Preliminary Orbit Determination

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Where we are going

Every note so far began with the sentence "a satellite is at **r** with velocity **v**". Nobody hands us that. What we have are **measurements**: a radar's range and angles, a telescope's right ascension and declination, a pair of GPS fixes. **Orbit determination** is the work of turning them into the six numbers that fix an orbit. It comes in two stages.

- **Preliminary** (or *initial*) **orbit determination**, the subject of most of this note, finds a state from the *smallest* set of observations that can determine it, with no first guess, using the two-body model and a few lines of algebra. The answer is rough, but it is all you have on the first pass over a new object.
- **Refinement** uses *all* the observations and a better dynamical model, and adjusts the state until the predicted observations match the measured ones as closely as possible, in the sense of least squares. It needs a first guess, and the preliminary answer is that guess. Section 9 gives the idea.

The preliminary methods are classified by what the observer measured:

Data at each epoch	Method	Section
three positions at three times	Gibbs; Herrick–Gibbs when they are close	Section 2, Section 3
two positions and the time between them	Lambert	Section 4

Data at each epoch	Method	Section
three directions at three times (angles only)	Laplace; Gauss	Section 6, Section 7
range, azimuth, elevation and their rates	direct conversion	topocentric note

What you should be able to answer at the end

(1) Why does an angles-only orbit need three observations, while a radar observation with rates needs one? (2) How does Gibbs's method find the velocity from three positions *without using the times*, and what does that cost? (3) Why does Gibbs's method fail when the positions are close together, and what does Herrick–Gibbs do instead? (4) How is Lambert's problem an orbit-determination method? (5) How do Laplace's and Gauss's methods turn three lines of sight into a state, where does the polynomial of degree eight come from, and which root is the right one? (6) Why is angles-only orbit determination so sensitive to measurement noise and to the length of the arc, and what makes a pass a good one? (7) What does least squares add, and what does the covariance matrix tell you?

The background is the two-body problem, the orbital elements (the algorithm from a state to the elements is used at the end of every example), the Lagrange coefficients and Lambert solver of the universal-variables note, and, for the observations themselves, the topocentric note: the site vector, the right ascension and declination of a line of sight, and the warning that a topocentric direction is not a geocentric one. The note uses that note's notation, and adds the symbols below. As elsewhere, a blue **Try it** box asks for a small calculation before you read on and a green **Check yourself** box asks a short question; each hides its answer.

Symbol	Meaning
t_1, t_2, t_3	the three observation epochs; $\tau_1 = t_1 - t_2 < 0$, $\tau_3 = t_3 - t_2 > 0$, $\tau = t_3 - t_1$
$\mathbf{r}_i$, $\mathbf{v}_i$	inertial position and velocity of the satellite at t_i; $r_i = \lVert\mathbf{r}_i\rVert$

Symbol	Meaning
$\mathbf{R}_i$	inertial position of the station at t_i: the site vector $\mathbf{r}_{\text{site}}$ of the topocentric note rotated through the Greenwich sidereal angle, $\mathbf{R}_i = \mathbf{R}_3(\theta_G(t_i))\,\mathbf{r}_{\text{site}}$
$\dot{\mathbf{R}}$, $\ddot{\mathbf{R}}$	velocity and acceleration of the station: $\omega_\oplus \times \mathbf{R}$ and $\omega_\oplus \times (\omega_\oplus \times \mathbf{R})$ (the site is fixed on the Earth, so only the rotation moves it)
ρ_i, $\mathbf{L}_i$	range from the station, and the unit line of sight in inertial components, $\mathbf{L}_i \equiv \hat{\rho}_i$ of the topocentric note; $\mathbf{r}_i = \mathbf{R}_i + \rho_i \mathbf{L}_i$
α_i, δ_i	topocentric right ascension and declination, $\mathbf{L}_i = (\cos\delta_i \cos\alpha_i,\ \cos\delta_i \sin\alpha_i,\ \sin\delta_i)$
F_i, G_i	Lagrange coefficients from t_2 to t_i: $\mathbf{r}_i = F_i \mathbf{r}_2 + G_i \mathbf{v}_2$
p, $\mathbf{e}$, $\mathbf{h}$	semi-latus rectum, eccentricity vector (to periapsis), angular momentum

1 What data do we need?

A two-body orbit is fixed by six numbers, so the observations must supply six independent numbers. The table of the introduction is just this count.

- **A radar with Doppler** gives ρ, A, ε and their rates, six numbers at *one* epoch. That is the state itself, and the topocentric note shows the conversion. Nothing more is needed.
- **A position** (three numbers) at one epoch is half a state; a second epoch of position gives six numbers, and the *time between* the two fixes closes the problem: that is Lambert's. Three positions are nine numbers, more than needed, and Gibbs's method uses them without the times.
- **A direction** (right ascension and declination) is *two* numbers; three epochs make six. At each epoch, one number is missing, the range ρ_i. Three directions therefore determine the orbit *only because the orbit is a Kepler orbit*: the dynamics supply the three missing numbers. This is the hard case, and it is where most of the note goes.

Common mistakes with the data

- **Geocentric for topocentric directions.** A telescope's right ascension and declination are *topocentric*. For a satellite at 1400 km range the topocentric and geocentric directions differ by tens of degrees (the parallax warning of the topocentric note). Every $\mathbf{L}_i$ below is the direction from the *station*.
- **One station vector.** $\mathbf{R}_i$ must be computed at *each* epoch, with the sidereal angle of that epoch; the Earth turns 0.25° per minute.
- **The wrong order of times.** $\tau_1 = t_1 - t_2$ is *negative*. Signs go wrong silently when τ_1 is entered as a positive number.

1.1 The example used throughout

We know the truth in every example, so each method's error can be reported (reality, of course, offers no such luxury). The station is Sriharikota, 13.72°N, 80.23°E, height 10 m. The epoch $t = 0$ is 14:00 UT on 14 October 2026, when the Greenwich sidereal angle is $\theta_G = 233.131°$ (Curtis's formula, as in the time note). The satellite moves on a two-body orbit with

$$a = 7800 \text{ km}, \quad e = 0.05826, \quad i = 43.081°,$$
$$\Omega = 290.224°, \quad \omega = 319.269°, \quad f(0) = 66.929°,$$

so that the period is 114.3 min, the perigee radius is 7345.5 km and the apogee radius 8254.5 km. At $t = 0$,

$$\mathbf{r}(0) = (4656.838,\ -5551.747,\ 2291.717) \text{ km}, \qquad r = 7600 \text{ km},$$
$$\mathbf{v}(0) = (3.621537,\ 4.413025,\ 4.604554) \text{ km/s}.$$

The pass is shown in Figure 1. The satellite rises in the south-west, passes the station on its north-west side with a peak elevation of 60.6° (at $t = -2$ s) and sets in the north-east: above 10° for 927 s, from $t = -418$ s to $t = +509$ s. At the five epochs we use, the station sees:

t [s]	range [km]	azimuth	elevation	α	δ
−300	2269.593	245.333°	20.887°	249.4397°	−17.1113°
−60	1394.200	288.152°	56.399°	278.7186°	21.4097°
0	1370.344	320.697°	60.603°	291.0129°	35.1436°
+60	1449.817	351.649°	56.185°	306.7986°	47.0505°
+300	2410.785	30.931°	25.355°	18.0587°	58.7162°

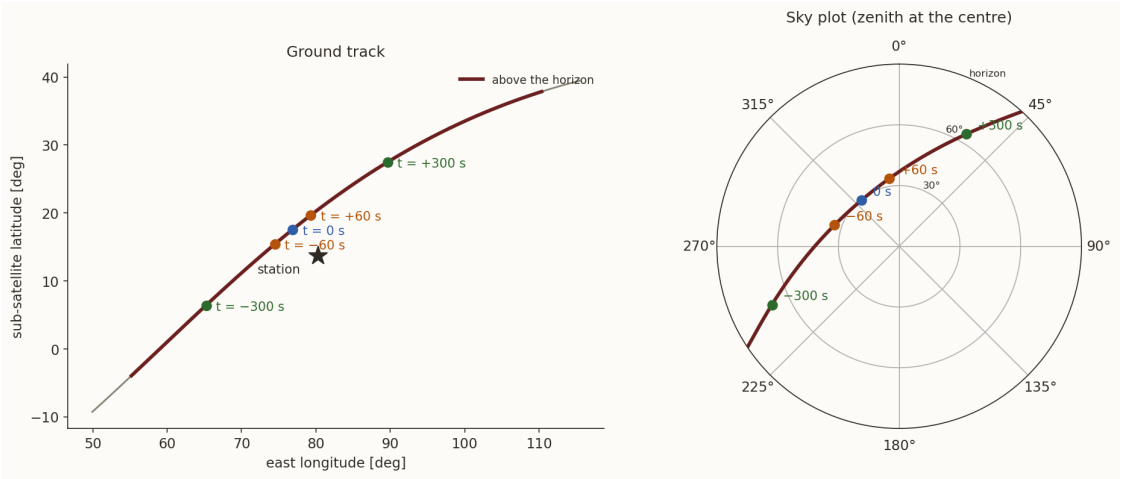


Figure 1: The pass used in every example. **Left:** the ground track (the grey parts are below the horizon), the station, and the five epochs. **Right:** the sky plot as seen from the station, zenith at the centre, azimuth round the edge. The satellite is 1370 km from the station at the middle of the pass and 2400 km at $t = \pm 300$ s. *Script: fig-od.py, helper _od.py.*

The azimuth, elevation and range of the table are computed with the helper module of the topocentric note (and checked against the line of sight built directly in inertial components by the verify script).

2 Three positions: the method of Gibbs

Suppose that a radar, or a pair of angle-and-range measurements converted with the topocentric note, has given the **position** of the satellite at three epochs, $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, and that we want the velocity $\mathbf{v}_2$ at the middle one. Gibbs's method uses only *geometry*: three points on a conic with a focus at the origin fix the conic. The times do not enter at all.

2.1 The conic as a linear equation

The orbit equation $r = p/(1 + e\cos f)$ is the same as $r + e\,r\cos f = p$. Let $\mathbf{e}$ be the eccentricity vector, of length e, pointing from the focus to periapsis. Then $e\,r\cos f = \mathbf{e}\cdot\mathbf{r}$, and

$$r + \mathbf{e}\cdot\mathbf{r} = p. \tag{1}$$

This is a **linear** equation in the unknown vector $\mathbf{e}$ and the unknown number p, for every point of the orbit. At the three observed points,

History: Gibbs. J. Willard Gibbs (the physicist of thermodynamics and vector analysis) published a vector treatment of orbit determination in 1889 (Gibbs 1889). The step from three position vectors to the orbit, which is the method of this section, is the part of it that

$$\mathbf{e} \cdot \mathbf{r}_i = p - r_i, \qquad i = 1, 2, 3. \qquad (*)$$

The plan is to combine the three equations $(*)$ so as to eliminate $\mathbf{e}$ and get p, then to eliminate p and get $\mathbf{e}$, and finally to get the velocity from p and $\mathbf{e}$.

