mathbf{m}}$ is $\hat{\mathbf{p}} = \cos\omega\,\hat{\mathbf{n}} + \sin\omega\,\hat{\mathbf{m}}$. Adding the components,

$$\hat{\mathbf{p}} = (\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i,\ \ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i,\ \ \sin\omega\sin i),$$

which is the vector found above.

Reading the result. The columns of $\mathbf{Q}$ are the perifocal unit vectors written in the inertial frame. This is true of any rotation matrix, since $\mathbf{Q}$ applied to the k-th unit vector picks out the k-th column. The third column is the orbit normal, and it contains no ω, because turning the apse line within the plane cannot move the plane. Its third entry, $\sin\omega\sin i$, is also the sign test of step 8 of the algorithm. The perigee is north of the equator exactly when $\sin\omega > 0$.

8. **A spent upper stage.** A launch from Sriharikota leaves its spent upper stage in a geosynchronous transfer orbit. Tracking gives its state in the geocentric equatorial frame at one instant,

   $$\mathbf{r} = (-799.0,\ 6769.9,\ -993.1)\ \text{km}, \qquad \mathbf{v} = (-9.4378,\ 0.4934,\ -3.1788)\ \text{km/s}.$$

   For this exercise take $\mu = 398\,600\ \text{km}^3/\text{s}^2$ and $R_\oplus = 6378$ km.

   (a) Find the specific angular momentum h, the eccentricity e, the semi-major axis a, the perigee and apogee altitudes, and the period.

(**b**) Find the inclination i, the right ascension of the ascending node Ω, the argument of perigee ω and the true anomaly f. Justify each quadrant choice.

(**c**) How long ago did the stage pass perigee? How long after the given instant does it reach apogee?

Solution. The plan follows the algorithm of Section 5.1. The angular momentum vector gives h, i and the node. The eccentricity vector gives e and ω. The energy gives a. Kepler's equation then turns angles into times.

(**a**) First the magnitudes and the radial velocity,

$$r = 6888.8 \text{ km}, \qquad v = 9.9710 \text{ km/s}, \qquad v_r = \frac{\mathbf{r}\cdot\mathbf{v}}{r} = 2.0378 \text{ km/s}.$$

Then

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = (-21\,030.2,\ 6832.8,\ 63\,498.7)\ \text{km}^2/\text{s}, \qquad h = 67\,238.7\ \text{km}^2/\text{s}.$$

The eccentricity vector, from step 7 of the algorithm, is

$$\mathbf{e} = \frac{1}{\mu}\Big[\Big(v^2 - \frac{\mu}{r}\Big)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v}\Big] = (0.24908,\ 0.68846,\ 0.00841), \qquad e = 0.7322.$$

The specific energy is $\varepsilon = v^2/2 - \mu/r = -8.1515\ \text{km}^2/\text{s}^2$, so

$$a = -\frac{\mu}{2\varepsilon} = 24\,449 \text{ km}.$$

As a check, $p = h^2/\mu = 11\,342.3$ km, and $p/(1-e^2)$ gives the same a. The apsides are at $r_p = a(1-e) = 6548$ km and $r_a = a(1+e) = 42\,351$ km, that is, at altitudes of 170 km and 35 973 km. The period is $T = 2\pi\sqrt{a^3/\mu} = 10.568$ h.

(**b**) The inclination needs no quadrant test, because i lies between 0° and 180°,

$$\cos i = \frac{h_K}{h} = 0.94438, \qquad i = 19.20^\circ.$$

The node vector is $\mathbf{n} = \hat{\mathbf{K}}\times\mathbf{h} = (-h_J,\ h_I,\ 0) = (-6832.8,\ -21\,030.2,\ 0)$, with $n = 22\,112.3$. Then $\cos\Omega = n_I/n = -0.30900$, whose principal value is 108.00°. But $n_J < 0$, so the ascending node is in the half of the equator with $J < 0$, and

$$\Omega = 360^\circ - 108.00^\circ = 252.00^\circ.$$

For the argument of perigee, $\cos\omega = \mathbf{n}\cdot\mathbf{e}/(ne)$ gives 178.00°. Here $e_K > 0$, so the perigee is north of the equator and the principal value stands, $\omega = 178.00^\circ$. For the true anomaly, $\cos f = \mathbf{e}\cdot\mathbf{r}/(er)$ gives 28.00°. Here $v_r > 0$, so the stage is moving away from perigee and again the principal value stands, $f = 28.00^\circ$.

(**c**) We need the time from perigee to $f = 28^\circ$. Time is uniform in the mean anomaly, so we go from f to E to M:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} \ \Rightarrow\ E = 11.20^\circ, \qquad M = E - e\sin E = 0.05326 \text{ rad}.$$

The mean motion is $\sqrt{\mu/a^3} = 1.6514\times10^{-4}$ rad/s, so the stage passed perigee

$$t = \frac{M}{\sqrt{\mu/a^3}} = 322 \text{ s} = 5.37 \text{ min}$$

before the given instant. Perigee to apogee takes half the period, 5.284 h, so the stage reaches apogee $5.284 - 0.090 = 5.19$ h after the given instant.

The numbers of this exercise are checked by exam-problems-verify.py.

All the numbers in this note and the worked examples are checked by orbital-elements-verify.py.