

Orbital Elements — Analytical Problems

Ten problems with no numbers: what the six elements know, and where they fail

Devendra Ghatе

October 10, 2026

Table of contents

How to use these problems	1
Problems	2
1. What the constants know	2
2. One state	3
3. Averages over one orbit	4
4. Symmetries	5
5. Latitude, right ascension and heading	6
6. No quadrant tests	7
7. A push out of the plane	7
8. Nearly circular orbits	8
9. North of the equator	9
10. What J_2 does to the elements	10

How to use these problems

The *Orbital Elements* deck converts states to elements and back, with numbers. These problems ask instead what the elements *mean*: which of them a single state already fixes, how they average, what symmetries do to them, how one impulse moves them, and why some of them disappear for circular or equatorial orbits. None needs a calculator.

Answers are folded under **Answer**. Every identity is checked against a direct state-to-elements conversion of 3000 random orbits by [codes/elements-analytic-verify.py](#).

Conventions (as in the deck). $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$ is the equatorial frame. $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h}$, $\mathbf{e} = \mathbf{v} \times \mathbf{h}/\mu - \hat{\mathbf{r}}$, $\varepsilon = v^2/2 - \mu/r$, $p = h^2/\mu$. The angles are i (from $\hat{\mathbf{K}}$ to $\mathbf{h}$), Ω (from $\hat{\mathbf{I}}$ to $\mathbf{n}$ about $\hat{\mathbf{K}}$), ω (from $\mathbf{n}$ to $\mathbf{e}$ about $\hat{\mathbf{h}}$), f (from $\mathbf{e}$ to $\mathbf{r}$ about $\hat{\mathbf{h}}$), and the argument of latitude $u = \omega + f$. γ is the flight-path angle, measured from the local horizontal.

#	topic	the insight
1	the constants $\mathbf{h}, \mathbf{e}, \varepsilon$	five independent numbers; for a given h the circle has the least energy
2	one state, (r, v, γ)	a is fixed by the speed alone; the empty focus lies on a circle
3	averages over an orbit	$\langle 1/r \rangle_t = 1/a$, but $\langle r \rangle_t = a(1 + e^2/2)$
4	symmetries	reversing the motion, and reflecting in the equator

#	topic	the insight
5	latitude and heading	$\sin \delta = \sin i \sin u$; $\cos i = \cos \delta \sin \beta$, so $i \geq \delta $
6	quadrant-free formulas	every quadrant test, absorbed into atan2
7	an out-of-plane impulse	a rigid rotation of the orbit about $\mathbf{r}$
8	nearly circular orbits	ω is ill-conditioned; ($e \cos \omega, e \sin \omega$) are not
9	time north of the equator	depends on e, ω only; greatest for $\omega = 270^\circ$
10	J_2 drifts	sun-synchronous orbits are retrograde; $\tan i = 2$; the equatorial orbit

Problems

1. What the constants know

- Prove that $\mathbf{h} \cdot \mathbf{e} = 0$ and $e^2 = 1 + 2\varepsilon h^2/\mu^2$.
- Hence show that the seven scalars in $(\mathbf{h}, \mathbf{e}, \varepsilon)$ hold only five independent numbers, and name the five classical elements they fix.
- Show that, of all orbits with a given h , the circle has the least energy, and that of all orbits with a given a , the circle has the greatest h .

Answer. (a) The vector $\mathbf{e}$ is built from two pieces, $\mathbf{v} \times \mathbf{h}$ and $\hat{\mathbf{r}}$. Both are perpendicular to $\mathbf{h}$: the first because it is a cross product with $\mathbf{h}$, and the second because $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is perpendicular to $\mathbf{r}$. So $\mathbf{h} \cdot \mathbf{e} = 0$, which says that $\mathbf{e}$ lies in the orbit plane.

For the magnitude, expand $\mathbf{e} \cdot \mathbf{e}$:

$$e^2 = \frac{|\mathbf{v} \times \mathbf{h}|^2}{\mu^2} - \frac{2(\mathbf{v} \times \mathbf{h}) \cdot \hat{\mathbf{r}}}{\mu} + 1.$$

We need two quantities.

- $|\mathbf{v} \times \mathbf{h}| = vh$, because $\mathbf{v}$ is perpendicular to $\mathbf{h}$.
- $(\mathbf{v} \times \mathbf{h}) \cdot \mathbf{r}$: a scalar triple product does not change when its factors are cycled, so $(\mathbf{v} \times \mathbf{h}) \cdot \mathbf{r} = \mathbf{h} \cdot (\mathbf{r} \times \mathbf{v}) = \mathbf{h} \cdot \mathbf{h} = h^2$. Hence $(\mathbf{v} \times \mathbf{h}) \cdot \hat{\mathbf{r}} = h^2/r$.

Putting these in,

$$e^2 = \frac{v^2 h^2}{\mu^2} - \frac{2h^2}{\mu r} + 1 = 1 + \frac{2h^2}{\mu^2} \left(\frac{v^2}{2} - \frac{\mu}{r} \right) = 1 + \frac{2\varepsilon h^2}{\mu^2}.$$

(b) Count the numbers: $\mathbf{h}$ has three, $\mathbf{e}$ three and ε one, seven in all. But (a) gives two relations among them. $\mathbf{h} \cdot \mathbf{e} = 0$ fixes one component of $\mathbf{e}$ once the others are known, and $e^2 = 1 + 2\varepsilon h^2/\mu^2$ fixes ε once $\mathbf{h}$ and $\mathbf{e}$ are known. That leaves five independent numbers. They are:

- from $\mathbf{h}$: the plane (i from h_K/h , and Ω from the node vector $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h}$), and the size $p = h^2/\mu$;
- from $\mathbf{e}$, which lies in that plane: e , and ω , its angle from $\mathbf{n}$.

ε adds nothing new: $\varepsilon = -\mu^2(1 - e^2)/(2h^2)$. The sixth element, f or the time of periapsis, says *where* the body is on the orbit. No constant of the motion can give it, because it changes with time.