2.2 The three positions must be coplanar

All three points lie in the orbital plane, so each cross product $\mathbf{r}_i \times \mathbf{r}_j$ is a multiple of the same unit vector $\hat{\mathbf{h}}$: write $\mathbf{r}_i \times \mathbf{r}_j = [ij]\,\hat{\mathbf{h}}$. Three vectors in a plane are linearly dependent, and the coefficients are exactly these brackets:

$$[23]\,\mathbf{r}_1 + [31]\,\mathbf{r}_2 + [12]\,\mathbf{r}_3 = \mathbf{0}. \qquad (2)$$

To see it, call the left-hand side $\mathbf{V}$ and cross it with $\mathbf{r}_1$: since $\mathbf{r}_2 \times \mathbf{r}_1 = -[12]\hat{\mathbf{h}}$ and $\mathbf{r}_3 \times \mathbf{r}_1 = [31]\hat{\mathbf{h}}$, $\mathbf{V} \times \mathbf{r}_1 = -[31][12]\hat{\mathbf{h}} + [12][31]\hat{\mathbf{h}} = \mathbf{0}$. In the same way $\mathbf{V} \times \mathbf{r}_2 = [23][12]\hat{\mathbf{h}} - [12][23]\hat{\mathbf{h}} = \mathbf{0}$. A vector with zero cross product with two non-parallel vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$, is zero, so $\mathbf{V} = \mathbf{0}$. (With noisy data the points are never exactly coplanar, and the size of $\hat{\mathbf{r}}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)/\|\mathbf{r}_2 \times \mathbf{r}_3\|$ is the natural test of the data: it is the sine of the angle by which $\mathbf{r}_1$ leaves the plane of the other two.)

2.3 The semi-latus rectum

Dot Equation 2 with $\mathbf{e}$ and use $(*)$:

$$\begin{aligned} &[23](p - r_1) + [31](p - r_2) + [12](p - r_3) = 0 \\ &\qquad\Longrightarrow\quad p\,([12] + [23] + [31]) = r_1[23] + r_2[31] + r_3[12]. \end{aligned}$$

Multiplying by $\hat{\mathbf{h}}$ turns the brackets back into vectors. Define

$$\begin{aligned} \mathbf{D} &= \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1, \\ \mathbf{N} &= r_1\,\mathbf{r}_2 \times \mathbf{r}_3 + r_2\,\mathbf{r}_3 \times \mathbf{r}_1 + r_3\,\mathbf{r}_1 \times \mathbf{r}_2, \end{aligned} \qquad (3)$$

so that $p\,\mathbf{D} = \mathbf{N}$. Both vectors point along $\hat{\mathbf{h}}$, hence

$$p = \frac{\|\mathbf{N}\|}{\|\mathbf{D}\|}. \qquad (4)$$

2.4 The eccentricity vector

For the other combination we use the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{e} = \mathbf{b}(\mathbf{a}\cdot\mathbf{e}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{e})$ and $(*)$:

$$\begin{aligned}(\mathbf{r}_1\times\mathbf{r}_2)\times\mathbf{e} &= \mathbf{r}_2\,(\mathbf{r}_1\cdot\mathbf{e}) - \mathbf{r}_1\,(\mathbf{r}_2\cdot\mathbf{e})\\ &= \mathbf{r}_2\,(p-r_1) - \mathbf{r}_1\,(p-r_2) = p\,(\mathbf{r}_2-\mathbf{r}_1) + r_2\,\mathbf{r}_1 - r_1\,\mathbf{r}_2.\end{aligned}$$

Add this to its two cyclic permutations. On the left, the three terms add to $\mathbf{D}\times\mathbf{e}$. On the right, the terms in p cancel in the cyclic sum ($\sum(\mathbf{r}_2-\mathbf{r}_1)=\mathbf{0}$), and what remains is

$$\mathbf{D}\times\mathbf{e} = \mathbf{S}, \qquad \mathbf{S} = \mathbf{r}_1\,(r_2-r_3) + \mathbf{r}_2\,(r_3-r_1) + \mathbf{r}_3\,(r_1-r_2). \tag{5}$$

Since $\mathbf{e}$ lies in the plane, it is perpendicular to $\mathbf{D}$, and $\mathbf{D}\times(\mathbf{D}\times\mathbf{e}) = -D^2\mathbf{e}$. So $\mathbf{D}\times\mathbf{S} = -D^2\mathbf{e}$, that is

$$\mathbf{e} = \frac{\mathbf{S}\times\mathbf{D}}{D^2}. \tag{6}$$

(For a circular orbit all r_i are equal, $\mathbf{S}=\mathbf{0}$ and $\mathbf{e}=\mathbf{0}$, as it should: Equation 5 contains the *differences* of the radii, which is what an eccentric orbit adds.) Figure 2 shows the vectors in the orbital plane: $\mathbf{S}$ is perpendicular to $\mathbf{e}$, because $\mathbf{D}\times\mathbf{e}$ is.

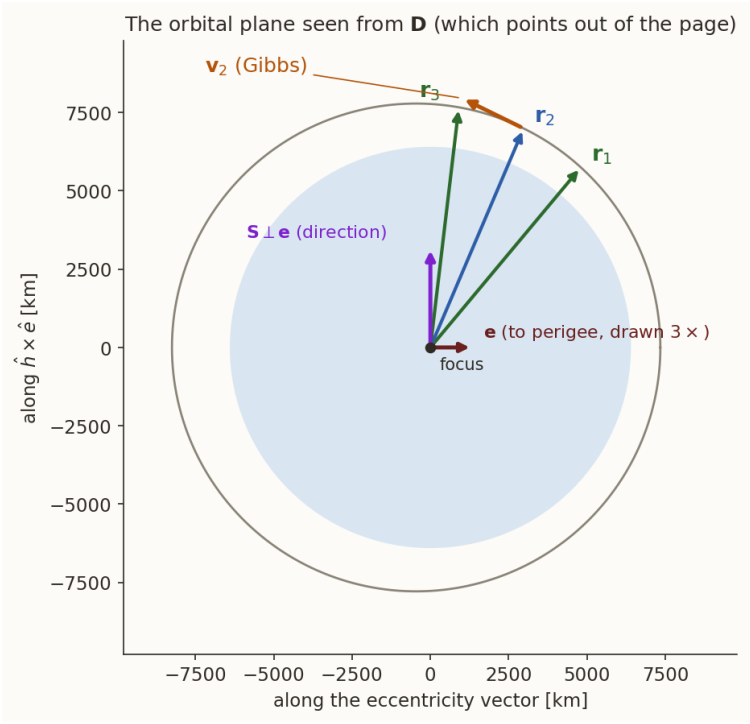


Figure 2: The orbital plane of the example, seen from the direction of **D** (out of the page). The three positions of the worked example below ($t = -300, 0, 300$ s), the eccentricity vector (to periapsis, drawn three times too long to be seen) and the direction of **S**, which is perpendicular to it. The arrow at $\mathbf{r}_2$ is the velocity that the method returns. *Script: fig-od.py.*

2.5 The velocity

In the perifocal frame, with $\hat{\mathbf{P}}$ along **e** and $\hat{\mathbf{Q}} = \hat{\mathbf{h}} \times \hat{\mathbf{P}}$, the velocity on a conic is $\mathbf{v} = (\mu/h)[-\sin f\, \hat{\mathbf{P}} + (e + \cos f)\hat{\mathbf{Q}}]$ (the perifocal note derives it as the hodograph, a circle centred at $(0, e\mu/h)$). Since $\hat{\mathbf{h}} \times \hat{\mathbf{P}} = \hat{\mathbf{Q}}$, $\hat{\mathbf{h}} \times \hat{\mathbf{Q}} = -\hat{\mathbf{P}}$, and $\hat{\mathbf{r}} = \cos f\, \hat{\mathbf{P}} + \sin f\, \hat{\mathbf{Q}}$, the bracket is $\hat{\mathbf{h}} \times (\mathbf{e} + \hat{\mathbf{r}})$:

$$\mathbf{v} = \frac{\mu}{h}\, \hat{\mathbf{h}} \times (\mathbf{e} + \hat{\mathbf{r}}), \qquad h = \sqrt{\mu p}. \tag{7}$$

Now put in what we have. $\hat{\mathbf{h}} = \mathbf{D}/D$, and by Equation 5, $\hat{\mathbf{h}} \times \mathbf{e} = \mathbf{S}/D$; also $\hat{\mathbf{h}} \times \hat{\mathbf{r}}_2 = (\mathbf{D} \times \mathbf{r}_2)/(D\, r_2)$. With $p = N/D$, the prefactor is $\mu/h = \sqrt{\mu/p} = \sqrt{\mu D/N}$, and $\sqrt{\mu D/N}\,/D = \sqrt{\mu/(ND)}$. Therefore

$$\boxed{\mathbf{v}_2 = \sqrt{\frac{\mu}{N\,D}}\left(\frac{\mathbf{D} \times \mathbf{r}_2}{r_2} + \mathbf{S}\right)} \tag{8}$$

with $N = \|\mathbf{N}\|$ and $D = \|\mathbf{D}\|$. Three positions in, one velocity out; the elements follow from $(\mathbf{r}_2, \mathbf{v}_2)$ by the algorithm of the elements note.

Algorithm: Gibbs

1. Form the cross products $\mathbf{r}_i \times \mathbf{r}_j$ and the radii; test coplanarity.
2. $\mathbf{N}$, $\mathbf{D}$ and $\mathbf{S}$ from Equation 3 and Equation 5.
3. $\mathbf{v}_2$ from Equation 8. (Also $p = N/D$, and $\mathbf{e}$ from Equation 6.)

