Mars Flyby as a Hyperbolic-Orbit Teaching Problem

Frames, miss distance, hyperbolic anomaly, time propagation, and gravity assist

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1 Purpose of the problem

This is a single connected exercise through which students learn the mechanics of a planar hyperbolic planetary flyby. The central sequence is

Sun frame → Mars frame → hyperbolic orbit → time propagation → hyperbolic turning → Sun frame.

The problem is entirely two-dimensional. The objectives are:

1. **Frame dependence of the speed gain.** In the Mars-centred frame an ideal unpowered flyby preserves the magnitude of the hyperbolic excess velocity, $|\mathbf{v}_{\infty 1}| = |\mathbf{v}_{\infty 2}|$; in the heliocentric frame the spacecraft speed *can* change, because the Mars-relative velocity vector has been rotated before $\mathbf{V}_M$ is added back.
2. **Time propagation on a hyperbola,** through the hyperbolic eccentric anomaly E and the hyperbolic Kepler equation $M = e \sinh E - E$.
3. **The asymptotic miss distance Δ,** the perpendicular distance by which the incoming asymptotic straight line would miss the centre of Mars if Mars exerted no gravity — the magnitude of the impact parameter in B-plane terminology. It exhibits **gravitational focusing**: Δ can be much larger than the actual periapsis radius.

2 Problem statement

A spacecraft performs a planar flyby of Mars. Use the patched-conic approximation: during the close encounter treat Mars as the only gravitating body; outside the encounter describe the motion heliocentrically. Choose a heliocentric Cartesian frame such that, at the nominal encounter epoch, $+x$ points radially outward from the Sun through Mars and $+y$ lies along the instantaneous direction of motion of Mars, which is assumed circular.

Symbol	Value	Meaning
$\mathbf{V}_M$	$\begin{bmatrix}0 & 24.13\end{bmatrix}^\mathsf{T}$ km/s	heliocentric velocity of Mars
$\mathbf{V}_1$	$\begin{bmatrix}1.80 & 21.73\end{bmatrix}^\mathsf{T}$ km/s	incoming heliocentric velocity of the spacecraft
μ_M	4.2828×10^4 km^3/s^2	gravitational parameter of Mars
R_M	3396 km	radius of Mars
r_M	2.2794×10^8 km	heliocentric orbital radius of Mars
Δ	8242.25 km	asymptotic miss distance

Here Δ is the perpendicular distance between the centre of Mars and the incoming asymptotic straight-line trajectory. During the flyby, Mars bends the Mars-relative velocity vector **counter-clockwise**.

3 Part A — From the Sun frame to the Mars frame

The spacecraft velocity relative to Mars before the encounter is $\mathbf{v}_{\infty 1} = \mathbf{V}_1 - \mathbf{V}_M$.

1. Calculate $\mathbf{v}_{\infty 1}$ and its magnitude v_∞.
2. Explain physically what v_∞ represents.
3. Is v_∞ the velocity at periapsis? Explain.

4 Part B — Classify the Mars-centred orbit

The specific orbital energy relative to Mars is $\varepsilon = v^2/2 - \mu_M/r$. Far from Mars $r \to \infty$, so $\mu_M/r \to 0$.

1. Show that the incoming Mars-centred trajectory has $\varepsilon = v_\infty^2/2$.
2. Determine the numerical value and sign of ε, and hence classify the orbit as elliptical, parabolic, or hyperbolic.
3. Explain why an unpowered interplanetary spacecraft arriving from outside a planet's gravitationally bound region normally has a hyperbolic planet-centred orbit.

5 Part C — Introduce the miss distance Δ

Imagine temporarily that Mars has no gravity: the spacecraft would continue along the incoming asymptote with speed v_∞, passing the centre of Mars at perpendicular distance Δ.

C.1 Angular momentum. Using $h = |\mathbf{r} \times \mathbf{v}|$ and interpreting Δ as the perpendicular lever arm from Mars to the asymptotic velocity line, show that $h = \Delta v_\infty$ and calculate h.

C.2 Eccentricity. For any Keplerian conic, $e = \sqrt{1 + 2\varepsilon h^2/\mu_M^2}$. Use your ε and h to determine e, and verify that the orbit is a hyperbola.