(c) Since $e^2 \geq 0$, the relation in (a) gives $1 + 2\varepsilon h^2/\mu^2 \geq 0$, i.e.

$$\varepsilon \geq -\frac{\mu^2}{2h^2},$$

with equality only when $e = 0$. So, of all orbits with a given h , the circle has the least energy. For the second statement, write the energy in terms of a instead, $\varepsilon = -\mu/(2a)$. Then (a) becomes $e^2 = 1 - h^2/(\mu a)$, so

$$h^2 = \mu a (1 - e^2) \leq \mu a,$$

again with equality only for the circle. Of all orbits with a given a (a given energy), the circle has the greatest angular momentum. This is the fact behind the Hohmann optimality proof of the maneuvers deck, where the cost decreases with h and increases with ε .

2. One state

A body at distance r moves with speed v at flight-path angle γ . Put $\nu = rv^2/\mu$ (take $\nu < 2$).

(a) Show that

$$a = \frac{r}{2 - \nu}, \quad e^2 = 1 - \nu(2 - \nu) \cos^2 \gamma, \quad \tan f = \frac{\nu \sin \gamma \cos \gamma}{\nu \cos^2 \gamma - 1}.$$

(b) For fixed r and v , show that e is least when the velocity is horizontal, and that its least value is $|\nu - 1|$. Where on the orbit is the body then?

(c) Show that the empty focus is at distance $2a - r$ from the body, on the line from the body to the attracting focus reflected in the normal to $\mathbf{v}$. Hence, as the direction of $\mathbf{v}$ varies at fixed speed, the empty focus moves on a circle. Use this to rederive (b), and find the direction of $\mathbf{v}$ that gives the greatest e .

Answer. We are given the state in the form (r, v, γ) and want a , e and f . The plan:

- a from the energy (*vis-viva*), which needs only r and v ;
- e from Problem 1, which needs ε and h ;
- f from the orbit equation and the radial velocity, which give $e \cos f$ and $e \sin f$.

Why ν ? In all three, the speed enters only through the combination $rv^2/\mu = v^2/(\mu/r)$. That is the square of the speed measured in units of the local circular speed. So $\nu = 1$ is circular speed, $\nu = 2$ is escape speed, and $\nu < 2$ means an ellipse. Writing everything in terms of ν makes the formulas dimensionless.

(a) *Semi-major axis.* *Vis-viva*, $v^2 = \mu(2/r - 1/a)$, gives $1/a = 2/r - v^2/\mu = (2 - \nu)/r$.

Eccentricity. Only the transverse part of $\mathbf{v}$ contributes to h , so $h = rv \cos \gamma$. The energy is $\varepsilon = v^2/2 - \mu/r = \frac{\mu}{2r}(\nu - 2)$. Then

$$\frac{2\varepsilon h^2}{\mu^2} = 2 \cdot \frac{\mu}{2r}(\nu - 2) \cdot \frac{r^2 v^2 \cos^2 \gamma}{\mu^2} = (\nu - 2) \frac{rv^2}{\mu} \cos^2 \gamma = (\nu - 2) \nu \cos^2 \gamma,$$

and Problem 1 gives $e^2 = 1 - \nu(2 - \nu) \cos^2 \gamma$.

True anomaly. We want two equations, one for $e \cos f$ and one for $e \sin f$, so that we get both the angle and its quadrant. The orbit equation $r = p/(1 + e \cos f)$, with $p = h^2/\mu$, gives

$$e \cos f = \frac{h^2}{\mu r} - 1 = \frac{r^2 v^2 \cos^2 \gamma}{\mu r} - 1 = \nu \cos^2 \gamma - 1.$$

The radial velocity $v_r = v \sin \gamma = (\mu/h) e \sin f$ gives

$$e \sin f = \frac{h v \sin \gamma}{\mu} = \frac{rv^2}{\mu} \sin \gamma \cos \gamma = \nu \sin \gamma \cos \gamma.$$

Their ratio is the stated $\tan f$, and their signs give its quadrant.

(b) At fixed r and v , ν is fixed and only γ varies. In $e^2 = 1 - \nu(2 - \nu)\cos^2\gamma$ the factor $\nu(2 - \nu)$ is positive for $0 < \nu < 2$, so e is least when $\cos^2\gamma$ is greatest, $\cos^2\gamma = 1$: horizontal velocity. Then $e^2 = 1 - 2\nu + \nu^2 = (\nu - 1)^2$, so $e_{\min} = |\nu - 1|$.

Where is the body? With $\gamma = 0$, $e\sin f = 0$ and $e\cos f = \nu - 1$. So it is at periapsis if $\nu > 1$ and at apoapsis if $\nu < 1$. That makes sense: moving horizontally and faster than circular, the body is at the lowest point of its orbit. The orbit is a circle only if $\nu = 1$ and $\gamma = 0$.

(c) Let P be the body, F the attracting focus and F^* the empty one. Two facts about the ellipse locate F^* .

- *Its distance.* The sum of the distances from the foci is $PF + PF^* = 2a$, so $PF^* = 2a - r$.
- *Its direction.* The reflection property: the tangent at P makes equal angles with PF and PF^* . So the direction from P to F^* is the outward radial direction $\hat{\mathbf{r}}$ reflected in the tangent line. Reflecting a unit vector in the line along $\hat{\mathbf{v}}$ keeps its component along $\hat{\mathbf{v}}$ and reverses the rest, which gives $2(\hat{\mathbf{r}} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \hat{\mathbf{r}}$.

So

$$\mathbf{F}^* = \mathbf{r} + (2a - r)[2(\hat{\mathbf{r}} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \hat{\mathbf{r}}].$$

Now keep the speed fixed and turn the direction of $\mathbf{v}$. Then a is fixed, by (a), and so is the distance $2a - r$. Only the direction changes, so F^* moves on the circle of radius $2a - r$ about P .

This gives (b) without any algebra. The distance between the foci is $FF^* = 2ae$. So to make e small we want the point of that circle nearest to F , and F is at distance r from the centre P of the circle. The nearest point is at distance $|r - (2a - r)|$, so

$$2ae \geq |r - (2a - r)| \implies e \geq \left|1 - \frac{r}{a}\right| = |1 - (2 - \nu)| = |\nu - 1|.$$

Equality needs F^* between P and F on the line PF , which happens when the reflected direction is $-\hat{\mathbf{r}}$, i.e. when $\hat{\mathbf{r}} \cdot \hat{\mathbf{v}} = 0$: horizontal velocity, as in (b).

