The N-Body Problem

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1 Where we are going

Every space mission starts with a prediction. To send a probe to Mars we must know where Mars will be seven months after launch; to fly past Jupiter we must know where Jupiter will be six years from now, to within a few hundred kilometres. So the first problem of spaceflight mechanics is an old one: **explain, and then predict, the positions of the bodies of the solar system at any time.**

This note attacks that problem in the most direct way possible. We replace every body by a point mass, let every point attract every other by Newton's law of gravitation, and write down the equations of motion. The result is the celebrated N**-body problem**. We will find that it is easy to write down and impossible to solve in general, and the reasons *why* it cannot be solved will tell us which simplifications to make next.

Two kinds of boxes run through the note. A blue **Try it** box asks you to do something small: compute a number, write out a sum, run a script. Do it before reading on; the point that follows is made by your own answer. A green **Check yourself** box asks a short question that tests whether the previous paragraphs went in. Each has a hidden answer, which you should open only after you have an answer of your own.

When you finish, you should be able to answer the following without hand-waving.

1. Why may the Sun and the planets be replaced by points, and when does that argument fail?
2. Which quantities stay constant when N bodies attract each other, and why?
3. Why can the two-body problem be solved with pen and paper while the three-body problem cannot?
4. What does it mean for an orbit to be *chaotic*, and why does chaos not stop us from flying missions?
5. Which simplification turns the N-body problem into something we can solve, and how good is it?

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

2 From observations to a law

2.1 Kepler's three laws

The story starts with data. Tycho Brahe measured the positions of the planets, with the naked eye, to one or two minutes of arc, night after night for some

twenty years. Johannes Kepler inherited these tables and spent years fitting curves to them. The result was three laws.

1. **First law.** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Second law.** The line joining the planet to the Sun sweeps out equal areas in equal times.
3. **Third law.** The square of the period T of a planet is proportional to the cube of the semi-major axis a of its orbit: $T^2 \propto a^3$, with the same constant for every planet.

Before we use the third law, let us test it on real numbers.

The table gives the semi-major axis a (in astronomical units, 1 AU $\approx 1.496 \times 10^8$ km, the mean Earth–Sun distance) and the period T (in years) of six planets. Compute T^2/a^3 for each.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn
a (AU)	0.387	0.723	1.000	1.524	5.203	9.537
T (yr)	0.241	0.615	1.000	1.881	11.86	29.46

What you should find

$T^2/a^3 = 1.002,\ 1.001,\ 1.000,\ 1.000,\ 0.999,\ 1.001$. Over a factor of 25 in distance and 120 in period, the ratio is the same to 0.2%. In these units the constant is 1 simply because the year and the AU are defined by the Earth's orbit.

Kepler's laws are a superb *description*, but they are kinematic: they say what the orbits look like, not why. They give no rule for computing the orbit of a new body from its position and velocity, and they say nothing about bodies that do not orbit the Sun.

The four large moons of Jupiter also obey $T^2 \propto a^3$. For Io, $a = 421\,700$ km and $T = 1.769$ days. Compute T^2/a^3 in $\mathrm{yr}^2/\mathrm{AU}^3$. Is it 1? What would Kepler have needed in order to explain the number you get?

Answer

Converting, $a = 2.819 \times 10^{-3}$ AU and $T = 4.844 \times 10^{-3}$ yr, so $T^2/a^3 \approx 1047$. Callisto gives the same 1047. So the law holds about Jupiter, but with a constant 1047 times larger than about the Sun. Kepler had no way to explain this. It needs a *force law* in which the constant depends on the attracting body; Newton's law gives $T^2/a^3 = 4\pi^2/(GM)$ (to be proved in the two-body note), so 1047 is the ratio of the mass of the Sun to the mass of Jupiter. With Kepler's law and a stopwatch, you have weighed Jupiter.

History: Tycho and Kepler. Tycho Brahe (1546–1601) built the observatory of Uraniborg on the island of Hven and later moved to Prague. Kepler (1571–1630) joined him there in 1600. When Tycho died in 1601 Kepler kept the observations, over the objections of Tycho's heirs.

Kepler published the first two laws in *Astronomia Nova* (1609), after a long struggle with the orbit of Mars, whose eccentricity (0.093) is large enough to rule out a circle at Tycho's accuracy. The third law appeared ten years later, in *Harmonices Mundi*

2.2 Newton's explanation

Newton supplied the missing law. He showed that if every body attracts every other with a force proportional to the product of their masses and inversely proportional to the square of their distance, then Kepler's three laws follow, with the constant in the third law fixed by the mass of the attracting body. The rest of this note works out what Newton's law implies when there are N bodies instead of two.

3 The model: bodies as points

The solar system contains the Sun, eight planets, their moons, and millions of asteroids and comets, n bodies in all. Each is an extended, rotating body whose density varies inside it. We cannot write equations for that. We need an idealisation, and the simplest one is to treat each body as a **point mass**: all of its mass concentrated at one point.

Why should that be acceptable? The natural guess is that a body looks like a point when it is small compared with the distances involved. Let us test that guess.

Compute the ratio (radius of the body)/(distance to the body it influences) for the three pairs below. Take $R_\odot = 6.96 \times 10^5$ km (Sun), $R_\oplus = 6378$ km (Earth), 1 AU for the Earth–Sun distance, 384 400 km for the Earth–Moon distance, and a satellite at 420 km altitude (the International Space Station).

(a) the Sun, seen from the Earth; (b) the Earth, seen from the Moon; (c) the Earth, seen from the space station.

What you should find

(a) $R_\odot/1\ \text{AU} = 0.0047$; (b) $R_\oplus/384\,400\ \text{km} = 0.017$; (c) $R_\oplus/(6378 + 420)\ \text{km} = 0.94$.