C.3 Periapsis radius. With semilatus rectum $p = h^2/\mu_M$ and conic equation $r = p/(1 + e\cos f)$, periapsis ($f = 0$) gives $r_p = p/(1 + e)$. Calculate r_p and the periapsis altitude $h_p = r_p - R_M$. Compare r_p with Δ and explain why $r_p < \Delta$; this is the gravitational-focusing effect.

6 Part D — Direct relation between miss distance and periapsis

At periapsis the velocity is perpendicular to the radius, so $h = r_p v_p$; at infinity $h = \Delta v_\infty$. Hence $\Delta v_\infty = r_p v_p$. Combining with the vis-viva result $v_p^2 = v_\infty^2 + 2\mu_M/r_p$, show that

$$\boxed{\Delta^2 = r_p^2 + \frac{2\mu_M r_p}{v_\infty^2}}$$

With $A = \mu_M/v_\infty^2$ this reads $\Delta^2 = r_p^2 + 2Ar_p$. Complete the square to obtain

$$\boxed{r_p = \sqrt{\Delta^2 + A^2} - A}$$

and verify that it reproduces the value of r_p found in Part C.

D.1 Collision miss distance. What asymptotic miss distance would just produce a grazing trajectory at the surface of Mars? Set $r_p = R_M$ and determine Δ_{impact}. Explain why a trajectory with $\Delta > R_M$ can still collide with Mars.

7 Part E — Hyperbolic semi-major axis and periapsis velocity

For a conic $\varepsilon = -\mu_M/2a$, and since the orbit is hyperbolic the conventional signed semi-major axis satisfies $a < 0$.

1. Calculate a and show that $a = -\mu_M/v_\infty^2$.
2. Verify the hyperbolic periapsis relation $r_p = a(1-e)$. With the positive quantity $A = -a = |a|$ this becomes $r_p = A(e - 1)$.
3. Calculate the periapsis speed $v_p = \sqrt{v_\infty^2 + 2\mu_M/r_p}$.
4. Explain physically why $v_p > v_\infty$ even though the spacecraft returns to the speed v_∞ as it recedes to infinity.

8 Part F — Hyperbolic eccentric anomaly

Set $t = 0$ at periapsis, so that $f = 0$ and $E = 0$; here E denotes the **hyperbolic eccentric anomaly**, and $A = -a > 0$. In a Mars-centred perifocal frame whose $+x$ axis points toward periapsis, a convenient parametric representation of the hyperbola is

$$\boxed{x = A(e - \cosh E), \qquad y = A\sqrt{e^2 - 1}\,\sinh E, \qquad r = A(e\cosh E - 1).}$$

F.1 Relation between f and E. Use the half-angle identity $\tan(f/2) = y/(r + x)$ with the expressions above. Show first that $r + x = A(e - 1)(1 + \cosh E)$, and hence prove

$$\boxed{\tan\frac{f}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{E}{2}} \qquad \iff \qquad \boxed{\tanh\frac{E}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{f}{2}}$$

The second form allows E to be calculated when f is known.

9 Part G — Derive the hyperbolic Kepler equation

The specific angular momentum is $h = x\dot{y} - y\dot{x}$. Differentiating the parametric equations,

$$\frac{dx}{dE} = -A\sinh E, \qquad \frac{dy}{dE} = A\sqrt{e^2 - 1}\,\cosh E,$$

and with $\dot{x} = (dx/dE)\dot{E}$, $\dot{y} = (dy/dE)\dot{E}$, show that

$$h = A^2\sqrt{e^2 - 1}\,(e\cosh E - 1)\,\dot{E}.$$

Show also, from $p = A(e^2 - 1)$, that $h = \sqrt{\mu_M A(e^2 - 1)}$. Equating the two expressions proves

$$\dot{E} = \frac{\sqrt{\mu_M/A^3}}{e\cosh E - 1}, \qquad \text{so with} \qquad \boxed{n = \sqrt{\frac{\mu_M}{A^3}}} \qquad \text{one has} \qquad \frac{dt}{dE} = \frac{e\cosh E - 1}{n}.$$

Integrating from periapsis ($E = 0$, $t = 0$) gives $nt = \int_0^E (e\cosh u - 1)\,du$, and therefore the hyperbolic Kepler equation

$$\boxed{M = e\sinh E - E, \qquad M = nt, \qquad t = \sqrt{\frac{A^3}{\mu_M}}\,(e\sinh E - E)\,.}$$

10 Part H — Time from periapsis to $f = 90^\circ$

This part quantitatively tests one of the assumptions of the patched-conic flyby model. Starting at periapsis ($f = 0$), find the time required to reach $f = 90^\circ$:

1. Use $\tanh(E/2) = \sqrt{(e-1)/(e+1)}\,\tan(f/2)$ to calculate E at $f = 90^\circ$.
2. Calculate the hyperbolic mean anomaly $M = e\sinh E - E$ and the mean motion $n = \sqrt{\mu_M/A^3}$.
3. Calculate $t = M/n$ and express the result in seconds and minutes.
4. Find r at $f = 90^\circ$ from $r = p/(1+e\cos f)$, and verify it independently from $r = A(e\cosh E - 1)$.