The **greatest** e comes from the far point of the circle, where $FF^* = r + (2a - r) = 2a$ and $e = 1$. That needs the reflected direction to be $+\hat{\mathbf{r}}$, which happens for $\hat{\mathbf{v}} = \pm\hat{\mathbf{r}}$. A radial launch gives the degenerate, straight-line ellipse.

3. Averages over one orbit

For an ellipse (a, e) with semi-minor axis b , show that the time averages over one period are

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{a}, \quad \left\langle \frac{1}{r^2} \right\rangle = \frac{1}{ab}, \quad \langle r \rangle = a\left(1 + \frac{e^2}{2}\right), \quad \langle r^2 \rangle = a^2\left(1 + \frac{3e^2}{2}\right), \quad \langle v^2 \rangle = \frac{\mu}{a},$$

while the average of r over true anomaly is b . Show that the body spends the fraction $\frac{1}{2} - \frac{e}{\pi}$ of each period inside the circle $r = a$. In what sense is a the “mean distance”?

Answer. A time average is $\langle g \rangle = \frac{1}{T} \oint g dt$. We cannot do these integrals in t directly, because $r(t)$ has no closed form. So we change variable to one of the anomalies, and choose the one that makes each integrand simplest.

- *Eccentric anomaly.* $r = a(1 - e\cos E)$ is linear in $\cos E$. Kepler’s equation $M = E - e\sin E$ gives $dM = (1 - e\cos E)dE = (r/a)dE$, and M grows uniformly in time. So

$$\langle g \rangle = \frac{1}{2\pi} \oint g \frac{r}{a} dE.$$

- *True anomaly.* Angular momentum gives $dt = r^2 df/h$. This is the better choice when g has a factor $1/r^2$.

The five time averages.

- $g = 1/r$: the factor r cancels, $\langle 1/r \rangle = \frac{1}{2\pi} \oint \frac{dE}{a} = \frac{1}{a}$. This is the simplest average of all.

- $g = r$: $\langle r \rangle = \frac{a}{2\pi} \oint (1 - e \cos E)^2 dE = a(1 + \frac{e^2}{2})$, using $\oint \cos E dE = 0$ and $\oint \cos^2 E dE = \pi$.
- $g = r^2$: $\langle r^2 \rangle = \frac{a^2}{2\pi} \oint (1 - e \cos E)^3 dE$. Expand the cube; the odd powers of $\cos E$ integrate to zero, leaving $\frac{a^2}{2\pi}(2\pi + 3e^2\pi) = a^2(1 + \frac{3e^2}{2})$.
- $g = 1/r^2$: use the true anomaly. $\langle 1/r^2 \rangle = \frac{1}{T} \oint \frac{1}{r^2} \frac{r^2 df}{h} = \frac{2\pi}{hT}$. The areal velocity $h/2$ times the period T is the area πab , so $hT = 2\pi ab$ and $\langle 1/r^2 \rangle = 1/(ab)$.
- $g = v^2$: *vis-viva* writes $v^2 = \mu(2/r - 1/a)$ in terms of $1/r$, whose average we already have: $\langle v^2 \rangle = \mu(2/a - 1/a) = \mu/a$. This is the virial theorem: the mean kinetic energy $\frac{1}{2}\langle v^2 \rangle = \mu/(2a)$ is $-\varepsilon$.

The average over true anomaly. This one is not a time average, so no dt appears:

$$\langle r \rangle_f = \frac{1}{2\pi} \oint \frac{p df}{1 + e \cos f} = \frac{p}{\sqrt{1 - e^2}} = a\sqrt{1 - e^2} = b,$$

using the standard integral $\oint \frac{df}{1 + e \cos f} = \frac{2\pi}{\sqrt{1 - e^2}}$.

Inside $r = a$. From $r = a(1 - e \cos E)$, the body is inside $r = a$ exactly when $\cos E > 0$, i.e. $|E| < \frac{\pi}{2}$. At $E = \pm \frac{\pi}{2}$, Kepler's equation gives $M = \pm(\frac{\pi}{2} - e)$. M is uniform in time, so the fraction of the period is $(\pi - 2e)/(2\pi) = \frac{1}{2} - \frac{e}{\pi}$. The body is in the inner half of its range for less than half the time, because it moves fast there.

“Mean distance”. The three averages of r are in the order $\langle r \rangle_f = b < a < a(1 + \frac{e^2}{2}) = \langle r \rangle_t$. Kepler's “mean distance” was $(r_p + r_a)/2 = a$. That is also the average of r over the eccentric anomaly, $\frac{1}{2\pi} \oint a(1 - e \cos E) dE = a$, and the reciprocal of the time average of $1/r$. That last one is the average that enters the energy, which is why a alone fixes ε and T .

4. Symmetries

- (a) The velocity of a body is reversed at some instant. Show that its new elements are

$$(a, e, 180^\circ - i, \Omega + 180^\circ, 180^\circ - \omega, 360^\circ - f).$$

- (b) The state is reflected in the equatorial plane, $(x, y, z, \dot{x}, \dot{y}, \dot{z}) \mapsto (x, y, -z, \dot{x}, \dot{y}, -\dot{z})$. Show that the new elements are $(a, e, i, \Omega + 180^\circ, \omega + 180^\circ, f)$.
- (c) An observer sees the orbit only as a curve in space, without the sense of motion. Which elements can the observer determine?

Answer. Every element is computed from $\mathbf{h}$, $\mathbf{e}$ and the state. So for each symmetry we first ask what happens to the vectors $\mathbf{h}$ and $\mathbf{e}$, and then read off the elements.

- (a) Replace $\mathbf{v}$ by $-\mathbf{v}$ and keep $\mathbf{r}$.

- $\mathbf{h}' = \mathbf{r} \times (-\mathbf{v}) = -\mathbf{h}$: the angular momentum reverses.
- $\mathbf{e}' = (-\mathbf{v}) \times (-\mathbf{h})/\mu - \hat{\mathbf{r}} = \mathbf{e}$: the two sign changes cancel, so the eccentricity vector is unchanged.

The speed and r are unchanged, so the energy is too, and $a' = a$, $e' = e$. Now the angles.