The first two are small, so the point-mass picture looks safe. The third is not small at all: from a low orbit the Earth is a disc 140° across, filling about a third of the whole sky. By the "small body" argument, we could **not** treat the Earth as a point for satellites.

The way out of case (c) is a theorem rather than an approximation. **A spherically symmetric body attracts every point outside it exactly as if all its mass were at its centre** (Newton's shell theorem). Size does not matter, only symmetry. The planets are very nearly spherically symmetric, so the point-mass model is good both for the planets about the Sun and for satellites about the Earth. What the Earth's flattening does to a satellite is the subject of a later note (gravity of the Earth).

History: the *Principia*. Newton's *Philosophiæ Naturalis Principia Mathematica* (1687) derives Kepler's laws from the inverse-square law in Book I, and in Book III applies the law to the planets, the Moon, the comets and the tides. Kepler's laws were derived from data; Newton's were derived from a physical principle, and they predicted the data.

4 Newton's law of gravitation

4.1 The force between two points

Fix an inertial frame (we return to what that means in Section 5) and let $\mathbf{R}_i$ be the position of body i, of mass m_i, in that frame. For two bodies i and j define

$$\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i, \qquad r_{ij} = |\mathbf{r}_{ij}|, \qquad \hat{\mathbf{r}}_{ij} = \frac{\mathbf{r}_{ij}}{r_{ij}}.$$

So $\mathbf{r}_{ij}$ points **from** m_i **to** m_j, r_{ij} is the distance between them and $\hat{\mathbf{r}}_{ij}$ is the unit vector from m_i towards m_j. The gravitational force on m_i due to m_j is

$$\mathbf{F}_{ij} = \frac{G\,m_i m_j}{r_{ij}^2}\,\hat{\mathbf{r}}_{ij} = \frac{G\,m_i m_j}{r_{ij}^3}\,\mathbf{r}_{ij}, \tag{1}$$

where $G = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the gravitational constant. The sign is positive because gravity is *attractive* and $\hat{\mathbf{r}}_{ij}$ already points from m_i towards m_j. Many books carry a minus sign here: they measure the unit vector the other way, from the attracting body to the attracted one. The physics is the same.

History: the shell theorem. Newton proved it in Book I, Section XII of the *Principia*. It is the reason the pull of the whole Earth on the Moon can be computed as if the Earth were a point.

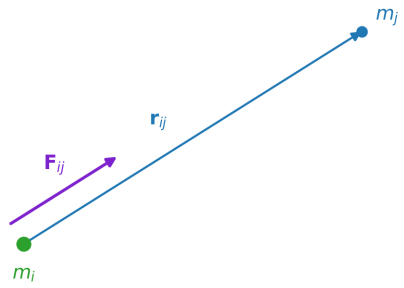


Figure 1: The force $\mathbf{F}_{ij}$ on m_i due to m_j acts at m_i and points along $\mathbf{r}_{ij}$, towards m_j. *Script: fig-gravity-force.py.*

Two quick checks. (a) Use Equation 1 to check the units of G. (b) Write down $\mathbf{F}_{ji}$, the force on m_j due to m_i, and show that $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. Which of Newton's laws is this?

Answer

(a) $[G] = [F][r^2]/[m^2] = \mathrm{kg\,m\,s^{-2}} \cdot \mathrm{m^2/kg^2} = \mathrm{m^3\,kg^{-1}\,s^{-2}}$.

(b) Swapping the labels, $\mathbf{F}_{ji} = Gm_j m_i\, \mathbf{r}_{ji}/r_{ji}^3$. Since $\mathbf{r}_{ji} = \mathbf{R}_i - \mathbf{R}_j = -\mathbf{r}_{ij}$ and $r_{ji} = r_{ij}$, we get $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. The forces are equal, opposite and along the line joining the bodies: Newton's third law, in its strong form. We will use both halves of this ("opposite" and "along the line") in Section 7.

4.2 Which body dominates? A surprise

It is tempting to think that the Moon moves under the Earth's gravity and that the Sun is a small correction. Let us check.

Compute the acceleration of the Moon (a) due to the Sun, $GM_\odot/(1\ \mathrm{AU})^2$, and (b) due to the Earth, $GM_\oplus/(384\,400\ \mathrm{km})^2$. Use $GM_\odot = 1.327 \times 10^{20}$ $\mathrm{m^3/s^2}$ and $GM_\oplus = 3.986 \times 10^{14}\ \mathrm{m^3/s^2}$. Which is larger?

What you should find

(a) $5.93\,\mathrm{mm/s^2}$; (b) $2.70\,\mathrm{mm/s^2}$. The **Sun pulls the Moon 2.2 times harder than the Earth does.**

So why does the Moon go round the Earth and not wander off? Because the Sun pulls the Earth almost exactly as hard as it pulls the Moon: the two are at nearly the same distance from the Sun. What changes the Moon's motion *relative to the Earth* is only the **difference** of the Sun's pulls on the two, which at the Moon's distance is about $3 \times 10^{-5}\ \mathrm{m/s^2}$, a hundredth of the Earth's pull. Two lessons follow, and both will return. First, it is relative motion that matters, so we should look for equations of relative motion. Second, "which body dominates" is a question about *differences* of accelerations; it is answered properly in the patched-conics note.

5 An inertial frame

Newton's second law, $m\ddot{\mathbf{R}} = \mathbf{F}$, holds only in an inertial frame. So before writing equations we must decide where to put the origin and the axes.

What is a good inertial frame for the solar system? Is the centre of the Sun a good origin?

Answer

No frame we can actually build is exactly inertial. In practice the origin is taken at the **solar-system barycentre** (the centre of mass of the whole system) and the axes are fixed with respect to distant quasars (the International Celestial Reference Frame, ICRF).