11 Part I — Is it reasonable to keep V_M constant during the flyby?

The same Mars heliocentric velocity was used when transforming into and out of the Mars-centred frame, yet Mars keeps moving around the Sun during the flyby. For a circular orbit its angular speed is $\omega_M = V_M/r_M$.

Use the time from Part H to estimate the angle $\Delta\theta_M = \omega_M t$ swept by Mars while the spacecraft moves from $f = 0$ to $f = 90^\circ$, in radians and degrees. Then estimate $|\Delta\mathbf{V}_M| \approx V_M \Delta\theta_M$ for small $\Delta\theta_M$, compare it with v_∞, and discuss whether $\mathbf{V}_M \approx$ constant during the close encounter is reasonable. Double the result to represent the interval from $f = -90^\circ$ to $f = +90^\circ$.

12 Part J — Algorithm for $\mathrm{r}(t)$ and $\mathrm{v}(t)$ on a hyperbolic orbit

Given hyperbolic elements $a < 0$, $e > 1$ and the time since periapsis, develop an algorithm for the state at time t, using $A = -a > 0$.

Step 1 — mean motion. $n = \sqrt{\mu_M/A^3}$.

Step 2 — mean anomaly. $M = n(t - t_p)$, or $M = nt$ if periapsis is at $t_p = 0$.

Step 3 — solve the hyperbolic Kepler equation $e\sinh E - E = M$. This cannot in general be inverted in elementary functions, so use Newton's method on $F(E) = e\sinh E - E - M$, with $F'(E) = e\cosh E - 1$:

$$\boxed{E_{k+1} = E_k - \frac{e\sinh E_k - E_k - M}{e\cosh E_k - 1}}$$

A convenient start is $E_0 = \sinh^{-1}(M/e)$; for small $|M|$, $e\sinh E - E \approx (e-1)E$ gives $E_0 \approx M/(e-1)$. Iterate until, say, $|E_{k+1} - E_k| < 10^{-12}$.

Step 4 — radius. $r = A(e\cosh E - 1)$.

Step 5 — true anomaly. $\tan(f/2) = \sqrt{(e+1)/(e-1)}\,\tanh(E/2)$; in a numerical implementation handle the quadrant of f carefully.

Step 6 — position. With $p = A(e^2-1)$ and $r = p/(1+e\cos f)$, the perifocal position is $\mathbf{r} = r\begin{bmatrix}\cos f & \sin f\end{bmatrix}^{\mathsf{T}}$.

Step 7 — velocity. With $h = \sqrt{\mu_M p}$,

$$\boxed{\mathbf{v} = \frac{\mu_M}{h}\begin{bmatrix}-\sin f \\ e+\cos f\end{bmatrix}} \qquad \text{equivalently} \qquad v_r = \frac{\mu_M}{h} e\sin f, \quad v_f = \frac{\mu_M}{h}(1+e\cos f),$$

so that $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_f\hat{\mathbf{e}}_f$.

J.8 Direct state equations in terms of E. Alternatively, once E is known,

$$x = A(e-\cosh E), \qquad y = A\sqrt{e^2-1}\sinh E,$$

$$\boxed{v_x = -\sqrt{\frac{\mu_M}{A}}\frac{\sinh E}{e\cosh E-1}, \qquad v_y = \sqrt{\frac{\mu_M}{A}}\frac{\sqrt{e^2-1}\cosh E}{e\cosh E-1}.}$$

Students should verify that the two methods give the same state.

13 Part K — Evaluate the state at $f = 90^\circ$

Using the results of Part H:

1. Calculate $\mathbf{r}$ and $\mathbf{v}$ in the perifocal frame at $f = 90^\circ$, the latter from $\mathbf{v} = (\mu_M/h)\begin{bmatrix}-\sin f & e+\cos f\end{bmatrix}^{\mathsf{T}}$.
2. Verify the same velocity using the equations in terms of E.
3. Check the specific energy numerically from $\varepsilon = v^2/2 - \mu_M/r$ and verify that it equals $v_\infty^2/2$.