- $\cos i' = h'_K/h = -\cos i$, so $i' = 180^\circ - i$.
- $\mathbf{n}' = \hat{\mathbf{K}} \times \mathbf{h}' = -\mathbf{n}$: the ascending node becomes the descending one, so $\Omega' = \Omega + 180^\circ$.
- For ω we need its sine and its cosine. $\cos \omega' = \mathbf{n}' \cdot \mathbf{e}/(ne) = -\cos \omega$. For the sine, the periapsis direction has K component $e_K = e \sin i \sin \omega$ (Problem 5 with $u = \omega$). Both e_K and $\sin i' = \sin i$ are unchanged, so $\sin \omega' = \sin \omega$. Hence $\omega' = 180^\circ - \omega$.
- $\cos f' = \mathbf{e} \cdot \mathbf{r}/(er) = \cos f$ is unchanged, but $\mathbf{r} \cdot \mathbf{v}' = -\mathbf{r} \cdot \mathbf{v}$, so the body is now moving the other way along the same curve: inbound where it was outbound. Hence $f' = -f$, i.e. $360^\circ - f$.

(b) Let $\mathbf{M} = \text{diag}(1, 1, -1)$ be the reflection, so $\mathbf{r}' = \mathbf{M}\mathbf{r}$ and $\mathbf{v}' = \mathbf{M}\mathbf{v}$. The one fact we need is how a cross product behaves under a reflection: $(\mathbf{M}\mathbf{a}) \times (\mathbf{M}\mathbf{b}) = -\mathbf{M}(\mathbf{a} \times \mathbf{b})$. (Check it on $\hat{\mathbf{I}} \times \hat{\mathbf{J}} = \hat{\mathbf{K}}$: the left side is still $\hat{\mathbf{K}}$, and $-\mathbf{M}\hat{\mathbf{K}} = \hat{\mathbf{K}}$.) So

$$\mathbf{h}' = -\mathbf{M}\mathbf{h} = (-h_I, -h_J, h_K),$$

and the angular momentum does *not* reflect like an ordinary vector. Then:

- $h'_K = h_K$, so $i' = i$.
- $\mathbf{n}' = \hat{\mathbf{K}} \times \mathbf{h}' = (-h'_J, h'_I, 0) = (h_J, -h_I, 0) = -\mathbf{n}$, so $\Omega' = \Omega + 180^\circ$.
- $\mathbf{e}' = (\mathbf{M}\mathbf{v}) \times (-\mathbf{M}\mathbf{h})/\mu - \mathbf{M}\mathbf{r} = \mathbf{M}(\mathbf{v} \times \mathbf{h})/\mu - \mathbf{M}\mathbf{r} = \mathbf{M}\mathbf{e}$: the eccentricity vector *is* an ordinary vector, and reflects.
- $\cos \omega' = \mathbf{n}' \cdot \mathbf{M}\mathbf{e}/(ne) = -\cos \omega$, because $\mathbf{n}$ has no K component. And $e'_K = -e_K$ gives $\sin \omega' = -\sin \omega$. So $\omega' = \omega + 180^\circ$.
- $\mathbf{M}\mathbf{e} \cdot \mathbf{M}\mathbf{r} = \mathbf{e} \cdot \mathbf{r}$ and $\mathbf{M}\mathbf{r} \cdot \mathbf{M}\mathbf{v} = \mathbf{r} \cdot \mathbf{v}$, so $f' = f$.

As a check, $u' = \omega' + f' = u + 180^\circ$, so $\sin \delta' = \sin i \sin(u + 180^\circ) = -\sin \delta$: the image is south whenever the body is north, as a reflection in the equator should be.

(c) Reversing the motion maps the curve onto itself. So the curve alone cannot tell apart the two sets of elements in (a). It does fix a , e , the plane (without a sense of rotation) and the direction of periapsis. Hence it fixes (i, Ω, ω) up to the map in (a). The one bit of information it cannot give is the sense of motion, which the elements store in whether i is below or above 90° .

5. Latitude, right ascension and heading

- (a) A body is at argument of latitude u on an orbit (i, Ω) . Show that its declination δ and right ascension α satisfy

$$\sin \delta = \sin i \sin u, \quad \tan(\alpha - \Omega) = \cos i \tan u,$$

with $\alpha - \Omega$ in the quadrant of $(\cos u, \cos i \sin u)$. Hence find the greatest latitude the orbit reaches.

- (b) Let β be the azimuth of the inertial velocity, measured from north towards east. Show that $\cos i = \cos \delta \sin \beta$.
- (c) Prove that a launch from latitude ϕ cannot go directly into an orbit with $i < |\phi|$, and that a launch due east gives $i = |\phi|$. Where on its orbit does a body move due east?

Answer. (a) We want the direction of the body in the equatorial frame. The 3-1-3 rotation of the deck does exactly this: in the orbit plane, measured from the node, the body is at $(\cos u, \sin u, 0)$, and

$$\hat{\mathbf{r}} = \mathbf{R}_3(\Omega) \mathbf{R}_1(i) (\cos u, \sin u, 0)^\top.$$

Do the two rotations one at a time. $\mathbf{R}_1(i)$ tilts the plane about the node line, giving $(\cos u, \cos i \sin u, \sin i \sin u)$. Its K component is the sine of the declination, so $\sin \delta = \sin i \sin u$. Then $\mathbf{R}_3(\Omega)$ turns everything about $\hat{\mathbf{K}}$ by Ω . That adds Ω to the azimuth in the IJ plane and leaves the K component alone. So $\alpha - \Omega$ is the azimuth of the pair $(\cos u, \cos i \sin u)$, which gives the stated $\tan(\alpha - \Omega)$ and its quadrant.

The greatest latitude. $|\sin \delta| = \sin i |\sin u| \leq \sin i$, with equality at $u = 90^\circ$ and $u = 270^\circ$. So the orbit reaches latitude i if $i \leq 90^\circ$, and $180^\circ - i$ if it is retrograde.