The barycentre beats the Sun's centre because the Sun itself moves about the barycentre, mostly under Jupiter's pull, by up to about a solar radius (Section 7.1 shows how to compute this). The barycentric frame is itself falling towards the centre of the Galaxy, at about 2×10^{-10} m/s^2. But the Galaxy's pull is very nearly *uniform* across the solar system, so it accelerates every body equally and drops out of their relative motion. "Inertial" always means *inertial enough for the accuracy you need*.

For now, we only assume that **some** inertial frame exists and that the $\mathbf{R}_i$ are measured in it. We will not choose the origin by hand: the equations of motion will tell us where to put it (Section 7.1).

6 The equations of motion

Body i feels the pull of every other body. Adding the forces Equation 1 and applying Newton's second law,

$$m_i \ddot{\mathbf{R}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{F}_{ij} = G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ij}^3} \, \mathbf{r}_{ij}, \qquad i = 1, 2, \dots, n. \tag{2}$$

The sum skips $j = i$ because a body does not attract itself.

Divide Equation 2 by m_i. On what does the acceleration of body i depend, and on what does it *not* depend?

Answer

$\ddot{\mathbf{R}}_i = G \sum_{j \neq i} m_j \mathbf{r}_{ij} / r_{ij}^3$. It depends on the positions of all the bodies and on the masses of the *other* bodies, but not on m_i. A feather and a planet at the same place fall with the same acceleration. This is why we will be able to compute a spacecraft's trajectory without knowing its mass.

Equation 2 is a set of $3n$ second-order equations, coupled because every right-hand side contains every position. To integrate them we need, at $t = 0$, every position and every velocity. Introducing the velocities $\mathbf{V}_i = \dot{\mathbf{R}}_i$ as unknowns turns the system into first-order form,

$$\dot{\mathbf{R}}_i = \mathbf{V}_i, \qquad \dot{\mathbf{V}}_i = G \sum_{j \neq i} \frac{m_j}{r_{ij}^3} \, \mathbf{r}_{ij}, \qquad \mathbf{R}_i(0) = \mathbf{R}_{i0}, \quad \mathbf{V}_i(0) = \mathbf{V}_{i0}, \tag{3}$$

which is $6n$ first-order ordinary differential equations (ODEs) with $6n$ initial conditions. The number $6n$ is called the **order** of the system.

What is the order of the system for the Sun and the eight planets? For the Sun, the planets, the Moon and Jupiter's four large moons? How many distinct pairs (i, j) must be evaluated per time step in each case?

Answer

$n = 9$: order 54, and $n(n-1)/2 = 36$ pairs. $n = 14$: order 84, and 91 pairs. The cost of each step grows like n^2, which is why nobody integrates the asteroid belt body by body.

7 The ten integrals

Before trying to solve Equation 3, it pays to ask a simpler question: **is anything constant?** A function of the positions and velocities that does not change along a solution is called an **integral of the motion** (or a constant of the motion). Integrals are worth hunting for, for three reasons.

- Each independent integral removes one unknown. If a system of order k had $k-1$ independent integrals, its trajectories would be fixed curves in the state space and only the timing would remain.
- Integrals tell us things about *every* solution at once, without solving anything.
- A numerical solution must conserve them. If it does not, the numerical solution is wrong.

We look for integrals by the standard trick: combine the equations of motion so that one side becomes an exact time derivative.

7.1 Linear momentum and the centre of mass

Sum Equation 2 over all bodies:

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{R}}_i = \sum_{i=1}^{n} \sum_{j \neq i} \mathbf{F}_{ij}.$$

The double sum on the right contains every ordered pair (i, j) with $i \neq j$. Group each pair with its mirror image (j, i):

$$\sum_{i=1}^{n} \sum_{j \neq i} \mathbf{F}_{ij} = \sum_{\text{pairs } i<j} \left(\mathbf{F}_{ij} + \mathbf{F}_{ji} \right) = \mathbf{0},$$

because $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. The internal forces cancel in pairs.

Write the double sum out in full for $n = 3$. How many terms are there, and which ones cancel?

What you should find

Six terms: $\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{31} + \mathbf{F}_{32}$. They cancel as $\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0}$, $\mathbf{F}_{13} + \mathbf{F}_{31} = \mathbf{0}$ and $\mathbf{F}_{23} + \mathbf{F}_{32} = \mathbf{0}$. In general there are $n(n-1)$ terms in $n(n-1)/2$ cancelling pairs.

So $\sum_i m_i \ddot{\mathbf{R}}_i = \mathbf{0}$. Integrating once and then once more,

$$\sum_{i=1}^{n} m_i \dot{\mathbf{R}}_i = \mathbf{C}_1, \qquad \sum_{i=1}^{n} m_i \mathbf{R}_i = \mathbf{C}_1 t + \mathbf{C}_2, \tag{4}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are constant vectors: six scalar constants. The first is the conservation of total linear momentum. To read the second, define the total mass and the **centre of mass** (or **barycentre**),

$$M = \sum_{i=1}^{n} m_i, \qquad \mathbf{R}_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{R}_i.$$

Then Equation 4 says $\mathbf{R}_{\mathrm{cm}}(t) = (\mathbf{C}_1 t + \mathbf{C}_2)/M$: **the centre of mass moves in a straight line at constant speed.** A frame moving with it is therefore also inertial, and in that frame $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{0}$. This is the origin the equations chose for us, and it is the barycentric frame of Section 5.

Take only the Sun and Jupiter, with $M_\odot/M_J = 1047.35$ and Jupiter at 778.5×10^6 km from the Sun. How far from the centre of the Sun is their barycentre? Compare with the Sun's radius, 6.96×10^5 km.