This demonstrates explicitly that the spacecraft accelerates as it approaches Mars and decelerates as it departs, while its Mars-centred orbital energy remains constant.

14 Part L — Hyperbolic asymptote and turning angle

Along an asymptote $r \to \infty$ in $r = p/(1+e\cos f)$, so the denominator vanishes: $1 + e\cos f_\infty = 0$, hence

$$\boxed{\cos f_\infty = -\frac{1}{e}}$$

Determine f_∞ for this flyby. The total turning angle is $\delta = 2f_\infty - \pi$; show that this is equivalent to $\sin(\delta/2) = 1/e$, and hence calculate δ.

15 Part M — Turning angle from the miss distance

From Part C, $h = \Delta v_\infty$. Using $e = \sqrt{1 + 2\varepsilon h^2/\mu_M^2}$ with $\varepsilon = v_\infty^2/2$, show that

$$\boxed{e = \sqrt{1 + \left(\frac{\Delta v_\infty^2}{\mu_M}\right)^2}}$$

and, since $\sin(\delta/2) = 1/e$, prove that

$$\boxed{\tan\frac{\delta}{2} = \frac{\mu_M}{\Delta v_\infty^2}, \qquad \text{so} \qquad \delta = 2\tan^{-1}\left(\frac{\mu_M}{\Delta v_\infty^2}\right).}$$

Discuss the limiting cases: what happens to δ as Δ becomes very large, and as Δ becomes small? Why does a smaller miss distance produce a stronger gravity assist, and why can Δ not be made arbitrarily small for a real spacecraft?

16 Part N — Outgoing Mars-relative velocity

For an ideal unpowered flyby $|\mathbf{v}_{\infty 2}| = |\mathbf{v}_{\infty 1}| = v_\infty$; only the direction changes. Since the deflection is counter-clockwise,

$$\mathbf{v}_{\infty 2} = R(\delta)\,\mathbf{v}_{\infty 1}, \qquad R(\delta) = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix}.$$

Calculate $\mathbf{v}_{\infty 2}$ and verify numerically that $|\mathbf{v}_{\infty 2}| = v_\infty$.

17 Part O — Transform back to the heliocentric frame

The outgoing heliocentric velocity is $\mathbf{V}_2 = \mathbf{V}_M + \mathbf{v}_{\infty 2}$. Calculate $\mathbf{V}_2$, compare $|\mathbf{V}_1|$ with $|\mathbf{V}_2|$, and state whether the spacecraft has gained or lost heliocentric speed. Explain why the heliocentric speed can change even though $|\mathbf{v}_{\infty 1}| = |\mathbf{v}_{\infty 2}|$.

18 Part P — Optional: orient the hyperbola in the encounter frame

Parts F–K were written in a perifocal frame whose $+x$ axis points toward periapsis. Let the incoming asymptotic direction angle in the Mars-centred encounter frame be $\alpha_1 = \operatorname{atan2}(v_{\infty 1,y}, v_{\infty 1,x})$ and the outgoing one $\alpha_2 = \alpha_1 + \delta$ for the counter-clockwise encounter. Because the hyperbola is symmetric about its periapsis axis, the periapsis velocity direction bisects the two asymptotic directions, $\alpha_p = (\alpha_1 + \alpha_2)/2$. For counter-clockwise motion the periapsis radius is 90° clockwise from the periapsis velocity, so the orientation of the perifocal $+x$ axis is

$$\boxed{\omega = \alpha_p - 90^\circ.}$$

Any perifocal state may then be rotated into the encounter frame by $\mathbf{r}_M = R(\omega)\mathbf{r}_{\mathrm{pf}}$ and $\mathbf{v}_M = R(\omega)\mathbf{v}_{\mathrm{pf}}$ — a natural bridge from scalar hyperbolic-orbit calculations to full vector state propagation.

19 Instructor solution

19.1 Incoming velocity relative to Mars

$$\mathbf{v}_{\infty 1} = \mathbf{V}_1 - \mathbf{V}_M = \begin{bmatrix} 1.80 \\ 21.73 - 24.13 \end{bmatrix} = \begin{bmatrix} 1.80 \\ -2.40 \end{bmatrix} \text{ km/s},$$

$$v_\infty = \sqrt{1.80^2 + 2.40^2} = \sqrt{3.24 + 5.76} = \sqrt{9} = \boxed{3.00 \text{ km/s}},$$

the hyperbolic excess speed relative to Mars.