(b) The heading β is defined in the local horizontal frame, and i is defined by $\cos i = h_K/h$. So we compute $\mathbf{h}$ in the local frame and take its K component. Take the local unit vectors $\hat{\mathbf{r}}$, east $\hat{\mathbf{E}} = \hat{\mathbf{K}} \times \hat{\mathbf{r}}/\cos \delta$, and north $\hat{\mathbf{N}} = \hat{\mathbf{r}} \times \hat{\mathbf{E}}$. Their K components are $\sin \delta$, 0 and $\cos \delta$. Write $\mathbf{v} = v_r \hat{\mathbf{r}} + v_E \hat{\mathbf{E}} + v_N \hat{\mathbf{N}}$. Since $\hat{\mathbf{r}} \times \hat{\mathbf{E}} = \hat{\mathbf{N}}$ and $\hat{\mathbf{r}} \times \hat{\mathbf{N}} = -\hat{\mathbf{E}}$,

$$\mathbf{h} = r \hat{\mathbf{r}} \times \mathbf{v} = r(v_E \hat{\mathbf{N}} - v_N \hat{\mathbf{E}}), \quad h_K = r v_E \cos \delta.$$

Only the eastward velocity contributes to h_K . The horizontal velocity is $v \cos \gamma$, and its east part is $v_E = v \cos \gamma \sin \beta$. Also $h = r v \cos \gamma$. So

$$\cos i = \frac{h_K}{h} = \frac{r v \cos \gamma \sin \beta \cos \delta}{r v \cos \gamma} = \cos \delta \sin \beta.$$

(c) A launch from latitude ϕ starts with $\delta = \phi$. By (b), $|\cos i| = \cos \phi |\sin \beta| \leq \cos \phi$. For a prograde orbit this says $\cos i \leq \cos |\phi|$, i.e. $i \geq |\phi|$. Equality needs $\sin \beta = 1$: a launch due east. (The azimuth here is inertial. The Earth's rotation tilts the launcher's own heading slightly.)

The same formula, applied at every point of the orbit, tells us where the body moves due east: where $\sin \beta = 1$, i.e. where $\cos \delta = \cos i$, the extreme latitudes $u = 90^\circ$ and $u = 270^\circ$. This is just the statement that a launch due east puts the launch site at the most northerly (or southerly) point of the orbit.

6. No quadrant tests

Show that

$$\begin{aligned}\Omega &= \text{atan2}(h_I, -h_J), & i &= \text{atan2}(\sqrt{h_I^2 + h_J^2}, h_K), \\ \omega &= \text{atan2}(h e_K, h_I e_J - h_J e_I), & u &= \text{atan2}(h r_K, h_I r_J - h_J r_I), \\ f &= \text{atan2}(h(\mathbf{r} \cdot \mathbf{v}), h^2 - \mu r)\end{aligned}$$

reproduce the deck's algorithm (arccos followed by a quadrant test) without any test. Where has each test gone? What do these formulas return for $i = 0$ or $e = 0$, and why is that dangerous? Why is the formula for i better than $\arccos(h_K/h)$ when i is small?

Answer. The deck finds each angle with arccos, which only returns values in $[0^\circ, 180^\circ]$, and then a test decides whether the true angle is 360° minus that. $\text{atan2}(y, x)$ returns the angle whose sine and cosine are in the ratio $y : x$, over the whole circle, as long as y and x are the sine and cosine multiplied by the *same positive* factor. So the plan for each angle is: write its sine and its cosine with a common positive factor, and drop the factor. The sign of the sine, which is what each test checked, is then built into atan2 .

We need the node vector, $\mathbf{n} = \hat{\mathbf{K}} \times \mathbf{h} = (-h_J, h_I, 0)$, whose length is $n = h \sin i$.

- Ω : $\cos \Omega = n_I/n = -h_J/(h \sin i)$ and $\sin \Omega = n_J/n = h_I/(h \sin i)$. The common positive factor is $1/(h \sin i)$. The deck's test " $n_J < 0$ " is now the sign of h_I .
- i : $\cos i = h_K/h$ and $\sin i = \sqrt{h_I^2 + h_J^2}/h$. Here the sine is never negative, so i is in $[0^\circ, 180^\circ]$ automatically, and there was never a test (deck: "why i needs no test").
- ω : $\cos \omega = \mathbf{n} \cdot \mathbf{e}/(ne) = (h_I e_J - h_J e_I)/(h e \sin i)$. For the sine, Problem 5 with $u = \omega$ says that the periapsis direction has K component $\sin i \sin \omega$, so $e_K = e \sin i \sin \omega$ and $\sin \omega = e_K/(e \sin i)$. Multiply both by $h e \sin i > 0$. The test " $e_K < 0$ " is now the sign of $h e_K$.
- u : the same argument with $\mathbf{r}$ in place of $\mathbf{e}$.
- f : the orbit equation gives $e \cos f = p/r - 1 = (h^2 - \mu r)/(\mu r)$, and the radial velocity gives $e \sin f = h v_r / \mu = h(\mathbf{r} \cdot \mathbf{v})/(\mu r)$. Multiply both by $\mu r > 0$. The test " $\mathbf{r} \cdot \mathbf{v} < 0$ " is now the sign of the first argument.

Degeneracies. The common factor we dropped was $h \sin i$ or e , and those vanish exactly when an element ceases to exist. For $i = 0$, $h_I = h_J = 0$, so the formulas for Ω , ω and u all become $\text{atan2}(0, 0)$. For $e = 0$, $\mathbf{e} = \mathbf{0}$, $h^2 = \mu r$ and $\mathbf{r} \cdot \mathbf{v} = 0$, so the formulas for ω and f do too. That is as it should be. The danger is that $\text{atan2}(0, 0)$ returns 0 without complaint, so a program prints a confident $\Omega = 0$ or $\omega = 0$ unless it tests $\sin i$ and e first. That is why the deck's tolerance tests are still needed.

Small i . Near $i = 0$, $\cos i \approx 1 - i^2/2$. In double precision $\cos i$ cannot be told apart from 1 once $i \lesssim 10^{-8}$, so $\arccos(h_K/h)$ returns 0; and well before that it has lost half its digits, because arccos is very steep near

1. The atan2 form uses $\sqrt{h_I^2 + h_J^2} \approx h i$ directly, and keeps full relative precision.

7. A push out of the plane

A small impulse Δv_W is applied along $\hat{\mathbf{h}}$ when the body is at argument of latitude u , on an orbit of any eccentricity.