What you should find

With the origin at the barycentre, $M_\odot \mathbf{R}_\odot + M_J \mathbf{R}_J = \mathbf{0}$, so the distance from the Sun's centre is $778.5 \times 10^6 \times M_J/(M_\odot + M_J) = 742\,600$ km, which is 1.07 solar radii. The Sun–Jupiter barycentre lies just *outside* the Sun. The Sun is not fixed: it wobbles about the barycentre once every Jupiter year (and, because of the other planets, in a more complicated pattern).

7.2 Angular momentum and the invariable plane

The internal forces do more than cancel in pairs: each pair also acts **along the line joining the two bodies**. A pair of equal and opposite forces along the same line exerts no net torque. So we expect the total angular momentum to be conserved too. Let us prove it.

Take the cross product of Equation 2 with $\mathbf{R}_i$ and sum over i:

$$\sum_{i=1}^{n} m_i \, \mathbf{R}_i \times \ddot{\mathbf{R}}_i = \sum_{i=1}^{n} \sum_{j \neq i} \mathbf{R}_i \times \mathbf{F}_{ij}.$$

Group the right-hand side into pairs again. Using Equation 1 and $\mathbf{r}_{ji} = -\mathbf{r}_{ij}$, the pair (i, j) contributes

$$\mathbf{R}_i \times \mathbf{F}_{ij} + \mathbf{R}_j \times \mathbf{F}_{ji} = \frac{G m_i m_j}{r_{ij}^3} \left(\mathbf{R}_i - \mathbf{R}_j\right) \times \mathbf{r}_{ij} = -\frac{G m_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \times \mathbf{r}_{ij} = \mathbf{0},$$

since $\mathbf{R}_i - \mathbf{R}_j = -\mathbf{r}_{ij}$ and the cross product of a vector with itself vanishes. Every pair gives zero, so the right-hand side vanishes. On the left,

$$\frac{d}{dt}\left(\mathbf{R}_i \times \dot{\mathbf{R}}_i\right) = \dot{\mathbf{R}}_i \times \dot{\mathbf{R}}_i + \mathbf{R}_i \times \ddot{\mathbf{R}}_i = \mathbf{R}_i \times \ddot{\mathbf{R}}_i,$$

so the left-hand side is an exact derivative and

$$\mathbf{H} = \sum_{i=1}^{n} m_i \, \mathbf{R}_i \times \dot{\mathbf{R}}_i = \mathbf{C}_3. \tag{5}$$

The total angular momentum $\mathbf{H}$ is constant in magnitude and in direction: three more scalar constants.

Where in the proof did we use that the force acts *along the line joining* the bodies, rather than merely that $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$?

Answer

In the step $(\mathbf{R}_i - \mathbf{R}_j) \times \mathbf{r}_{ij} = \mathbf{0}$. If the pair of forces were equal and opposite but *not* along $\mathbf{r}_{ij}$, they would form a couple and $\mathbf{R}_i \times \mathbf{F}_{ij} + \mathbf{R}_j \times \mathbf{F}_{ji} = (\mathbf{R}_i - \mathbf{R}_j) \times \mathbf{F}_{ij}$ would not vanish. Momentum would still be conserved; angular momentum would not.

Because $\mathbf{H}$ is fixed, it defines a fixed plane: the plane through the barycentre perpendicular to $\mathbf{H}$. It is called the **invariable plane** (or Laplace plane). Two other planes are used as references in astronomy, and it helps to name them now.

- The **ecliptic** is the plane of the Earth's orbit about the Sun.
- The **solar equator** is the plane through the Sun's centre perpendicular to the Sun's rotation axis.

The invariable plane is inclined at about 1.58° to the ecliptic and about 6° to the solar equator. (The more familiar 23.44° is a different angle: the tilt of the *Earth's* equator to the ecliptic.)

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(a) The ecliptic (grey), the total orbital angular momentum $\mathbf{H}$ of the eight planets, and the invariable plane perpendicular to it (green). The vertical scale is exaggerated by a factor of 2.8 so that the 1.58° tilt can be seen. Drag to rotate. *Script: fig-invariable-plane.py*.

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(b)

Figure 2

Which planets decide where $\mathbf{H}$ points? In the two-body note we will show that a planet of mass m on an orbit of semi-major axis a and eccentricity e about the Sun carries angular momentum $m\sqrt{GM_\odot\, a(1-e^2)}$. Since $GM_\odot$ is common to all planets, each planet's share is proportional to $m\sqrt{a(1-e^2)}$.

Using the masses (in Earth masses) $m_J = 317.8$, $m_S = 95.2$, $m_U = 14.5$, $m_N = 17.1$ and a from the table in Section 2 (with $a_U = 19.19$, $a_N = 30.07$ AU), and ignoring e, estimate the share of the planets' orbital angular momentum carried by each giant planet. What carries the Earth?

What you should find

$317.8\sqrt{5.203} = 725$, $95.2\sqrt{9.537} = 294$, $14.5\sqrt{19.19} = 64$, $17.1\sqrt{30.07} = 94$, against 1 for the Earth. Jupiter carries about 62%, Saturn 25%, Neptune 8%, Uranus 5%, and the four inner planets together less than 0.3%. So the invariable plane is essentially Jupiter's orbital plane tilted a little towards Saturn's. Indeed, Jupiter's orbit is inclined only 0.32° to it.

7.3 Energy

Two vector integrals came from adding the forces and their moments. The third integral is a scalar. To find it we first show that the gravitational force is the **gradient of a scalar function**. That fact matters on its own: one scalar function of the positions is far easier to handle than $3n$ force components, and it is what makes an energy integral possible.