2.6 Example 1: three positions 300 s apart

A radar has fixed the satellite of the scenario at $t = -300,\ 0,\ +300$ s, with positions rounded to 1 m:

$$\mathbf{r}_1 = \begin{bmatrix} 3393.672 \\ -6628.680 \\ 835.129 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 4656.838 \\ -5551.747 \\ 2291.717 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 5542.775 \\ -4023.494 \\ 3563.179 \end{bmatrix} \text{ km}.$$

The angles between neighbouring positions are 16.81° and 16.31° (33.12° between $\mathbf{r}_1$ and $\mathbf{r}_3$). The coplanarity measure is -1.3×10^{-7}, an angle of 7×10^{-6} degrees. Then

$$\mathbf{N} = \begin{bmatrix} -6.66000 \\ -2.45350 \\ 7.58961 \end{bmatrix} \times 10^9 \text{ km}^3, \qquad \mathbf{D} = \begin{bmatrix} -856\,755.6 \\ -315\,622.4 \\ 976\,342.4 \end{bmatrix} \text{ km}^2,$$

$$\mathbf{S} = \begin{bmatrix} 58\,016.6 \\ -32\,787.0 \\ 40\,311.4 \end{bmatrix} \text{ km}^2,$$

with $N = 1.03912 \times 10^{10}$ km^3, $D = 1.33675 \times 10^6$ km^2 and $S = 7.7884 \times 10^4$ km^2. So $p = N/D = 7773.515$ km (the true value is 7773.521 km), and $e = \|\mathbf{S} \times \mathbf{D}\|/D^2 = 0.058264$ (true 0.058264). Equation 8 gives

$$\mathbf{v}_2 = (3.621536,\ 4.413022,\ 4.604552) \text{ km/s},$$

against the true $(3.621537,\ 4.413025,\ 4.604554)$ km/s: an error of 0.0030 m/s, which is the effect of rounding the positions by a metre. The elements of $(\mathbf{r}_2, \mathbf{v}_2)$ are $a = 7799.994$ km, $e = 0.05826$, $i = 43.0811^\circ$, $\Omega = 290.2235^\circ$, $\omega = 319.2679^\circ$ and $f = 66.9301^\circ$, which are the true ones to the digits given in the scenario.

Compute $\mathbf{D}$, $\mathbf{N}$ and p from the three positions above, and check that $\mathbf{D}$ is parallel to the orbit's $\mathbf{h} = \mathbf{r}_2 \times \mathbf{v}_2$.

What you should find

D and **N** as in the text, $p = 7773.515$ km, and the angle between **D** and $\mathbf{r}_2 \times \mathbf{v}_2$ is zero to the digits of the data ($\cos\theta = 1.0000000$).

Gibbs's method uses no times. Where is the information in the times, and is any of it needed?

Answer

For an exact Kepler orbit the times are not needed: the geometry of three points on a conic already fixes the orbit. (The three positions are nine numbers. The shape and orientation of the orbit take five, the three places on it three more, and the ninth number is the coplanarity condition; no number is left over for the time.) But the times are not useless: they must agree with Kepler's equation, and that is a **check on the data**. Propagating $(\mathbf{r}_2, \mathbf{v}_2)$ of the example by ∓ 300 s should return $\mathbf{r}_1$ and $\mathbf{r}_3$; it does, to 1.2 m and 1.5 m, the rounding of the data. And when the data are noisy, the times are information that Gibbs throws away: Section 4 shows that a method which uses them gets a better velocity from *fewer* positions.

3 Close positions: Herrick–Gibbs

Gibbs's formula is exact, so one might expect it to be best. It is not, when the three positions are close together. Figure 3 (left) shows why this happens and what to do instead.

3.1 Why Gibbs fails

Look at $\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1$. On a circular orbit of radius r with neighbouring positions ψ apart, each of the first two terms has length $r^2 \sin\psi$ and the third, pointing the other way, $r^2 \sin 2\psi$, so

$$D = r^2\,(2\sin\psi - \sin 2\psi) \approx r^2\psi^3, \qquad \text{while each term is} \approx r^2\psi.$$

The sum is ψ^2 times smaller than its terms: three numbers of size $r^2\psi$ that nearly cancel. A position error $\delta r/r$ shows up in D as a relative error of about $(\delta r/r)/\psi^2$. At 1.1° between $\mathbf{r}_1$ and $\mathbf{r}_3$ ($\psi = 0.55°$), $\|\mathbf{D}\|/\|\mathbf{r}_1 \times \mathbf{r}_2\| = 9 \times 10^{-5}$, and a 1 m rounding of positions 7600 km from the centre (1.3×10^{-7}) is magnified about ten thousand times. The same cancellation sits in **N** and **S**. Figure 3 (right) shows that the ratio is indeed ψ^2 over the whole range, for the orbit of the example.

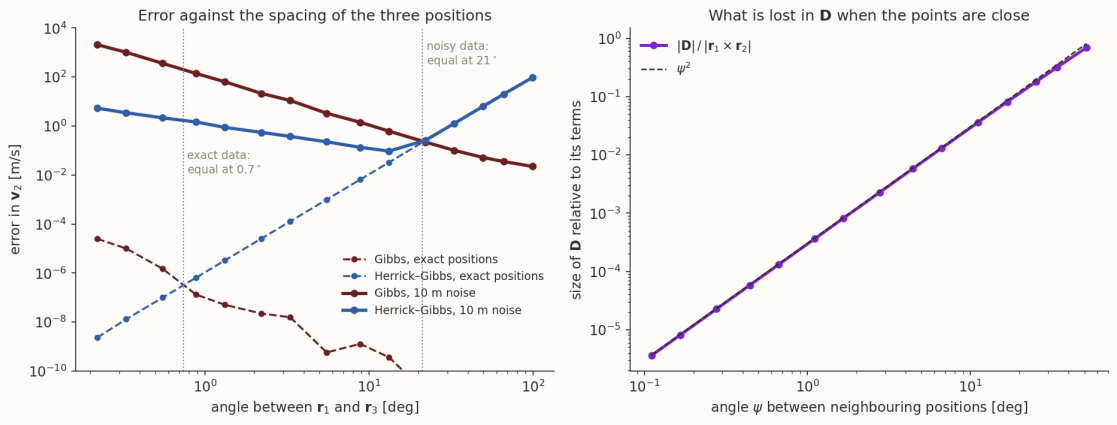


Figure 3: **Left:** error in the velocity at the middle epoch, against the angle between $\mathbf{r}_1$ and $\mathbf{r}_3$ ($t = -\tau, 0, +\tau$). Dashed: exact positions (double-precision arithmetic only). Solid: positions with 10 m of Gaussian noise per coordinate, median of 400 trials. **Right:** the size of $\mathbf{D}$ relative to its largest term, which falls as ψ^2. *Script: fig-od.py, experiment _odmc.py.*

3.2 The idea of Herrick–Gibbs

When the points are close, the time spacing is short, and the velocity can be had by a **Taylor series in time** instead of the conic. The new ingredient is the times t_1, t_2, t_3. The derivation has two steps.

Step 1: interpolation. Expand the positions about t_2,

$$\mathbf{r}_i - \mathbf{r}_2 = \tau_i \mathbf{v}_2 + \tfrac{1}{2}\tau_i^2 \mathbf{a}_2 + \tfrac{1}{6}\tau_i^3 \dot{\mathbf{a}}_2 + \tfrac{1}{24}\tau_i^4 \ddot{\mathbf{a}}_2 + O(\tau^5), \qquad i = 1, 3.$$

Eliminate the acceleration $\mathbf{a}_2$ (which we have not got yet) by forming $\tau_3^2(\mathbf{r}_1 - \mathbf{r}_2) - \tau_1^2(\mathbf{r}_3 - \mathbf{r}_2)$; the terms in $\tau_1^2\tau_3^2$ cancel, and

$$\begin{aligned} \mathbf{v}_2 = {} & \frac{\tau_3^2(\mathbf{r}_1 - \mathbf{r}_2) - \tau_1^2(\mathbf{r}_3 - \mathbf{r}_2)}{\tau_1\tau_3(\tau_3 - \tau_1)} \\ & + \frac{\tau_1\tau_3}{6}\,\dot{\mathbf{a}}_2 + \frac{\tau_1\tau_3(\tau_1 + \tau_3)}{24}\,\ddot{\mathbf{a}}_2 + O(\tau^4). \end{aligned} \tag{9}$$

The first term is the velocity of the parabola through the three points, and the others are what it misses.

Step 2: the missing terms are known. The jerk $\dot{\mathbf{a}}_2$ and the snap $\ddot{\mathbf{a}}_2$ are derivatives of the acceleration, and the acceleration is **a function of position alone**, $\mathbf{a}_i = -\mu\mathbf{r}_i/r_i^3$, so we know $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ from the three positions. Expand them about t_2 too, $\mathbf{a}_i = \mathbf{a}_2 + \tau_i\dot{\mathbf{a}}_2 + \frac{1}{2}\tau_i^2\ddot{\mathbf{a}}_2 + \ldots$, and form $\tau_3\mathbf{a}_1 - (\tau_1 + \tau_3)\mathbf{a}_2 + \tau_1\mathbf{a}_3$: the terms in $\mathbf{a}_2$ cancel, and

$$\tau_3\mathbf{a}_1 - (\tau_1 + \tau_3)\mathbf{a}_2 + \tau_1\mathbf{a}_3 = 2\tau_1\tau_3\,\dot{\mathbf{a}}_2 + \tfrac{1}{2}\tau_1\tau_3(\tau_1 + \tau_3)\,\ddot{\mathbf{a}}_2 + O(\tau^5).$$

One twelfth of this is *exactly* the two correction terms of Equation 9. So

$$\boxed{\mathbf{v}_2 = \frac{\tau_3^2(\mathbf{r}_1 - \mathbf{r}_2) - \tau_1^2(\mathbf{r}_3 - \mathbf{r}_2)}{\tau_1\tau_3(\tau_3 - \tau_1)} - \frac{\mu}{12}\left[\tau_3\frac{\mathbf{r}_1}{r_1^3} - (\tau_1 + \tau_3)\frac{\mathbf{r}_2}{r_2^3} + \tau_1\frac{\mathbf{r}_3}{r_3^3}\right]} \tag{10}$$

with an error $O(\tau^4)$: for equal spacing, $+\frac{7}{360}\tau^4\,\mathbf{r}_2^{(5)}$. (The verify script confirms the whole chain symbolically: the combination reproduces the derivative of *any* polynomial history of degree ≤ 4 exactly, the interpolation alone is exact only through degree 2, and the leading error is the one quoted.) The conic never appeared: Equation 10 is a statement about smooth curves, and gravity entered only through $\mathbf{a} = -\mu\mathbf{r}/r^3$.

3.3 Example 2: three positions 10 s apart

The scenario at $t = -10, 0, +10$ s, positions rounded to 1 m: $\mathbf{r}_1 = (4620.412, -5595.624, 2245.568)$, $\mathbf{r}_2 = (4656.838, -5551.747, 2291.717)$, $\mathbf{r}_3 = (4692.842, -5507.365, 2337.657)$ km. The angle between $\mathbf{r}_1$ and $\mathbf{r}_3$ is 1.10°.

	$\mathbf{v}_2$ [km/s]	error [m/s]
truth	(3.621537, 4.413025, 4.604554)	
Gibbs, Equation 8	(3.620416, 4.411655, 4.603080)	2.30
Herrick–Gibbs, Equation 10	(3.621544, 4.413030, 4.604514)	0.040

With *exact* positions both are good to 10^{-6} m/s; the difference is entirely the metre of rounding, amplified by Gibbs's cancellation and not by Herrick–Gibbs's.

History. The combination of a Taylor series in the times with the gravitational acceleration at the three points is usually called the *Herrick–Gibbs* method in the textbooks (Vallado 2013), a close-spaced companion of Gibbs's method.