- Show that, to first order, the impulse rotates the whole orbit rigidly about the radius vector through the angle $\varphi = \Delta v_W / v_\theta = r \Delta v_W / h$. Hence show that a , e and f do not change.
- Show that

$$\Delta i = \frac{r \cos u}{h} \Delta v_W, \quad \Delta \Omega = \frac{r \sin u}{h \sin i} \Delta v_W, \quad \Delta \omega = -\cos i \Delta \Omega.$$

- (c) Where should the push be made to change only i , and where to change only Ω ? Why does ω change, and what happens to $\varpi = \Omega + \omega$ as $i \rightarrow 0$?

Answer. (a) What does the impulse do? It leaves $\mathbf{r}$ unchanged and adds $\Delta v_W \hat{\mathbf{h}}$, perpendicular to the orbit plane, to $\mathbf{v}$. Let $\hat{\mathbf{s}} = \hat{\mathbf{h}} \times \hat{\mathbf{r}}$ be the transverse unit vector, so that $\mathbf{v} = v_r \hat{\mathbf{r}} + v_\theta \hat{\mathbf{s}}$ and $h = r v_\theta$.

Compare this with a rotation through a small angle φ about $\hat{\mathbf{r}}$. Such a rotation leaves $\mathbf{r}$ and the radial part $v_r \hat{\mathbf{r}}$ alone, and turns $v_\theta \hat{\mathbf{s}}$ into $v_\theta (\cos \varphi \hat{\mathbf{s}} + \sin \varphi \hat{\mathbf{h}}) \approx v_\theta \hat{\mathbf{s}} + v_\theta \varphi \hat{\mathbf{h}}$. So, to first order, **the impulse is a rotation of the velocity about $\hat{\mathbf{r}}$, through**

$$\varphi = \frac{\Delta v_W}{v_\theta} = \frac{r \Delta v_W}{h}.$$

Now use a symmetry. The gravity of a point mass looks the same after any rotation about the centre. So if we take the old orbit and rotate it as a whole through φ about the line along $\hat{\mathbf{r}}$, we get another Kepler orbit. At the instant of the burn, that rotated orbit passes through the same $\mathbf{r}$, with the old velocity rotated, which is the new velocity. A Kepler orbit is fixed by a single state, so **the new orbit is the old one, rotated rigidly about $\mathbf{r}$.**

A rotation keeps lengths and angles. So the size and shape of the orbit, a and e , are unchanged, and so is the angle f between $\mathbf{e}$ and $\mathbf{r}$. Only the orientation (i, Ω, ω) changes. (Exactly, the speed rises slightly, $v'^2 = v^2 + \Delta v_W^2$, so the energy changes by $\frac{1}{2} \Delta v_W^2$. That is second order.)

(b) The orientation of the plane is carried by $\hat{\mathbf{h}}$, so we follow it under the rotation. A small rotation by φ about $\hat{\mathbf{r}}$ changes a vector $\mathbf{a}$ by $\varphi \hat{\mathbf{r}} \times \mathbf{a}$, so

$$\Delta \hat{\mathbf{h}} = \varphi \hat{\mathbf{r}} \times \hat{\mathbf{h}} = -\varphi \hat{\mathbf{s}}.$$

We need two components of $\hat{\mathbf{s}}$. It is the in-plane direction at argument of latitude $u + 90^\circ$. So by Problem 5 its K component is $\sin i \sin(u + 90^\circ) = \sin i \cos u$, and its component along the node direction $\hat{\mathbf{n}}$ is $\cos(u + 90^\circ) = -\sin u$.

- *Inclination.* $\hat{h}_K = \cos i$, so $\Delta(\cos i) = -\sin i \Delta i$ must equal $\Delta \hat{h}_K = -\varphi \sin i \cos u$. Hence $\Delta i = \varphi \cos u$.
- *Node.* The horizontal part of $\hat{\mathbf{h}}$ is $\sin i (\sin \Omega, -\cos \Omega, 0)$. When Ω changes, this turns about $\hat{\mathbf{K}}$, and its change along $\hat{\mathbf{n}} = (\cos \Omega, \sin \Omega, 0)$ is $\sin i \Delta \Omega$. The change along $\hat{\mathbf{n}}$ is also $-\varphi \hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = \varphi \sin u$. So $\sin i \Delta \Omega = \varphi \sin u$.
- *Periapsis.* We know f is unchanged, so $\Delta \omega = \Delta u$. And the body has not moved, so its declination is unchanged. Differentiating $\sin \delta = \sin i \sin u$ gives $\cos i \sin u \Delta i + \sin i \cos u \Delta u = 0$. Substitute $\Delta i = \varphi \cos u$ and cancel $\cos u$: $\Delta u = -\varphi \cos i \sin u / \sin i = -\cos i \Delta \Omega$.

With $\varphi = r \Delta v_W / h$ these are the three stated formulas.

(c) For i alone we need $\sin u = 0$: burn at the nodes, $u = 0^\circ$ or 180° . The sign of $\cos u$ is opposite at the two nodes, so the push must be in opposite directions there. For Ω alone we need $\cos u = 0$: burn at $u = \pm 90^\circ$, the extreme latitudes, as in Problem 9 of the maneuvers set.

Why does ω change? The periapsis has simply turned with the rest of the orbit. But ω is measured from the node, and the node has moved. Add the two changes:

$$\Delta \varpi = \Delta \Omega + \Delta \omega = (1 - \cos i) \Delta \Omega = \frac{1 - \cos i}{\sin i} \sin u \varphi = \tan \frac{i}{2} \sin u \varphi \rightarrow 0 \quad (i \rightarrow 0).$$

For a near-equatorial orbit the push swings Ω and ω by large and nearly opposite amounts (because $\Delta \Omega \propto 1/\sin i$), while their sum ϖ , the quantity that survives at $i = 0$, hardly moves.

8. Nearly circular orbits

For $e \ll 1$, let $\xi = e \cos \omega$, $\eta = e \sin \omega$ and $\lambda = \omega + M$.

- From Kepler's equation, show that $r = a(1 - e \cos M) + O(e^2)$ and $f = M + 2e \sin M + O(e^2)$.
- Hence show that

$$r = a(1 - \xi \cos \lambda - \eta \sin \lambda) + O(e^2), \quad u = \lambda + 2(\xi \sin \lambda - \eta \cos \lambda) + O(e^2).$$

So the motion depends smoothly on (ξ, η, λ) as $e \rightarrow 0$, although ω does not exist at $e = 0$.