The gradient of a distance. All gradients below are taken with respect to the position of the body that *feels* the force, with the other body held fixed:

$$\nabla_i \equiv \frac{\partial}{\partial \mathbf{R}_i}.$$

Let X_{ik} be the k-th Cartesian component of $\mathbf{R}_i$, so $\mathbf{R}_i = X_{i1}\hat{\mathbf{i}} + X_{i2}\hat{\mathbf{j}} + X_{i3}\hat{\mathbf{k}}$. Then

History: Laplace's plane. Laplace proposed the invariable plane as a reference plane that, unlike the ecliptic, does not move. The ecliptic itself drifts slowly because the other planets perturb the Earth's orbit. A modern determination (Souami and Souchay, 2012) found that Pluto and Ceres must be included to

$$r_{ij}^2 = \mathbf{r}_{ij} \cdot \mathbf{r}_{ij} = \sum_{k=1}^{3} \left(X_{jk} - X_{ik} \right)^2 .$$

Differentiating both sides with respect to X_{ik},

$$2 r_{ij} \frac{\partial r_{ij}}{\partial X_{ik}} = 2 \left(X_{jk} - X_{ik} \right) (-1) \qquad \Longrightarrow \qquad \frac{\partial r_{ij}}{\partial X_{ik}} = - \frac{X_{jk} - X_{ik}}{r_{ij}}.$$

Collecting the three components,

$$\nabla_i \, r_{ij} = - \frac{\mathbf{r}_{ij}}{r_{ij}} = - \hat{\mathbf{r}}_{ij}.$$

This is geometrically obvious once written down: moving body i *towards* j (along $\hat{\mathbf{r}}_{ij}$) shortens the distance fastest. One more application of the chain rule gives

$$\frac{\partial}{\partial X_{ik}} \left(\frac{1}{r_{ij}} \right) = - \frac{1}{r_{ij}^2} \frac{\partial r_{ij}}{\partial X_{ik}} = \frac{X_{jk} - X_{ik}}{r_{ij}^3} \qquad \Longrightarrow \qquad \nabla_i \left(\frac{1}{r_{ij}} \right) = \frac{\mathbf{r}_{ij}}{r_{ij}^3}.$$

The force as a gradient. Comparing with Equation 1,

$$\mathbf{F}_{ij} = \frac{G m_i m_j}{r_{ij}^3} \, \mathbf{r}_{ij} = \nabla_i \left(\frac{G m_i m_j}{r_{ij}} \right) = - \nabla_i \Phi_{ij}, \tag{6}$$

where

$$\Phi_{ij} = - \frac{G m_i m_j}{r_{ij}}.$$

Φ_{ij} is the **mutual potential energy** of the pair; note that it contains *both* masses. (The gravitational *potential* of m_j, a field defined per unit mass, is $\phi_j = -G m_j / r_{ij}$, so that $\Phi_{ij} = m_i \phi_j$.) Summing over j, the total force on body i is

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} = - \nabla_i \Phi_i, \qquad \Phi_i = \sum_{j \neq i} \Phi_{ij} = - G \sum_{j \neq i} \frac{m_i m_j}{r_{ij}},$$

and the equation of motion becomes $m_i \ddot{\mathbf{R}}_i = - \nabla_i \Phi_i$.

The energy integral. Take the dot product of $m_i\ddot{\mathbf{R}}_i = -\nabla_i\Phi_i$ with $\dot{\mathbf{R}}_i$ and sum over the bodies:

$$\sum_{i=1}^{n} m_i\,\dot{\mathbf{R}}_i \cdot \ddot{\mathbf{R}}_i = -\sum_{i=1}^{n} \dot{\mathbf{R}}_i \cdot \nabla_i\Phi_i. \tag{7}$$

The left-hand side is an exact derivative: it is dT/dt, where

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\,\dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i$$

is the total kinetic energy. We would like the right-hand side to be the time derivative of a single function too. The natural guess, $\sum_i \Phi_i$, is wrong by a factor of two, and it is worth seeing why before going on.

Take $n = 2$. Write out Φ_1 and Φ_2. What is $\Phi_1 + \Phi_2$? What is the potential energy of the pair, i.e. the work needed to bring the two masses together from infinite separation?

What you should find

$\Phi_1 = \Phi_{12} = -Gm_1m_2/r_{12}$ and $\Phi_2 = \Phi_{21} = -Gm_2m_1/r_{12}$, so $\Phi_1 + \Phi_2 = -2Gm_1m_2/r_{12}$. But the pair has a single bond, of energy $-Gm_1m_2/r_{12}$. The sum over bodies counts each pair twice, once from each end.

So we define the potential energy of the system as a sum over **pairs**:

$$V = -G\sum_{\text{pairs } i<j} \frac{m_im_j}{r_{ij}} = -\frac{G}{2}\sum_{i=1}^{n}\sum_{j\neq i} \frac{m_im_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} \Phi_i. \tag{8}$$

V depends on time only through the positions, so by the chain rule

$$\frac{dV}{dt} = \sum_{i=1}^{n} \nabla_i V \cdot \dot{\mathbf{R}}_i.$$

Which terms of V depend on $\mathbf{R}_i$? Exactly the pairs that contain body i, i.e. the terms Φ_{ij} for $j \neq i$, each appearing **once** in the pair sum. Hence

$$\nabla_i V = \sum_{j\neq i} \nabla_i\Phi_{ij} = \nabla_i\Phi_i.$$

(The factor $\frac{1}{2}$ in Equation 8 and the double counting in the double sum cancel exactly.) Substituting into Equation 7,

$$\frac{dT}{dt} = -\sum_{i=1}^{n} \dot{\mathbf{R}}_i \cdot \nabla_i V = -\frac{dV}{dt} \qquad \Longrightarrow \qquad T + V = C_4. \tag{9}$$

The total energy is constant: the tenth scalar constant.

A student integrates Equation 3 numerically for 10^4 years and finds that $T + V$ has changed by one part in 10^6. What can be concluded? What if it had stayed constant to 10^{-14}?