3.4 Which one when?

Figure 3 shows two regimes. With exact data, Gibbs is better for every spacing larger than 0.74° (below that, double precision itself starts to be exhausted), and Herrick–Gibbs's τ^4 truncation error takes over beyond. With 10 m of noise per coordinate, which is the realistic case, the curves cross at 21°: below that Herrick–Gibbs is better, above it Gibbs. Herrick–Gibbs's best error is 0.09 m/s at about 13°, and Gibbs's error keeps decreasing as the points spread out (0.10 m/s at 33°, 0.02 m/s at 99°) because noise matters less for widely separated points. The rule of thumb is therefore *Herrick–Gibbs for points within a few tens of degrees of each other, Gibbs beyond*, and *the noise level decides the dividing line*.

4 Two positions and the time: Lambert

If the time of flight Δt between two position fixes is known, the velocity follows from **Lambert's problem**: find the Kepler orbit that joins $\mathbf{r}_1$ to $\mathbf{r}_3$ in time Δt. The Lambert note formulates and solves it, and the universal-variables note gives the solver used here; as an orbit-determination method it needs no new theory, only a change of viewpoint: the solver returns $\mathbf{v}_1$ and $\mathbf{v}_3$, and the state at any other time is a propagation away.

Example 3. Take the outer two positions of Example 1, $\mathbf{r}_1$ and $\mathbf{r}_3$ ($t = \mp 300$ s, rounded to 1 m), and $\Delta t = 600$ s. The transfer angle is 33.117° and the universal-variable equation converges to $z = 0.3184$. The velocities returned differ from the true ones by 0.0011 m/s at $\mathbf{r}_1$ and 0.0010 m/s at $\mathbf{r}_3$; propagating $\mathbf{v}_1$ forward by 300 s gives the state at $t = 0$ to 0.33 m and 0.0011 m/s. That is better than the three-position Gibbs result of Example 1, though it used one position less: the **time** was information.

With noise the gap widens. For 10 m of noise per coordinate, the median velocity error at $t = 0$ is 0.037 m/s from Lambert (two positions, propagated) against 0.11 m/s from Gibbs (three positions) in 400 trials. The price is that *the clock must be right*: a timing error δt between the two fixes moves the satellite along its track by $v\,\delta t$, 7.3 m for a millisecond, and Lambert's method reads it as a position error. (It also has to be told the direction of motion and the number of complete revolutions; for a satellite tracked through one pass, zero.)

Run the universal-variable Lambert solver on the rounded positions $\mathbf{r}_1, \mathbf{r}_3$ of Example 1 with $\Delta t = 600$ s, then propagate to $t = 0$ and compare with the true state.

What you should find

$\mathbf{v}_1 = (4.748842,\ 2.704460,\ 5.041449)$ km/s against the truth $(4.748841,\ 2.704460,\ 5.041448)$; at $t = 0$ the error is 0.33 m in position and 1.1×10^{-3} m/s in velocity.

5 Angles only: the problem

A telescope gives two numbers per epoch, the right ascension and declination of the line of sight. With the station's position $\mathbf{R}_i$ at the epoch (the site vector of the topocentric note, turned by the sidereal angle), the satellite is somewhere along the line of sight,

$$\mathbf{r}_i = \mathbf{R}_i + \rho_i \mathbf{L}_i, \qquad \mathbf{L}_i = \begin{bmatrix} \cos\delta_i \cos\alpha_i \\ \cos\delta_i \sin\alpha_i \\ \sin\delta_i \end{bmatrix}, \tag{11}$$

and the one thing the telescope cannot say is how far: the range ρ_i. Three epochs leave **three unknown ranges**. If they were known, Equation 11 would give three positions and Gibbs's method the velocity, so the whole problem is to find ρ_1, ρ_2, ρ_3. The equations that determine them come from the dynamics: the three positions must be coplanar with the Earth's centre (one condition), and the intervals between them must agree with Kepler's equation (two more). The two classical methods use that information in different ways.

- **Laplace's method** (Section 6) is *local*: it differentiates the observed direction in time and uses the equation of motion at the middle epoch.
- **Gauss's method** (Section 7) is *global*: it writes the three positions as combinations of one another through the Lagrange coefficients, which is an exact statement about Kepler motion.

Both end in the same polynomial of degree eight for r_2. A point to keep in mind throughout: the range is not *measured*; it is *inferred from the curvature* of the line-of-sight track on the sky. A small curvature is a small signal, and Section 8 is about what that does.

6 Laplace's method

6.1 Derivatives of the line of sight

From the three directions we construct $\mathbf{L}_2$, $\dot{\mathbf{L}}_2$ and $\ddot{\mathbf{L}}_2$ by fitting a parabola, $\mathbf{L}(t) \approx \mathbf{L}_2 + \dot{\mathbf{L}}_2\,(t-t_2) + \tfrac{1}{2}\ddot{\mathbf{L}}_2\,(t-t_2)^2$, through the three observations. Requiring it to pass through $\mathbf{L}_1$ at $t=\tau_1$ and through $\mathbf{L}_3$ at $t=\tau_3$ gives two linear equations for $\dot{\mathbf{L}}_2$ and $\ddot{\mathbf{L}}_2$ whose solution is

$$\begin{aligned} \dot{\mathbf{L}}_2 &= \frac{\tau_3^2(\mathbf{L}_1-\mathbf{L}_2) - \tau_1^2(\mathbf{L}_3-\mathbf{L}_2)}{\tau_1\tau_3(\tau_3-\tau_1)}, \\ \ddot{\mathbf{L}}_2 &= \frac{2\,[\,\tau_3(\mathbf{L}_1-\mathbf{L}_2) - \tau_1(\mathbf{L}_3-\mathbf{L}_2)\,]}{\tau_1\tau_3(\tau_1-\tau_3)}. \end{aligned} \tag{12}$$

For equal spacing h these are the familiar central differences $(\mathbf{L}_3-\mathbf{L}_1)/2h$ and $(\mathbf{L}_1-2\mathbf{L}_2+\mathbf{L}_3)/h^2$. The first is the same quadratic interpolation as in Equation 9; the price is the error of the interpolation, which grows with the arc as h^2 and is the weak point of the method.

History: Laplace. Laplace's method, from his memoir on the orbits of comets (Laplace 1784), differentiates the observed direction of a comet and uses Newton's law at one instant. It needs the derivatives of the observed direction, which is why it asks for closely spaced, accurate observations. Gauss's method, published in 1809 (Gauss 1809), takes no derivatives of the obser-

6.2 Eliminating the range rates

Differentiate $\mathbf{r} = \mathbf{R} + \rho\mathbf{L}$ twice and set $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3 = -\mu(\mathbf{R}+\rho\mathbf{L})/r^3$:

$$\ddot{\mathbf{R}} + \ddot{\rho}\,\mathbf{L} + 2\dot{\rho}\,\dot{\mathbf{L}} + \rho\,\ddot{\mathbf{L}} = -\frac{\mu}{r^3}\,(\mathbf{R}+\rho\mathbf{L}).$$

Collect the unknown ranges on the left,

$$\ddot{\rho}\,\mathbf{L} + 2\dot{\rho}\,\dot{\mathbf{L}} + \rho\Big(\ddot{\mathbf{L}} + \frac{\mu}{r^3}\mathbf{L}\Big) = -\Big(\ddot{\mathbf{R}} + \frac{\mu}{r^3}\mathbf{R}\Big). \qquad (**)$$

This is a vector equation (three scalar ones) for the three unknowns $\rho, \dot{\rho}, \ddot{\rho}$, with the *unknown* r in the coefficients. The trick is to dot it with a vector that is perpendicular to both $\mathbf{L}$ and $\dot{\mathbf{L}}$, which kills the first two terms: $\mathbf{n} = \mathbf{L} \times \dot{\mathbf{L}}$. Then only ρ survives on the left, because $\mathbf{L} \cdot \mathbf{n} = 0$ and $\ddot{\mathbf{L}} \cdot \mathbf{n} = D$, where

$$D = \mathbf{L} \cdot (\dot{\mathbf{L}} \times \ddot{\mathbf{L}}) = \ddot{\mathbf{L}} \cdot (\mathbf{L} \times \dot{\mathbf{L}})$$

(the same triple product, cyclically permuted). Dotting $(**)$ with $\mathbf{n}$ gives $\rho\, D = -\ddot{\mathbf{R}} \cdot \mathbf{n} - \mu\,(\mathbf{R} \cdot \mathbf{n})/r^3$, that is

$$\begin{aligned} &\rho = A + \frac{B}{r^3}, \\ &A = -\frac{D_1}{D}, \quad B = -\frac{\mu D_2}{D}, \quad D_1 = \ddot{\mathbf{R}} \cdot \mathbf{n}, \quad D_2 = \mathbf{R} \cdot \mathbf{n}. \end{aligned} \qquad (13)$$

All quantities are at the middle epoch, and A, B are known from the data. Equation 13 is the first of two relations between the range ρ and the unknown distance r.

6.3 The second relation and the polynomial of degree eight

The second relation is pure geometry: $r^2 = \|\mathbf{R} + \rho\mathbf{L}\|^2$, so, with $C = \mathbf{L} \cdot \mathbf{R}$,

$$r^2 = \rho^2 + 2C\rho + R^2, \qquad \text{i.e.} \qquad \rho = -C + \sqrt{r^2 - R^2 + C^2}. \qquad (14)$$

(The root with the *minus* sign is negative whenever $C > 0$, that is whenever the satellite is above the horizon, and would put the satellite behind the station.) Substituting Equation 13 into the first form and multiplying by r^6:

$$r^8 - (A^2 + 2AC + R^2)\, r^6 - 2B\,(A + C)\, r^3 - B^2 = 0. \qquad (15)$$

It is a polynomial of degree eight in r (because ρ is a function of r^3 and the relation is quadratic in ρ). We want its positive real root.

Which root? Descartes's rule of signs counts the sign changes of the coefficients: the leading one is $+1$, the coefficient of r^6 is $-(A^2 + 2AC + R^2) = -\|\mathbf{R} + A\mathbf{L}\|^2 \le 0$ and that of r^0 is $-B^2 \le 0$. If the coefficient $-2B(A + C)$ of r^3 is also negative or zero, there is exactly one sign change and so **exactly one positive root**. A graphical argument says the same (Figure 4, left). The

function Equation 13 is *decreasing* in r when $B > 0$, while the geometric Equation 14 is *increasing*, so they cross at most once. In every example of this note, for every arc, $B > 0$ and the root is unique. (When $B < 0$ the argument fails, the sign of B depending on the geometry, and Descartes's rule allows up to three positive roots. One then keeps the roots with $\rho > 0$ and chooses among them with whatever else is known about the object, for instance that it is above the surface of the Earth. The verify script lists every positive root of every example.)