- (c) Show that a small change of the state changes ω by $\delta\omega = (\xi \delta\eta - \eta \delta\xi)/e^2$, and explain why ω is ill-conditioned for small e while ξ , η and λ are not.

Answer. The difficulty with ω for a nearly circular orbit is that it is the direction of a very short vector. So the idea is to describe the orbit by the components of that vector, $\xi = e \cos \omega$ and $\eta = e \sin \omega$, instead of by its length and direction. To use them we need r and u to first order in e (a), then we rewrite them in the new variables (b).

(a) *Radius.* Kepler's equation is $E = M + e \sin E$. To first order in e we may put $E = M$ on the right, so $E = M + e \sin M + O(e^2)$. Then $r = a(1 - e \cos E) = a(1 - e \cos M) + O(e^2)$, because the difference between $\cos E$ and $\cos M$ is already multiplied by e .

True anomaly. Start from $\cos f = (\cos E - e)/(1 - e \cos E)$. To first order, $1/(1 - e \cos E) \approx 1 + e \cos E$, so

$$\cos f \approx (\cos E - e)(1 + e \cos E) \approx \cos E - e + e \cos^2 E = \cos E - e \sin^2 E.$$

Write $f = E + \epsilon$ with ϵ small. Then $\cos(E + \epsilon) \approx \cos E - \epsilon \sin E$. Comparing the two, $\epsilon = e \sin E$. So $f = E + e \sin E + O(e^2) = M + 2e \sin M + O(e^2)$, using the expansion of E .

(b) The formulas in (a) contain $e \cos M$ and $e \sin M$. Put $M = \lambda - \omega$ and expand:

$$\begin{aligned} e \cos M &= e \cos \omega \cos \lambda + e \sin \omega \sin \lambda = \xi \cos \lambda + \eta \sin \lambda, \\ e \sin M &= e \cos \omega \sin \lambda - e \sin \omega \cos \lambda = \xi \sin \lambda - \eta \cos \lambda. \end{aligned}$$

The radius follows at once. For the argument of latitude, $u = \omega + f = \omega + M + 2e \sin M = \lambda + 2e \sin M$. So

$$r = a(1 - \xi \cos \lambda - \eta \sin \lambda) + O(e^2), \quad u = \lambda + 2(\xi \sin \lambda - \eta \cos \lambda) + O(e^2).$$

Neither formula contains ω on its own, only ξ , η and λ , and both are smooth through $\xi = \eta = 0$. To first order the orbit is a circle shifted by a times the small vector (ξ, η) , traversed with a small along-track wobble. This is why the non-singular elements of the deck use $(e \cos \varpi, e \sin \varpi)$ and a mean longitude.

(c) ω is the polar angle of the point (ξ, η) , $\omega = \text{atan2}(\eta, \xi)$. Differentiating,

$$\delta\omega = \frac{\xi \delta\eta - \eta \delta\xi}{\xi^2 + \eta^2} = \frac{\xi \delta\eta - \eta \delta\xi}{e^2}.$$

A small error of size $|\delta\mathbf{e}|$, perpendicular to (ξ, η) , turns ω by $|\delta\mathbf{e}|/e$, and that grows without bound as $e \rightarrow 0$. Yet, by (b), the position changes only by $O(a|\delta\mathbf{e}|)$.

The two statements agree because $u = \omega + f$ is measured from the node, which does not depend on $\mathbf{e}$. A large error in ω comes with an equal and opposite error in f (or M). So the physical state is well conditioned; only the split of u into ω and f is not.

9. North of the equator

- (a) Show that the fraction of each period that a body spends north of the equatorial plane depends only on e and ω (for $0 < i < 180^\circ$), and that it is $\frac{1}{2}$ for $\omega = 0^\circ$ or 180° .
(b) Show that the half of the orbit with $|f| < 90^\circ$ takes the fraction

$$\tau_{\text{near}} = \frac{\arccos e - e\sqrt{1 - e^2}}{\pi}$$

of the period.

- (c) Hence find the fraction for $\omega = 90^\circ$ and for $\omega = 270^\circ$, and prove that 270° gives the largest fraction of all. At what latitude is the apoapsis then? Why do Molniya orbits use this ω ?

Answer. (a) “North” means $z > 0$. With $z = r \sin i \sin u$ and $\sin i > 0$, that is $\sin u > 0$, i.e. $0^\circ < u < 180^\circ$. In terms of the true anomaly, $f = u - \omega$ runs from $-\omega$ to $180^\circ - \omega$. How long the body takes over that arc, as a fraction of T , depends only on e (through Kepler’s equation) and on where the arc starts, ω . It does not depend on a , i or Ω .

For $\omega = 0^\circ$ the arc runs from periapsis to apoapsis, which takes half the period by symmetry. For $\omega = 180^\circ$ it runs from apoapsis to periapsis, which again takes half.

(b) We need the time from $f = -90^\circ$ to $f = 90^\circ$. Time is uniform in the mean anomaly, so find M at $f = 90^\circ$. There $r = p = a(1 - e^2)$. With $r = a(1 - e \cos E)$ this gives $\cos E = (1 - r/a)/e = e$, so $E = \arccos e$ and $\sin E = \sqrt{1 - e^2}$. Kepler’s equation then gives $M = \arccos e - e\sqrt{1 - e^2}$. By symmetry the arc $|f| < 90^\circ$ takes $2M/(2\pi)$ of the period, which is τ_{near} .

(c) For $\omega = 90^\circ$ the northern arc is $-90^\circ < f < 90^\circ$, the half of the orbit nearer periapsis, which takes τ_{near} . For $\omega = 270^\circ$ it is $90^\circ < f < 270^\circ$, the other half, which takes $1 - \tau_{\text{near}}$.

Is τ_{near} less than $\frac{1}{2}$? At $e = 0$ it is $(\pi/2)/\pi = \frac{1}{2}$, and

$$\frac{d}{de} (\arccos e - e\sqrt{1 - e^2}) = -\frac{1}{\sqrt{1 - e^2}} - \sqrt{1 - e^2} + \frac{e^2}{\sqrt{1 - e^2}} = -2\sqrt{1 - e^2} < 0,$$

so $\tau_{\text{near}} < \frac{1}{2}$ for every $e > 0$. The half near periapsis is passed quickly.