Answer

Since the exact solution conserves $T + V$, a change of 10^{-6} is pure numerical error, and it gives a rough measure of how inaccurate the solution is. The converse does not hold: conservation to 10^{-14} is necessary but not sufficient. An integrator can conserve energy well and still put a planet at the wrong place along its orbit (a timing error does not change the energy).

7.4 A closer look at conservative forces

A force that is the gradient of a scalar is called **conservative**. The name comes from what it does to work. Take a point mass m_i moving from point A to point B in the field of a fixed m_j. Along any path, $\mathbf{F}_{ij} \cdot d\mathbf{r} = -\nabla_i \Phi_{ij} \cdot d\mathbf{r} = -d\Phi_{ij}$, so the work done by gravity is

$$W = \int_A^B \mathbf{F}_{ij} \cdot d\mathbf{r} = \Phi_{ij}(A) - \Phi_{ij}(B).$$

It depends only on the end points, not on the route.

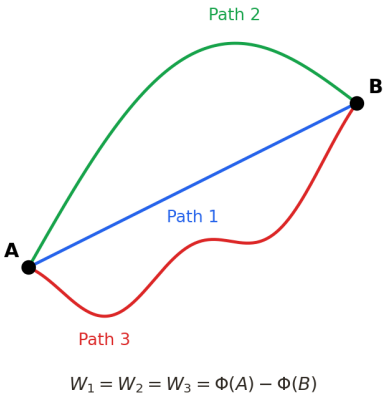


Figure 3: The work done by a conservative force in moving from A to B is the same along every path: $W_1 = W_2 = W_3 = \Phi(A) - \Phi(B)$. *Script: fig-conservative-paths.py.*

Does this mean that if I move a satellite from one orbit to another, every route costs the same?

Answer

The same *work*, yes; the same *fuel*, no. The gap between those two statements is where mission design lives.

The work done by gravity is $-\Delta V$, fixed by the end points. The two orbits also fix the energy at each end, so the energy the engine must supply is fixed too. What is **not** fixed is the velocity change Δv needed to deliver that energy, and Δv is what costs propellant. A burn along the velocity raises the energy per unit mass by $\Delta\varepsilon = \frac{1}{2}(v+\Delta v)^2 - \frac{1}{2}v^2 = v\,\Delta v + \frac{1}{2}\Delta v^2$, so the same energy is bought with less Δv where the speed v is already large. Where you burn, how you change the plane, and how long the burn lasts all change the bill, though the work does not change.

Since the curl of a gradient is zero, a conservative force is also **irrotational**: $\nabla_i \times \mathbf{F}_{ij} = -\nabla_i \times (\nabla_i \Phi_{ij}) = \mathbf{0}$. It is instructive to check this directly, without the potential. Let a point mass M sit at the origin; its field (force per unit mass) at $\mathbf{r} = (x, y, z)$ is

$$\mathbf{g} = -GM\,\frac{\mathbf{r}}{r^3}, \qquad r = \sqrt{x^2+y^2+z^2}, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Show that the z-component of $\nabla \times \mathbf{g}$, which is $\partial g_y/\partial x - \partial g_x/\partial y$, vanishes. You need only the derivative $\partial(r^{-3})/\partial x = -3x/r^5$.

What you should find

$$\frac{\partial g_y}{\partial x} = -GM\,y\,\frac{\partial}{\partial x}\left(r^{-3}\right) = \frac{3GM\,xy}{r^5},$$
$$\frac{\partial g_x}{\partial y} = -GM\,x\,\frac{\partial}{\partial y}\left(r^{-3}\right) = \frac{3GM\,xy}{r^5},$$

so their difference is zero. The other two components follow by relabelling the axes.

More generally, **every** central field $\mathbf{g} = g(r)\,\hat{\mathbf{r}}$ is irrotational, whatever the function g. Irrotationality says the field comes from a potential; it does not single out the inverse square. We will meet properties that *do* single out the inverse square in the two-body note.

8 Counting: why two bodies are easy and three are not

We now have ten scalar integrals, valid for every n:

integral	equation	scalar constants
centre-of-mass velocity	$\sum m_i \dot{\mathbf{R}}_i = \mathbf{C}_1$	3
centre-of-mass position	$\sum m_i \mathbf{R}_i - \mathbf{C}_1 t = \mathbf{C}_2$	3
angular momentum	$\mathbf{H} = \mathbf{C}_3$	3
energy	$T + V = C_4$	1

The system has order $6n$. Each independent integral lowers the order by one. Two further reductions are classical (eliminating the time, and eliminating the orientation of the line in which the plane of the motion cuts the invariable plane, known as the *node*), and together they reduce the order from $6n$ to $6n - 12$.

- $n = 2$**.** Order 12. Removing the centre of mass (six constants) leaves the relative motion of the two bodies, a problem of order 6. In the two-body note we will find that the relative motion has, in addition to angular momentum and energy, one more integral that is special to the inverse-square law (the *eccentricity vector*). With it the problem is solved completely, and the solution is Kepler's ellipse.
- $n = 3$**.** Order 18. After all reductions an order-6 problem remains, and there are no more integrals to use.

So for $n \geq 3$ there is no general solution in closed form: no formula, built from the ten integrals and known functions, that gives the positions at time t for arbitrary initial conditions. (Special solutions do exist; we will see one shortly.)

The ten integrals hold for any n. Why do they nearly solve the two-body problem, but leave the three-body problem wide open?

Answer

Because the order grows with n and the number of integrals does not. For $n = 2$, ten integrals against an order of 12 leaves little to do, and the extra inverse-square integral finishes the job. For $n = 3$ the order is 18, and ten integrals (plus the two classical reductions) leave a sixth-order problem with nothing to reduce it further.