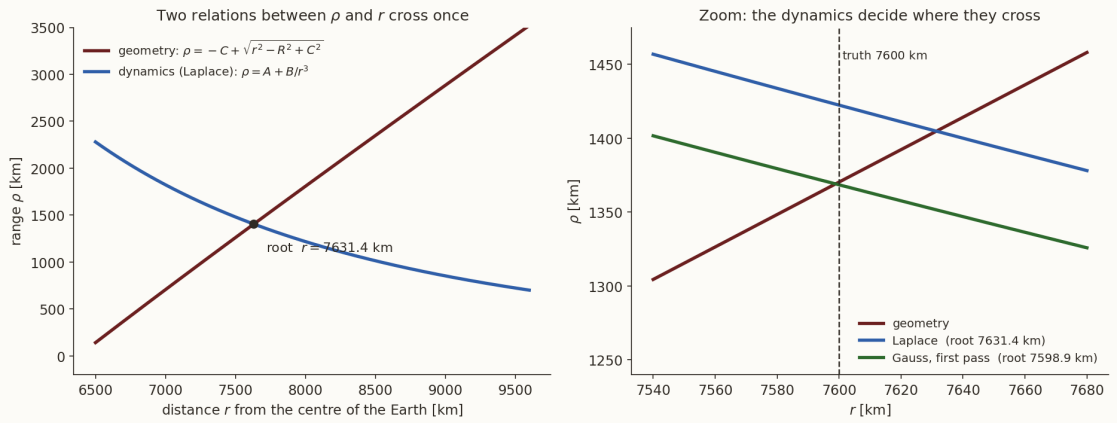


Figure 4: **Left:** the two relations between the range ρ and the geocentric distance r in the worked example, Equation 14 (maroon, increasing) and Equation 13 (blue, decreasing). They cross once. **Right:** a zoom on the crossing, with the corresponding curve of Gauss's method (green, first pass). The three roots, 7631.4 km (Laplace), 7598.9 km (Gauss) and the truth, 7600 km, differ because the *dynamical* relations differ, not the geometry. *Script: fig-od.py*.

6.4 The range rate and the state

With r known, ρ comes from Equation 13. For $\dot\rho$, dot $(**)$ with $\mathbf{m} = \mathbf{L} \times \ddot{\mathbf{L}}$, which is perpendicular to $\mathbf{L}$ and $\ddot{\mathbf{L}}$, so that both the $\ddot\rho$ term and the ρ term vanish (the ρ term is $\rho\,(\dddot{\mathbf{L}} + \mu\mathbf{L}/r^3)\cdot\mathbf{m} = 0$). What is left is $2\dot\rho\,\dot{\mathbf{L}}\cdot\mathbf{m} = -(\ddot{\mathbf{R}} + \mu\mathbf{R}/r^3)\cdot\mathbf{m}$, and $\dot{\mathbf{L}}\cdot(\mathbf{L}\times\ddot{\mathbf{L}}) = -D$, so

$$\dot\rho = \frac{(\ddot{\mathbf{R}} + \mu\mathbf{R}/r^3)\cdot\mathbf{m}}{2D}, \qquad \mathbf{r}_2 = \mathbf{R}_2 + \rho\,\mathbf{L}_2, \qquad \mathbf{v}_2 = \dot{\mathbf{R}}_2 + \dot\rho\,\mathbf{L}_2 + \rho\,\dot{\mathbf{L}}_2. \tag{16}$$

The last is the derivative of $\mathbf{r} = \mathbf{R} + \rho\mathbf{L}$.

Algorithm: Laplace

1. $\mathbf{R}_2, \dot{\mathbf{R}}_2, \ddot{\mathbf{R}}_2$ at the middle epoch; $\dot{\mathbf{L}}_2, \ddot{\mathbf{L}}_2$ from Equation 12.
2. D, D_1, D_2, then A and B from Equation 13; $C = \mathbf{L}_2\cdot\mathbf{R}_2$.

3. The positive root r of Equation 15; $\rho = A + B/r^3$.
4. $\dot\rho$, $\mathbf{r}_2$, $\mathbf{v}_2$ from Equation 16.

6.5 Example 4: three directions 60 s apart

A telescope has measured, at $t = -60, 0, +60$ s, the topocentric right ascension and declination (rounded to 10^{-6} degrees, which is 3.6 mas):

t [s]	α [deg]	δ [deg]	$\mathbf{R}$ [km]
−60	278.718649	21.409699	(4235.280, −4524.291, 1502.908)
0	291.012940	35.143647	(4255.034, −4505.718, 1502.908)
+60	306.798619	47.050460	(4274.707, −4487.058, 1502.908)

The station's velocity and acceleration at $t = 0$ are $\dot{\mathbf{R}} = (0.328562,\ 0.310282,\ 0)$ km/s and $\ddot{\mathbf{R}} = (-2.2626,\ 2.3959,\ 0) \times 10^{-5}$ km/s^2. The unit vectors are $\mathbf{L}_1 = (0.141122,\ -0.920236,\ 0.365034)$, $\mathbf{L}_2 = (0.293214,\ -0.763333,\ 0.575628)$, $\mathbf{L}_3 = (0.408134,\ -0.545591,\ 0.731954)$, and Equation 12 gives

$$\dot{\mathbf{L}}_2 = (2.2251,\ 3.1220,\ 3.0577) \times 10^{-3}\ \mathrm{s}^{-1},$$
$$\ddot{\mathbf{L}}_2 = (-1.0325,\ 1.6900,\ -1.5075) \times 10^{-5}\ \mathrm{s}^{-2}.$$

Then $D = 9.7466 \times 10^{-9}$ s^{-3}, $D_1 = 1.0268 \times 10^{-7}$ km s^{-3}, $D_2 = -15.381$ km/s, so

$$A = -10.535 \text{ km}, \qquad B = 6.2903 \times 10^{14} \text{ km}^4, \qquad C = 5552.115 \text{ km},$$

and the polynomial Equation 15 has the coefficients $-A^2 - 2AC - R^2 = -4.0549 \times 10^7$ km^2, $-2B(A + C) = -6.9717 \times 10^{18}$ km^5 and $-B^2 = -3.9568 \times 10^{29}$ km^8. The coefficient of r^3 is negative, so there is one positive root (the other seven are one negative real and three complex-conjugate pairs):

$$r = 7631.402 \text{ km}.$$

Then $\rho = 1404.806$ km, $\dot\rho = 0.45780$ km/s, and

$$\mathbf{r}_2 = (4666.943,\ -5578.052,\ 2311.554) \text{ km},$$
$$\mathbf{v}_2 = (3.58862,\ 4.34669,\ 4.55894) \text{ km/s}.$$

The truth is $\rho = 1370.344$ km, $\dot\rho = 0.48430$ km/s, $r = 7600$ km. The method has returned the state to 34.5 km and 87 m/s, and the elements $a = 7679$ km,

$e = 0.0553$ (true 7800, 0.0583) are rough. **That error is not noise** (the data are rounded to 3.6 mas): it is the error of fitting a parabola to a line of sight that moves 34° across the sky. Replace the interpolated derivatives by the exact ones, obtained from a high-order finite difference of the true line of sight, and the algebra above returns the truth to 4×10^{-7} km. The error is 1.0 km for the same method with ± 10 s, 34 km for ± 60 s, 128 km for ± 120 s and 408 km for ± 240 s: it grows roughly as the square of the arc.

At $r = 7600$ km, what range does the geometry Equation 14 give, and what range does the dynamics Equation 13 give? Which way must r move to make them agree?

What you should find

Geometry: $\rho = 1370.3$ km (that is the true range, since $r = 7600$ km is the true distance). Dynamics: $\rho = A + B/r^3 = 1422.4$ km, 52 km too large; the dynamical curve falls with r and the geometric one rises, so r must *increase*, to 7631.4 km, where both give 1404.8 km.

Why can't one simply shorten the arc until the parabola fits perfectly?

Answer

Because the *signal* that carries the range shrinks even faster than the interpolation error: the range is inferred from the curvature of the line-of-sight track, which is proportional to h^2, and the measurement noise does not shrink. With the arc ± 10 s the interpolation error is 1 km (noise-free) but a 2″ noise makes the error 77 km. Section 8 has the numbers.

7 Gauss's method

7.1 From the Lagrange coefficients to a relation between the positions

History: Gauss and Ceres. Giuseppe Piazzi, observing at Palermo, found the small planet Ceres on 1 January 1801 and followed it for 41 days before it disappeared in the glare of the Sun. The arc was short, and the problem of predicting where to look next, from a few observations, was set to the mathemati-

Kepler motion carries the state at t_2 to the state at any other time by the Lagrange coefficients (the universal-variables note): $\mathbf{r}_1 = F_1 \mathbf{r}_2 + G_1 \mathbf{v}_2$ and $\mathbf{r}_3 = F_3 \mathbf{r}_2 + G_3 \mathbf{v}_2$, where the coefficients are those of the intervals τ_1 and τ_3. These are two equations for the unknowns $\mathbf{r}_2$ and $\mathbf{v}_2$ in terms of $\mathbf{r}_1$ and $\mathbf{r}_3$. Eliminating $\mathbf{v}_2$ (multiply the first by G_3, the second by G_1, subtract) gives $G_3 \mathbf{r}_1 - G_1 \mathbf{r}_3 = (F_1 G_3 - F_3 G_1)\, \mathbf{r}_2$; eliminating $\mathbf{r}_2$ in the same way gives $F_1 \mathbf{r}_3 - F_3 \mathbf{r}_1 = (F_1 G_3 - F_3 G_1)\, \mathbf{v}_2$. So

$$\begin{aligned} \mathbf{r}_2 &= c_1\, \mathbf{r}_1 + c_3\, \mathbf{r}_3, \qquad & \mathbf{v}_2 &= \frac{F_1\, \mathbf{r}_3 - F_3\, \mathbf{r}_1}{F_1 G_3 - F_3 G_1}, \\ c_1 &= \frac{G_3}{F_1 G_3 - F_3 G_1}, \qquad & c_3 &= -\frac{G_1}{F_1 G_3 - F_3 G_1}. \end{aligned} \tag{17}$$

The first says that the middle position is a combination of the outer two, with weights c_1, c_3 (both close to $\frac{1}{2}$ for equal spacing, as the middle point is roughly the midpoint of the chord). The weights depend on the orbit only through F and G.