Why 270° is the maximum. Every northern arc is a half-orbit, 180° of true anomaly. So we ask which half-orbit takes the longest. Let $\tau(c)$ be the fraction of the period spent on the half $c - 90^\circ < f < c + 90^\circ$, centred on $f = c$. Since $dt/df = r^2/h$,

$$\frac{d\tau}{dc} = \frac{r^2(c + 90^\circ) - r^2(c - 90^\circ)}{hT} = \frac{p^2}{hT} \left[\frac{1}{(1 - e \sin c)^2} - \frac{1}{(1 + e \sin c)^2} \right],$$

using $\cos(c \pm 90^\circ) = \mp \sin c$. The bracket has the sign of $\sin c$. So τ rises for $0^\circ < c < 180^\circ$ and falls for $180^\circ < c < 360^\circ$: it is greatest for the half centred on apoapsis ($c = 180^\circ$), and least for the half centred on periapsis. The northern half is centred on $u = 90^\circ$, i.e. on $f = 90^\circ - \omega$, so $c = 180^\circ$ means $\omega = 270^\circ$.

The apoapsis is then at $u = \omega + 180^\circ = 90^\circ$, so by Problem 5 its latitude is $\delta = i$: the most northerly point of the orbit. A Molniya satellite ($e \approx 0.74$) therefore spends about 92% of each period in the north, hanging slowly over high northern latitudes. That only lasts if ω does not drift, which is what the critical inclination of Problem 10 provides.

10. What J_2 does to the elements

To first order, the Earth’s oblateness leaves a , e and i unchanged on average and makes the node and the periapsis drift at

$$\dot{\Omega} = -\frac{3}{2} k \cos i, \quad \dot{\omega} = \frac{3}{4} k (5 \cos^2 i - 1), \quad k = nJ_2 \left(\frac{R_\oplus}{p} \right)^2 > 0.$$

- A sun-synchronous orbit needs $\dot{\Omega}$ equal to the Sun’s eastward rate, which is positive. Show that it must be retrograde, that its inclination increases with a (for a given e), and that there is a largest a for which one exists.
- Show that the apse line is fixed within the orbit plane only at $i = \arctan 2$ and at $i = 180^\circ - \arctan 2$.
- For an equatorial orbit neither Ω nor ω exists. Show that $\varpi = \Omega + \omega$ nevertheless advances at $\frac{3}{2}k$, and that $\xi = e \cos \varpi$, $\eta = e \sin \varpi$ obey $\dot{\xi} = -\frac{3}{2}k\eta$, $\dot{\eta} = \frac{3}{2}k\xi$. Why is this statement well posed even as $e \rightarrow 0$?
- Show that a polar orbit keeps its node fixed and its apse line regresses at $\frac{3}{4}k$.

Answer. All four parts come from reading the two rates. k is positive, so the signs of $\dot{\Omega}$ and $\dot{\omega}$ are set by $\cos i$ and by $5 \cos^2 i - 1$.

(a) $\dot{\Omega} = -\frac{3}{2}k \cos i$ must equal the Sun’s rate $\dot{\alpha}_\odot > 0$. With $k > 0$ that needs $\cos i < 0$: the orbit must be retrograde. To see how i depends on a , write out k . With $n = \sqrt{\mu} a^{-3/2}$ and $p = a(1 - e^2)$,

$$k = nJ_2 \left(\frac{R_\oplus}{p} \right)^2 = \sqrt{\mu} J_2 R_\oplus^2 a^{-7/2} (1 - e^2)^{-2}.$$

Setting $\dot{\Omega} = \dot{\alpha}_\odot$ and solving for $\cos i$,

$$\cos i = -\frac{2\dot{\alpha}_\odot}{3k} = -\frac{2\dot{\alpha}_\odot(1 - e^2)^2 a^{7/2}}{3\sqrt{\mu} J_2 R_\oplus^2}.$$

So $|\cos i|$ grows like $a^{7/2}$: as a grows, i climbs from just above 90° towards 180° . The higher the orbit, the weaker the J_2 effect, and the closer to the equator the orbit must be to drift fast enough. A solution needs $|\cos i| \leq 1$, which fails beyond the a at which the right-hand side reaches -1 . (The deck finds $a = 12\,352$ km for circular orbits.)

(b) The apse line is fixed when $\dot{\omega} = 0$, i.e. $\cos^2 i = \frac{1}{5}$. To put this in terms of $\tan i$, use $\tan^2 i = (1 - \cos^2 i)/\cos^2 i = (4/5)/(1/5) = 4$. So $\tan i = \pm 2$, which gives $i = \arctan 2$ (63.4°) or $180^\circ - \arctan 2$ (116.6°).

(c) At $i = 0$, Ω and ω do not exist separately, but the formulas still give their sum:

$$\dot{\varpi} = \dot{\Omega} + \dot{\omega} = k \left(-\frac{3}{2} + \frac{3}{4} \cdot 4 \right) = \frac{3}{2}k.$$

For an equatorial orbit ϖ is the angle of the periapsis from $\hat{\mathbf{I}}$, so $(\xi, \eta) = (e \cos \varpi, e \sin \varpi)$ are the components of the eccentricity vector in the equatorial plane. Since $\dot{e} = 0$, differentiating gives

$$\dot{\xi} = -e \sin \varpi \dot{\varpi} = -\frac{3}{2}k \eta, \quad \dot{\eta} = e \cos \varpi \dot{\varpi} = \frac{3}{2}k \xi.$$

So the eccentricity vector rotates rigidly at $\frac{3}{2}k$. Unlike Ω , ω and ϖ , the pair (ξ, η) needs neither a node nor a periapsis to be defined. The equations are linear, and they stay valid through $e = 0$, where they simply say that a circular orbit stays circular. This is the cure for the degeneracies that Problems 6 and 8 exposed.

(d) At $i = 90^\circ$, $\cos i = 0$, so $\dot{\Omega} = 0$ and $\dot{\omega} = \frac{3}{4}k(0 - 1) = -\frac{3}{4}k$. The plane stays fixed in inertial space, while the periapsis regresses within it. This is the $i = 90^\circ$ row of the deck's table, where "a true polar orbit does not precess at all" refers to the plane.