History: the impossibility theorems. Heinrich Bruns proved in 1887 that the three-body problem has no integrals that are algebraic functions of the positions and velocities other than the ten classical ones. Henri Poincaré, in his 1890 prize memoir for King Oscar II of Sweden, extended this to a wide class of analytic integrals, and in the process discovered what we now call

9 Solving numerically, and what chaos means

If no formula exists, we can still integrate Equation 3 numerically with any standard ODE integrator (Runge–Kutta methods, for example), from given initial conditions. Figure 4 shows two such solutions for three equal masses, in units where $G = 1$ and each mass is 1.

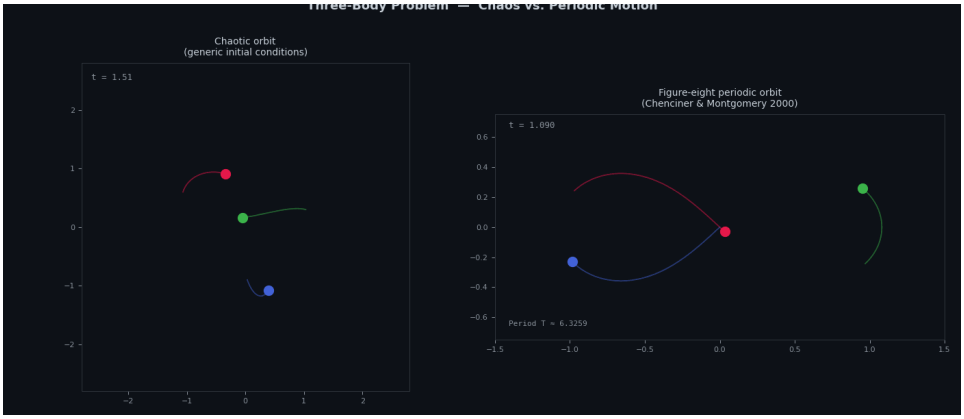


Figure 4: Two solutions of the three-body problem with equal masses. **Left:** generic initial conditions; the motion never repeats. **Right:** the figure-eight, a special periodic solution in which the three bodies chase each other round one curve. *Script: three-body-animation.py.*

The left panel looks messy, but messy is not the same as unpredictable. To see what *is* special about it, run it twice.

Figure 5 shows the left-panel system integrated twice. In run B, body 1 starts 10^{-6} to the right of where it starts in run A (in units where the bodies start about 1 apart). Read off from the right panel: (a) roughly how large the difference between the two runs is at $t = 10$ and at $t = 40$; (b) how large the integrator error is, measured by the change in energy. Then run fig-nbody-sensitivity.py with a smaller offset (give it the argument 10^{-9}; its header explains how), and note the separation it prints at $t = 45$.

What you should find

(a) A few times 10^{-4} at $t = 10$ and a few times 10^{-2} just before $t = 40$: the offset has grown ten-thousandfold. At the close encounter near $t = 40$ it jumps to order 1, and by $t = 45$ the two copies of body 1 are more than 3 units apart. (b) The energy error stays below 10^{-9} throughout, so the divergence is **not** numerical error: it is a property of the equations.

With an offset of 10^{-9} the printed separation at $t = 45$ is 0.013 instead of 3.3. A thousand times smaller starting error postpones the divergence, but the offset is still amplified about 10^7 times. In both runs bodies 1 and 3 end up as a tight binary and body 2 is thrown out; what the offset changes is the direction in which the binary leaves (the left panel) and how tight it is.

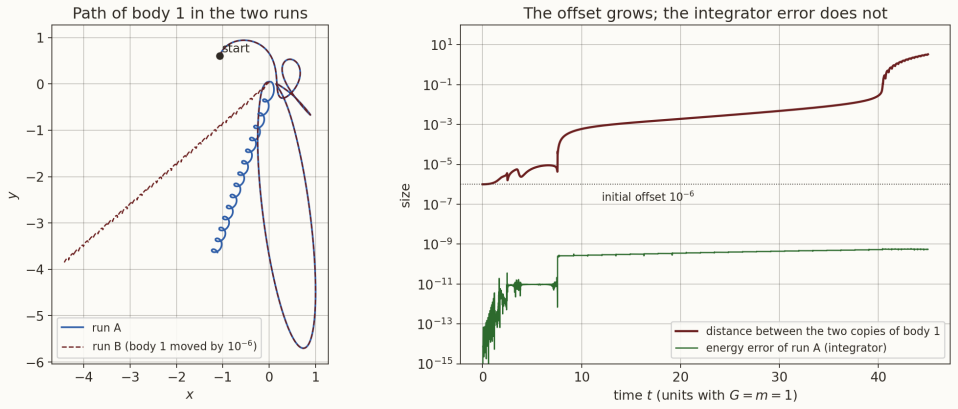


Figure 5: The chaotic case of Figure 4, integrated twice; run B starts with body 1 moved by 10^{-6}. **Left:** the path of body 1 in each run. **Right:** the distance between the two copies of body 1 (maroon) and the energy error of run A (green). Each jump in the maroon curve is a close encounter between two bodies. *Script: fig-nbody-sensitivity.py.*

This is what **chaos** means here. A solution is chaotic if solutions that start arbitrarily close together separate, on average exponentially fast, until the separation is as large as the system itself. The solution is still exactly determined by its initial conditions; but since initial conditions are only known approximately, prediction beyond some horizon becomes meaningless.

The solar system is chaotic too. But its chaos is slow: an error in the Earth's position grows by a factor of $e \approx 2.7$ only about every five million years.

If the solar system is chaotic, how can a mission team plan a Mars launch window twenty years in advance?