7.2 The three ranges

Put $\mathbf{r}_i = \mathbf{R}_i + \rho_i \mathbf{L}_i$ into $\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3$ and collect the unknown ranges on the left:

$$c_1\rho_1\,\mathbf{L}_1 - \rho_2\,\mathbf{L}_2 + c_3\rho_3\,\mathbf{L}_3 = -c_1\mathbf{R}_1 + \mathbf{R}_2 - c_3\mathbf{R}_3. \tag{18}$$

Given c_1 and c_3 this is a *linear* system for ρ_1, ρ_2, ρ_3, and it is solved by the same trick as before, dotting with the cross product of two of the $\mathbf{L}$'s to kill the other two terms. Let

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{L}_2 \times \mathbf{L}_3, \quad \mathbf{p}_2 = \mathbf{L}_1 \times \mathbf{L}_3, \quad \mathbf{p}_3 = \mathbf{L}_1 \times \mathbf{L}_2, \\ D_0 &= \mathbf{L}_1 \cdot \mathbf{p}_1, \qquad D_{ij} = \mathbf{R}_i \cdot \mathbf{p}_j. \end{aligned}$$

Then $\mathbf{L}_1 \cdot \mathbf{p}_1 = D_0$, $\mathbf{L}_2 \cdot \mathbf{p}_2 = \mathbf{L}_2 \cdot (\mathbf{L}_1 \times \mathbf{L}_3) = -D_0$, $\mathbf{L}_3 \cdot \mathbf{p}_3 = D_0$, and each product with the "wrong" $\mathbf{L}$ is zero. Dotting Equation 18 with $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in turn:

$$\begin{aligned} \rho_1 &= \frac{1}{D_0}\Big(-D_{11} + \frac{D_{21}}{c_1} - \frac{c_3}{c_1}D_{31}\Big), \\ \rho_2 &= \frac{1}{D_0}(-c_1D_{12} + D_{22} - c_3D_{32}), \\ \rho_3 &= \frac{1}{D_0}\Big(-\frac{c_1}{c_3}D_{13} + \frac{D_{23}}{c_3} - D_{33}\Big). \end{aligned} \tag{19}$$

(D_0 is the triple product of the three lines of sight; it is the determinant of the system, and it will matter in Section 8.) So everything hinges on c_1 and c_3, which depend on the orbit through F and G, which depend on the unknown state.

7.3 The first pass: series for F and G

Gauss's way out is to expand F and G in the time interval and keep the first terms. With $\mathbf{a} = -\mu\mathbf{r}/r^3$, the Taylor series of $\mathbf{r}(t_2 + \tau)$ is $\mathbf{r}_2 + \tau\mathbf{v}_2 - \frac{1}{2}\tau^2\mu\mathbf{r}_2/r_2^3 + \ldots$, so, with $u = \mu/r_2^3$,

$$F \approx 1 - \frac{u\,\tau^2}{2}, \qquad G \approx \tau - \frac{u\,\tau^3}{6}$$

(the next terms carry the radial velocity of the satellite and are smaller; here $v_r/r \approx 5\times10^{-5}\ \mathrm{s}^{-1}$). The determinant of Equation 17 is then, to first order in u,

$$F_1G_3 - F_3G_1 \approx \tau_3 - \tau_1 - u\Big[\frac{\tau_3^3}{6} - \frac{\tau_1^3}{6} + \frac{\tau_1^2\tau_3}{2} - \frac{\tau_3^2\tau_1}{2}\Big] = \tau - \frac{u\,\tau^3}{6},$$

because the bracket is $(\tau_3-\tau_1)^3/6$. (Expand $(\tau_3-\tau_1)^3$ to check.) Using this and $G_3 \approx \tau_3(1-u\tau_3^2/6)$, $G_1 \approx \tau_1(1-u\tau_1^2/6)$, we get from Equation 17, again to first order in u (use $1/(1-x)\approx 1+x$),

$$c_1 \approx \frac{\tau_3}{\tau}\Big[1+\frac{u}{6}(\tau^2-\tau_3^2)\Big], \qquad c_3 \approx -\frac{\tau_1}{\tau}\Big[1+\frac{u}{6}(\tau^2-\tau_1^2)\Big]. \tag{20}$$

Put this into the middle line of Equation 19. The terms free of u and those proportional to $u=\mu/r_2^3$ separate:

$$\begin{aligned}
\rho_2 &= A + \frac{\mu B}{r_2^3},\\
A &= \frac{1}{D_0}\Big(-\frac{\tau_3}{\tau}D_{12} + D_{22} + \frac{\tau_1}{\tau}D_{32}\Big),\\
B &= \frac{1}{6D_0}\Big[\frac{\tau_3}{\tau}(\tau_3^2-\tau^2)D_{12} + \frac{\tau_1}{\tau}(\tau^2-\tau_1^2)D_{32}\Big].
\end{aligned} \tag{21}$$

This is the same form as Equation 13, and it combines with the same geometric relation Equation 14 ($E=\mathbf{R}_2\cdot\mathbf{L}_2$ is the C of that equation) to the same polynomial,

$$r_2^8 - (A^2 + 2AE + R_2^2)\,r_2^6 - 2\mu B\,(A+E)\,r_2^3 - \mu^2B^2 = 0. \tag{22}$$

The root story is as for Laplace (Figure 4, right, shows both curves). With the root r_2: $u=\mu/r_2^3$, then c_1, c_3 from Equation 20, the ranges from Equation 19, the three positions from Equation 11, and $\mathbf{v}_2$ from Equation 17.

7.4 Iterating

This first pass is accurate only to the first order in $u\tau^2$, so its error grows as h^2 (below). Here is the improvement that makes Gauss's method the better method. Having a state $(\mathbf{r}_2, \mathbf{v}_2)$, **compute the exact Lagrange coefficients** by universal variables (the universal-variables note) rather than the series; recompute c_1, c_3 from Equation 17, the ranges from Equation 19, the three positions, and $\mathbf{v}_2$ from Equation 17; and repeat until the ranges stop changing.

At convergence the three positions and the velocity satisfy $\mathbf{r}_i = F_i\mathbf{r}_2 + G_i\mathbf{v}_2$ *exactly*, with the exact F_i, G_i of that very state: they lie on a Kepler orbit, and each lies on its observed line of sight. For noise-free data that orbit is the true one.

Algorithm: Gauss, iterated

1. $\mathbf{p}_j$, D_0 and $D_{ij} = \mathbf{R}_i \cdot \mathbf{p}_j$ from the data. A, B from Equation 21.
2. The positive root r_2 of Equation 22. c_1, c_3 from Equation 20. Ranges from Equation 19; $\mathbf{r}_i = \mathbf{R}_i + \rho_i \mathbf{L}_i$; $\mathbf{v}_2$ from Equation 17. This is the *first pass*.
3. *Iterate:* the exact F_i, G_i for $(\mathbf{r}_2, \mathbf{v}_2)$ over τ_1, τ_3; c_1, c_3 and the ranges again; new $\mathbf{r}_i$ and $\mathbf{v}_2$. Repeat until the ranges change by less than a tolerance.

7.5 Example 5: the same data

Use the data of Example 4. $D_0 = 2.1053 \times 10^{-3}$, and the matrix of $D_{ij} = \mathbf{R}_i \cdot \mathbf{p}_j$ (km) is

$$\begin{bmatrix} -900.350 & -1767.223 & -936.454 \\ -904.805 & -1775.746 & -940.934 \\ -909.240 & -1784.226 & -945.392 \end{bmatrix}.$$

Equation 21 gives $A = -10.114$ km and $B = 1.5182 \times 10^9$ km s^2, and Equation 22 has the coefficients -4.0553×10^7 km^2, -6.7077×10^{18} km^5 and -3.6623×10^{29} km^8: again one positive root, $r_2 = 7598.869$ km. Then $c_1 = c_3 = 0.5008176$, $\rho = (1392.943,\ 1369.103,\ 1448.501)$ km, and the **first pass** gives

$$\mathbf{r}_2 = (4656.474,\ -5550.799,\ 2291.002) \text{ km},$$
$$\mathbf{v}_2 = (3.61893,\ 4.40890,\ 4.60054) \text{ km/s},$$

with an error of 1.24 km and 6.3 m/s: against 34 km and 87 m/s for Laplace on the *same data*. The iteration reduces it:

iteration	ρ_2 [km]	$\lvert\Delta\mathbf{r}_2\rvert$ [km]	$\lvert\Delta\mathbf{v}_2\rvert$ [m/s]
0 (first pass)	1369.1027	1.2416	6.315
1	1370.9636	0.6193	3.156
2	1370.0353	0.3090	1.575
4	1370.2674	0.0770	0.392
8	1370.3395	0.0049	0.025
12	1370.3439	0.0004	0.002

True $\rho_2 = 1370.3443$ km. The error is halved at *every* iteration (Figure 5) until it reaches the floor set by the rounding of the angles, about 0.1 m, and the range alternates between too large and too small. After 32 iterations the ranges stop changing at the tolerance 10^{-12}; the position error is 0.12 m and the velocity error 0.7 mm/s, which is the effect of rounding the angles to 3.6 mas. The elements are $a = 7799.998$ km, $e = 0.05830$, $i = 43.0811°$, $\Omega = 290.2235°$, $\omega = 319.2685°$, $f = 66.9295°$: the true orbit.

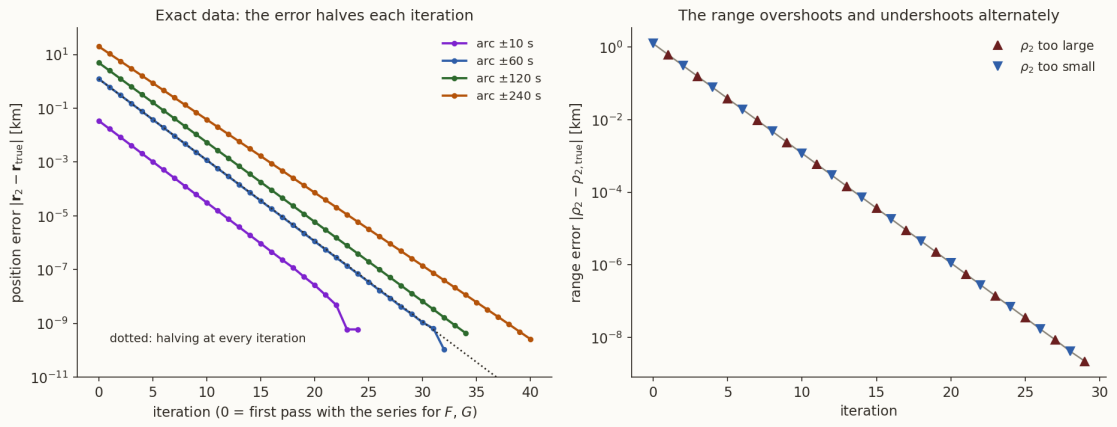


Figure 5: **Left:** position error of Gauss's method against the iteration, for exact lines of sight at four arcs (± 10, ± 60, ± 120 and ± 240 s). Iteration 0 is the first pass with the series. The error halves at each step in every case (the longer the arc, the larger the starting error and the more steps needed), until it reaches the floor of double precision. **Right:** the error in ρ_2 for the ± 60 s arc, with its sign. *Script: fig-od.py.*

For the data of Example 5, compute the exact coefficients from the true state, F_1, G_1, F_3, G_3 over ∓ 60 s, and compare c_1, c_3 with the series of Equation 20.