Answer

Twenty years is about 4×10^{-6} of the five-million-year growth time, so an error grows by a negligible factor over a mission's lifetime. Chaos limits predictions over millions of years, not decades. The planetary ephemerides that missions use are numerical solutions of the N-body problem, fitted to observations and updated regularly.

The right panel of Figure 4 shows the other side of the story. The figure-eight is an exact periodic solution: the three bodies follow one another round the same curve, forever. Such special solutions exist, but they are rare, and finding them takes either cleverness or a computer.

History: chaos in the solar system and the figure-eight. Jacques Laskar showed in 1989, by integrating averaged equations for the planets over 200 million years, that the inner solar system is chaotic, with errors growing by a factor e roughly every 5 million years.

The figure-eight solution was found numerically by Cris Moore in 1993, and its

10 The way forward

Where does this leave us? The N-body problem has no general closed-form solution. Numerical solutions are accurate, but they are expensive for many bodies, and they give numbers rather than understanding: a table of positions does not tell you *why* a burn in one place raises the orbit on the other side, or what changes if you burn a little earlier.

What extra assumption could reduce the N-body problem to something solvable, while keeping the essential dynamics of the solar system or of a satellite about the Earth?

Answer

Exploit how uneven the masses are. The Sun holds about 99.87% of the mass of the Sun and planets together; the Earth outweighs a satellite by a factor of order 10^{21}. So:

1. keep **two** bodies, and treat everything else as a small disturbance: the two-body problem, which can be solved in closed form;
2. treat them as **point masses**, which is exact for spherically symmetric bodies (shell theorem);
3. when one mass is negligible, ignore its pull on the other: the **restricted** problem.

Every model in the rest of the course is this idealisation, or a correction to it.

How small is "everything else"? A quick estimate settles it.

Jupiter is the most massive planet, $M_\odot/M_J = 1047$. At its closest, it is $5.20 - 1.00 = 4.20$ AU from the Earth. Compute the ratio of Jupiter's pull on the Earth to the Sun's pull on the Earth.

What you should find

$\dfrac{GM_J/(4.20\text{ AU})^2}{GM_\odot/(1\text{ AU})^2} = \dfrac{1}{1047 \times 4.20^2} = 5.4 \times 10^{-5}$. Even the largest planet, at its closest, pulls on the Earth about twenty thousand times more weakly than the Sun. (What actually perturbs the Earth's orbit about the Sun is the *difference* between Jupiter's pulls on the Earth and on the Sun, which is of the same order.)

The consequences are the following.

- **The Sun barely moves.** With $M_\odot/M_J \approx 1047$, the Sun's motion about the barycentre is about a thousand times smaller than Jupiter's. So we may treat the Sun as fixed at the barycentre, with the planets moving about it.

- **The planets barely disturb one another.** Each planet's motion is decided almost entirely by the Sun, so we may treat each planet as moving in the field of the Sun alone.

Both together give the **restricted two-body problem**: one small body moving in the field of one large one. It is what we solve next, in the two-body note, and it turns out to contain an astonishing amount: Kepler's three laws, escape, flybys, and the timing of every orbit.

11 Summary

1. The bodies of the solar system may be treated as **point masses**: because they are small compared with their separations, and, more fundamentally, because a spherically symmetric body attracts like a point (shell theorem).
2. Newton's law gives n coupled second-order vector equations Equation 2, or $6n$ first-order ODEs. Each body's acceleration is independent of its own mass.
3. Because the internal forces are equal, opposite and along the line of centres, the system has **ten integrals**: centre-of-mass velocity and position (6), angular momentum (3) and energy (1). The centre of mass moves uniformly, so the barycentric frame is inertial; $\mathbf{H}$ defines the invariable plane, 1.58° from the ecliptic.
4. Gravity is **conservative**: $\mathbf{F}_{ij} = -\nabla_i \Phi_{ij}$, the work done is independent of path, and the field is irrotational. The system's potential energy is a sum over pairs.
5. For $n = 2$ the integrals (plus one special to the inverse square) solve the problem. For $n \geq 3$ there are no further algebraic integrals (Bruns), and the problem has no general closed-form solution.
6. Numerical solutions are routine, but generic three-body motion is **chaotic**: nearby solutions diverge exponentially. The solar system is chaotic on a time scale of millions of years.
7. The extreme mass ratios of the solar system justify the **restricted two-body problem**, the subject of the next note.

12 Exercises

1. **Weighing the Earth with the Moon.** The Moon's orbit has $a = 384\,400$ km and sidereal period 27.32 days. Assuming $T^2/a^3 = 4\pi^2/(GM)$, compute GM for the Earth (ignore the Moon's own mass). How far off is the answer from 3.986×10^5 km^3/s^2, and why?
2. **The Earth–Moon barycentre.** With $M_\oplus / M_{\text{Moon}} = 81.3$, find the distance of the Earth–Moon barycentre from the centre of the Earth. Is it

inside or outside the Earth?

3. **Momentum without angular momentum.** Construct two forces $\mathbf{F}_{12} = -\mathbf{F}_{21}$ that are *not* along the line of centres. Show that $\sum m_i \dot{\mathbf{R}}_i$ is still conserved but $\mathbf{H}$ is not.
4. **The pair sum.** For $n = 4$, write V as a pair sum and as $\frac{1}{2}\sum_i \Phi_i$, and verify that they agree term by term. Then compute $\nabla_1 V$ and show it equals $\nabla_1 \Phi_1$.
5. **Order counting.** How many first-order ODEs describe the Earth–Moon–spacecraft system? Which of the ten integrals survive if the spacecraft fires its engine?
6. **Chaos and the integrator.** In Figure 5 the energy error jumps at the same times as the separation. Explain why close encounters are hard for an integrator with a fixed tolerance, and why that does not explain the divergence of the two runs.

All the numbers in this note, and the identities used in the derivations, are checked by n-body-verify.py.