What you should find

Exact: $c_1 = 0.5008196$, $c_3 = 0.5008171$ (they differ, since the true orbit is not symmetric about t_2); series: both 0.5008172. The two agree to about 2×10^{-6}, and the first-pass error of Example 5 comes from differences of that size, which Equation 19 multiplies by numbers of order 10^3 km and divides by $D_0 = 2 \times 10^{-3}$.

8 How good is it? Noise, arc and geometry

All results so far were for data that are exact or rounded to 3.6 mas. A real telescope measures to about 1″ to 2″, and the radars that give positions to metres or tens of metres are common. We add Gaussian noise of 2″ (on $\alpha \cos\delta$ and on δ, independently; 13 m across the line of sight at the 1370 km range of the middle epoch) to the lines of sight, run each method on 300 noisy sets for each arc $\pm h$ and record the position error at $t = 0$: the median

and the central 68 % (Figure 6; the numbers are reproducible, with fixed seeds).



Figure 6: **Left:** position error against the half-width of the arc, with the median (line) and the 16–84 % band, for Laplace, Gauss's first pass and the iterated Gauss, with 2″ of noise. The dashed lines are the errors of the same methods without noise. **Right:** how far the middle line of sight lies out of the great-circle plane through the outer two, which is $\approx D_0/\|\mathbf{L}_1 \times \mathbf{L}_3\|$; it grows as h^2, and has to be compared with the noise. *Script: fig-od.py, experiment _odmc.py.*

h [s]	D_0	out of plane	Laplace [km]	Gauss, first pass [km]	Gauss, iterated [km]
10	1.0×10^{-5}	21″	77	77	77
20	8.3×10^{-5}	85″	16	16	16
40	6.5×10^{-4}	342″	16	4.2	4.2
60	2.1×10^{-3}	772″	34	2.1	2.0
120	1.4×10^{-2}	3164″	128	4.9	0.50
240	6.4×10^{-2}	14 016″	408	20	0.13
480	0.19	92 769″	1012	81	0.031

Three things are visible.

1. **Short arcs are noise-limited.** For $h \leq 20$ s all three methods give the same error, set by the noise, and it falls about four-fold each time the arc doubles: the error in the range goes as σ/h^2. The reason is in the right panel of the figure: the range is carried by the *curvature* of the line-of-sight track, by how far the middle line of sight lies off the great circle through the outer two (that is the quantity D_0 divides by), and this is 21″ for $h = 10$ s (only ten times the noise) and grows as h^2. A 2″ noise on a 21″ signal is a 10 % error in it. This is the heart of angles-only

orbit determination: *the range is a second difference of the data*, and noise is amplified by $1/h^2$.

2. **Long arcs are truncation-limited, except for the iterated method.** Laplace's error is dominated by the interpolation error beyond $h \approx 30$ s, and Gauss's first pass by the series beyond 60 s (their noise-free errors are the dashed lines); both grow as h^2. The iterated Gauss method has neither, and its error keeps falling as the arc grows. That is why it is the method to use whenever the observations are spread over minutes.
3. **The arc is limited by the model.** The curves above assume the satellite moves on a *Kepler orbit*, and for a long arc this is untrue. The J_2 acceleration of the Earth's oblateness is about 10^{-3} of the central acceleration; over $\pm$240 s it displaces the satellite by about 210 m (and by about 840 m over $\pm$480 s) from the two-body orbit, so beyond a few minutes the *model* error exceeds the iterated-Gauss errors of the table. Orbit determination for longer arcs needs a force model, which is why the refinement of Section 9 uses numerical integration in practice.

Geometry matters as much as noise. D_0 appears as a divisor in every formula. It vanishes if the three lines of sight lie in a plane, which happens when the station lies *in the plane of the orbit*: then the track on the sky is a great circle, and the curvature that carries the range is zero. The table gives D_0 at $\pm$60 s for passes with different cross-track offsets from the same station.

peak elevation	D_0 at $\pm$60 s	distance of the station from the orbital plane
89.5°	3.2×10^{-5}	7 km
83.4°	6.0×10^{-4}	119 km
77.4°	1.1×10^{-3}	230 km
60.6° (our pass)	2.1×10^{-3}	566 km
39.5°	2.1×10^{-3}	1125 km
25.6°	1.5×10^{-3}	1682 km

A pass through the zenith has a 65 times smaller D_0 than our pass. Angles-only orbit determination works *best* on moderately high passes and worst on passes almost overhead, a counter-intuitive fact: the best-seen pass is the worst-conditioned.

A pass goes within 7 km of the zenith. Gauss's method returns a wild answer. What has gone wrong, and what would you do?

Answer

The lines of sight are nearly coplanar, so $D_0 \approx 3 \times 10^{-5}$ and the three ranges are determined as a ratio of small noisy numbers. The curvature of the sky

track, which carries the range, is almost zero. One would lengthen the arc (the out-of-plane signal grows as h^2), use another station or another pass, or wait for a pass that is not overhead.

9 Using all the data: least squares

This section can be skipped on a first reading; the preliminary methods are complete without it.

Gauss's method uses three observations. A pass supplies dozens, and discarding all but three throws away most of the information. The standard way to use them all is the method of least squares. Given a first guess of the state, from Gauss's method for instance, one adjusts the six numbers of the state until the observations the orbit would produce agree with those measured as closely as possible.

9.1 The model and the iteration

Let $\mathbf{x} = (\mathbf{r}_0, \mathbf{v}_0)$ be the state at a reference epoch, $\mathbf{y}$ the vector of the $2n$ measured angles ($\alpha_k \cos\delta_k$ and δ_k at the n epochs), and $\mathbf{h}(\mathbf{x})$ the angles that the state *predicts*: propagate $\mathbf{x}$ to t_k with the universal-variable propagator, subtract the station position, and read off the right ascension and declination as in the topocentric note. The **residual** is $\mathbf{b} = \mathbf{y} - \mathbf{h}(\mathbf{x})$. If $\mathbf{x}$ is the truth, $\mathbf{b}$ is the measurement noise.

Suppose $\mathbf{x}$ is close to the truth; then $\mathbf{h}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{h}(\mathbf{x}) + \mathsf{H}\,\Delta\mathbf{x}$, where $\mathsf{H} = \partial\mathbf{h}/\partial\mathbf{x}$ is a $2n \times 6$ matrix (computed here by central differences; one could use the state-transition matrix). The best correction minimises the sum of squares

$$J(\Delta\mathbf{x}) = \frac{1}{\sigma^2}\,\lVert \mathbf{b} - \mathsf{H}\,\Delta\mathbf{x} \rVert^2,$$

(σ is the standard deviation of each angle). Setting $\partial J/\partial\Delta\mathbf{x} = \mathbf{0}$ gives the **normal equations**

$$\mathsf{H}^{\mathsf{T}}\mathsf{H}\,\Delta\mathbf{x} = \mathsf{H}^{\mathsf{T}}\mathbf{b}, \qquad \mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}. \tag{23}$$

Repeat: recompute $\mathbf{b}$ and H at the new $\mathbf{x}$, solve again, and stop when $\Delta\mathbf{x}$ is negligible. This is the Gauss–Newton method, known in astrodynamics as the **differential correction**. It converges quickly if the first guess is good, and may not converge at all if it is not; this is the reason for the preliminary methods.

History: least squares. Legendre published the method in 1805 as an appendix to his work on determining the orbits of comets (Legendre 1805); Gauss published it in 1809 in the *Theoria motus* (Gauss 1809), and wrote there that he had used it since 1795; Legendre disputed this. Legendre's appendix sat in a book on the orbits of comets, though his worked example was the shape of the Earth.

9.2 What the covariance says

If the errors are independent with standard deviation σ and $\mathbf{x}$ is near the solution, Equation 23 gives $\Delta\mathbf{x} = (\mathsf{H}^{\mathsf{T}}\mathsf{H})^{-1}\mathsf{H}^{\mathsf{T}}\varepsilon$, with ε the noise in $\mathbf{b}$. Its covariance follows from this,

$$\mathsf{P} = \sigma^2\,(\mathsf{H}^{\mathsf{T}}\mathsf{H})^{-1}. \tag{24}$$

(Use $\langle\varepsilon\varepsilon^{\mathsf{T}}\rangle = \sigma^2 1$ and $\mathsf{A} = (\mathsf{H}^{\mathsf{T}}\mathsf{H})^{-1}\mathsf{H}^{\mathsf{T}}$: $\mathsf{P} = \mathsf{A}\,\sigma^2\mathsf{A}^{\mathsf{T}} = \sigma^2(\mathsf{H}^{\mathsf{T}}\mathsf{H})^{-1}$.) P is the **formal** uncertainty of the state, produced without knowing the truth: its eigenvectors are the directions in which the state is well or badly determined, and the square roots of its eigenvalues are the standard deviations along them.

9.3 Example 6: a whole pass

The satellite is observed every 5 s from $t = -120$ s to $+120$ s (49 epochs), with 2″ noise. First guess: Gauss's method on three observations 10 s apart ($t = -10, 0, +10$ s), which has an error of 153 km and 784 m/s (the short arc). Differential correction gives:

iteration	RMS residual	position error	velocity error
0 (the guess)	263.5″	152.9 km	784 m/s
1	28.7″	18.8 km	96 m/s
2	1.76″	0.169 km	0.87 m/s
3	1.71″	0.123 km	0.63 m/s

It converges in three iterations, to a residual of 1.7″ against the noise of 2″: the data fit to the noise level, neither better (which would mean over-fitting) nor worse (a wrong model). The expected value of the RMS residual for 98 residuals and 6 fitted parameters is $2''\sqrt{92/98} = 1.94''$.

The covariance Equation 24, expressed in a frame with one axis along the line of sight at $t = 0$, has the standard deviations

$$\sigma_{\parallel} = 253 \text{ m (along the line of sight)}, \qquad \sigma_{\perp} = 2.8 \text{ m and } 2.9 \text{ m (across)}.$$

The state is determined *a hundred times better across the line of sight than along it*: the angles fix where the satellite is on the sky to 3 m, and the range remains uncertain by 250 m. It is the same lesson as in Section 8, now in matrix form. The actual error of this run, in the same frame, is 123 m along and -1.9, 1.0 m across: consistent with the covariance.

To check that the covariance can be believed, we repeat the whole experiment 200 times with fresh noise (Figure 7). The scatter of the errors along and across the line of sight is 247 m and 2.7 m, 3.0 m, against the formal 253 m, 2.8 m, 2.9 m: **the formal covariance predicts the actual scatter**. As for the benefit of using all the data, with the same noise, Gauss's method on the first, middle and last observation of the pass has a median position error of 0.55 km and 3.0 m/s, against 0.16 km and 0.85 m/s for the least squares on all 49: three times better. It does not make the range as good as the cross-range, because *the arc does not contain the information*: to improve on that one needs a longer arc, or another station, or a second pass.

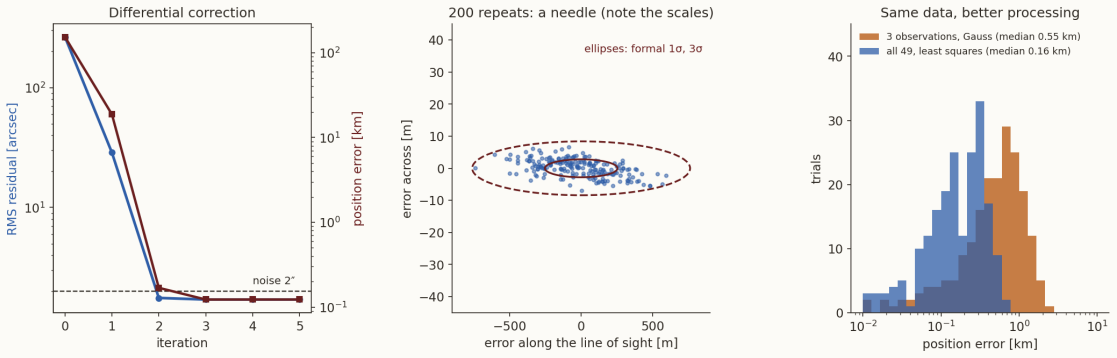


Figure 7: **Left:** differential correction on one noisy pass: the RMS residual (blue) falls to the noise level, 2″ (dashed), in two iterations, and the position error (maroon) with it. **Middle:** the position error of 200 noisy repeats, in the frame of the line of sight; note the different scales of the two axes (the cloud is 90 times longer than wide). The ellipses are the formal 1σ and 3σ of Equation 24. **Right:** the distribution of the position errors of Gauss's method on three observations and of least squares on all of them. *Script: fig-od.py, experiment _odmc.py.*

Why is it not a contradiction that least squares finds the state to 3 m across the line of sight with 2″ of angle noise, which is 13 m at this range?

Answer

Each angle has a 13 m error, but there are 98 of them, and averaging N independent errors reduces the result by $\sqrt{N}$. The ratio 13.3/2.8 says that the data are worth about $(13.3/2.8)^2 \approx 22$ independent measurements of the position across the line of sight (fewer than 98, because the same six numbers have to explain all of them). Along the line of sight there is no such averaging: the range is not in the angles, only in their curvature.

What this section left out. Perturbations (the oblate Earth, drag, the Sun and Moon) are absent from the model above, and for a real pass the propagation would be numerical, with the same differential correction on top. Observations of different kinds (ranges, Doppler, angles) are combined with weights, $\mathsf{W} = \mathsf{R}^{-1}$ in place of $1/\sigma^2$ in Equation 23. Bad measurements are rejected

by their residuals. New observations are often added one at a time, by a sequential filter, instead of in a batch; the Kalman filter (Kalman 1960) is the standard form. And a second pass, hours later, is far more informative than more observations of the first, because the along-track uncertainty grows in time while the cross-track one does not. All of this is beyond the preliminary methods, and a course of its own.

Summary

In one paragraph. An orbit needs six numbers, and the preliminary methods turn the smallest sufficient set of observations into the state. A radar with Doppler gives it directly. Three positions give it by **Gibbs**, with no use of the times, from the conic equation $r + \mathbf{e} \cdot \mathbf{r} = p$, which is linear in $\mathbf{e}$ and p (Equation 8); when they are close, the cancellation in Gibbs's $\mathbf{D}$ is bad, and **Herrick–Gibbs** uses a Taylor series in time with the gravitational acceleration at the three points (Equation 10). Two positions and the time give it by **Lambert**, with the time as additional information. Three directions give it by **Laplace** (the derivatives of the line of sight and the equation of motion at one instant) or **Gauss** (the Lagrange coefficients, and iterating them): both reduce to a polynomial of degree eight for r_2 with a single positive root for satellites seen from the ground; Gauss, iterated, is exact for exact data. The range is carried by the *curvature* of the line-of-sight track, so the error of the range falls as the square of the arc, and the geometry matters (a pass through the zenith is the worst one). Least squares uses all the observations, converges from the preliminary state, and its covariance says how well each direction is determined.

Method	Data	Uses the times?	Best when	Worst when
radar conversion (topocentric)	ρ, A, ε and rates, one epoch	no	a radar with Doppler is available	the rates are noisy
Gibbs	three positions	no	the points are far apart	they are close (cancellation)
Herrick–Gibbs	three positions	yes	the points are within a few tens of degrees	far apart (τ^4 error)
Lambert	two positions and the time	yes	two fixes and a clock	the clock is poor

Method	Data	Uses the times?	Best when	Worst when
Laplace	three directions	yes	the arc is short and the data are very accurate	long arcs (interpolation error $\propto h^2$)
Gauss	three directions	yes	the arc is moderate; with iteration, any arc	the station is nearly in the orbital plane ($D_0 \to 0$)
least squares	all of the above	yes	a first guess and many observations	a poor first guess

What we left out. (i) *Perturbations* in the preliminary stage: the two-body model is adequate for arcs of a few minutes and wrong beyond (the J_2 displacement is about 210 m over $\pm$240 s). (ii) *Aberration, light time, refraction*, which move the observed direction by seconds of arc and are corrected before the data enter (see the topocentric note). (iii) *Other angles-only methods*: iterating directly on the ranges (the "double-r" idea), and methods that handle several passes and orbits with more than one revolution. (iv) *Sequential filtering* and the statistics of observation errors. (v) *Ill-posedness*: the case of a station in the orbital plane, which the formulas of this note cannot handle.

Further reading. H. D. Curtis, *Orbital Mechanics for Engineering Students* (Curtis 2014), the chapter on preliminary orbit determination (Gibbs's, Lambert's and Gauss's methods, and the topocentric frame); D. A. Vallado, *Fundamentals of Astrodynamics and Applications* (Vallado 2013), the chapter on initial orbit determination (including Herrick–Gibbs, Laplace and Gauss, and their numerical behaviour); R. R. Bate, D. D. Mueller and J. E. White, *Fundamentals of Astrodynamics* (Bate, Mueller, and White 1971), the chapter on orbit determination from observations; P. R. Escobal, *Methods of Orbit Determination* (Escobal 1965), for the full treatment; and the original sources: Gauss (Gauss 1809), Laplace (Laplace 1784), Gibbs (Gibbs 1889).

The numbers of this note are checked by orbit-determination-verify.py, using the helper modules _od.py (the methods and the scenario, built on _topo.py and _uv.py) and _odmc.py (the Monte Carlo experiments, with fixed seeds); fig-od.py draws the figures.

Exercises

1. **Counting.** How many scalar observations are needed to determine a Kepler orbit from (a) range, azimuth and elevation with their rates, (b) range, azimuth and elevation without rates, (c) azimuth and elevation only, (d) range only? (Answer: (a) six numbers at one epoch; (b) three per epoch, two epochs plus the time between them (Lambert), or three epochs; (c) two per epoch, three epochs; (d) one per epoch, six epochs, and the problem is badly conditioned.)
2. **Gibbs on a circle.** Three positions on a circular orbit of radius r, spaced by an angle ψ. Show that $\mathbf{S} = \mathbf{0}$, $\mathbf{N} = r\,\mathbf{D}$, $D = r^2(2\sin\psi - \sin 2\psi)$, hence $p = r$, $\mathbf{e} = \mathbf{0}$, and $v_2 = \sqrt{\mu/r}$. (Hint: the three radii are equal, so $\mathbf{S} = \mathbf{0}$ and $\mathbf{N} = r\mathbf{D}$; Equation 8 then gives $\mathbf{v}_2 = \sqrt{\mu/(rD^2)}\,\mathbf{D}\times\mathbf{r}_2/r$, and $\|\mathbf{D}\times\mathbf{r}_2\| = D\,r$.)
3. **The dependence identity.** Prove Equation 2 in components for three vectors in the xy-plane, $\mathbf{r}_i = (x_i, y_i, 0)$, with $[ij] = x_i y_j - x_j y_i$.
4. **Herrick–Gibbs.** Show that the combination Equation 10 gives the exact velocity at t_2 for the polynomial history $\mathbf{r}(t) = \mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 t^2 + \mathbf{c}_3 t^3 + \mathbf{c}_4 t^4$, with $\mathbf{a} = \ddot{\mathbf{r}}$ in place of $-\mu\mathbf{r}/r^3$. (Hint: the combination is linear in $\mathbf{r}$ and $\mathbf{a}$; check it on each power of t. The t^3 and t^4 terms are the ones that need the acceleration.)
5. **The clock.** Two position fixes 600 s apart on an orbit with $v = 7.3$ km/s. A timing error of 1 ms in one of them is equivalent to what position error along the track? (Answer: 7.3 m.)
6. **Unique root.** Show that if $B > 0$ and $\rho > 0$ on the geometric branch, the curves Equation 13 and Equation 14 cross at most once. (Hint: differentiate both with respect to r.) What does $B < 0$ do? (Answer: both derivatives are then positive, and the curves can cross several times.)
7. **The size of the signal.** A 2″ error at a range of 2410 km is how many metres across the line of sight? (Answer: 23.4 m.) How many times larger than the 2″ noise is the out-of-plane signal 21″ of the first row of the table of Section 8? (Answer: 10.6 times; the range error of 77 km is the corresponding few-percent error in the curvature, magnified.)
8. **A zenith pass.** In the helper module, call the truth state with a cross-track offset of 0° instead of 5° (a pass within 8 km of the zenith) and run the iterated Gauss method at $\pm$60 s, first on exact lines of sight and then with 2″ of noise (300 trials). Compare with the scenario. (Answer: the exact lines of sight still give the exact orbit, to 4×10^{-9} km, but with 2″ of noise the median error is 123 km against about 2 km for the pass of the scenario (the table of Section 8), some 60 times worse; D_0 is 65 times smaller.)
9. **Series for F and G.** Show that $F_1 G_3 - F_3 G_1 = \tau - u\tau^3/6 + O(u^2)$

from $F \approx 1 - u\tau^2/2$, $G \approx \tau - u\tau^3/6$. Then check, for equal spacing $\tau_3 = -\tau_1 = 60$ s and $u = 9.08 \times 10^{-7}$ s^{-2}, that the exact value of $(F_1 G_3 - F_3 G_1)/\tau$ is 0.99782, which the series reproduces to 2×10^{-6}.

10. **Which method?** For each case, name a method and say why: (a) a radar fixes three positions 5 s apart; (b) a radar fixes three positions 5 min apart; (c) a telescope gives directions every 2 s for 2 min; (d) a GPS receiver gives a position every minute, and one needs the velocity at one of them. (Answer: (a) Herrick–Gibbs; (b) Gibbs; (c) iterated Gauss on the first, middle and last directions, then least squares on all; (d) Herrick–Gibbs with the neighbouring fixes, or Lambert if only two are used.)

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