19.2 Orbit classification

At infinity $\varepsilon = v_\infty^2/2 = 9/2 = \boxed{4.50 \text{ km}^2/\text{s}^2} > 0$, so the Mars-centred trajectory is a **hyperbola**.

19.3 Angular momentum from the miss distance

Far from Mars $h = |\mathbf{r} \times \mathbf{v}|$, and only the component of $\mathbf{r}$ perpendicular to the velocity contributes; that component has magnitude Δ. Hence

$$h = \Delta v_\infty = 8242.25 \times 3.00 = \boxed{24726.8 \text{ km}^2/\text{s}}.$$

19.4 Eccentricity from Δ

Since $2\varepsilon = v_\infty^2$ and $h = \Delta v_\infty$,

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu_M^2}} = \sqrt{1 + \frac{v_\infty^2 \, \Delta^2 v_\infty^2}{\mu_M^2}} = \sqrt{1 + \left(\frac{\Delta v_\infty^2}{\mu_M}\right)^2}.$$

With $\Delta v_\infty^2/\mu_M = 8242.25(9)/42828 \approx 1.73205$,

$$e = \sqrt{1 + (1.73205)^2} = \sqrt{4} = \boxed{2.000}.$$

19.5 Semilatus rectum and periapsis

$$p = \frac{h^2}{\mu_M} = \frac{(24726.8)^2}{42828} \approx \boxed{14276.0 \text{ km}}, \qquad r_p = \frac{p}{1+e} = \frac{14276.0}{3} \approx \boxed{4758.67 \text{ km}},$$

$$h_p = r_p - R_M = 4758.67 - 3396 \approx \boxed{1362.67 \text{ km}}.$$

Note the comparison $\Delta = 8242.25$ km against $r_p = 4758.67$ km: $r_p < \Delta$, because Mars's gravity focuses the trajectory inward.

19.6 Direct miss-distance relation

Equating $h = \Delta v_\infty$ at infinity with $h = r_p v_p$ at periapsis and using $v_p^2 = v_\infty^2 + 2\mu_M/r_p$,

$$\Delta^2 v_\infty^2 = r_p^2 v_p^2 = r_p^2 \left(v_\infty^2 + \frac{2\mu_M}{r_p} \right) = r_p^2 v_\infty^2 + 2\mu_M r_p,$$

so that, dividing by v_∞^2 and writing $A = \mu_M/v_\infty^2$,

$$\Delta^2 = r_p^2 + \frac{2\mu_M r_p}{v_\infty^2} = r_p^2 + 2Ar_p.$$

Adding A^2 to both sides gives $\Delta^2 + A^2 = (r_p + A)^2$, and taking the positive root,

$$\boxed{r_p = \sqrt{\Delta^2 + A^2} - A.}$$

19.7 Critical miss distance for impact

For a grazing trajectory $r_p = R_M$, so the focusing relation gives

$$\Delta_{\text{impact}} = R_M \sqrt{1 + \frac{2\mu_M}{R_M v_\infty^2}} \approx \boxed{6622.2 \text{ km}},$$

much larger than $R_M = 3396$ km. A straight-line trajectory that appears to miss the centre of Mars by, say, 5000 km would still be gravitationally focused into the planet — a useful physical meaning of the impact parameter.

19.8 Hyperbolic semi-major axis

From $\varepsilon = -\mu_M/2a$ and $2\varepsilon = v_\infty^2$,

$$a = -\frac{\mu_M}{2\varepsilon} = -\frac{\mu_M}{v_\infty^2} = -\frac{42828}{9} = \boxed{-4758.67 \text{ km}}, \qquad A = -a = 4758.67 \text{ km}.$$

Since $e = 2$, $A(e-1) = 4758.67$ km, which correctly reproduces r_p.

19.9 Periapsis velocity

$$v_p = \sqrt{v_\infty^2 + \frac{2\mu_M}{r_p}} = \sqrt{9 + \frac{2(42828)}{4758.67}} = \sqrt{9 + 18} = \sqrt{27} = \boxed{5.19615 \text{ km/s}}.$$

19.10 Hyperbolic eccentric anomaly at $f = 90^\circ$

With $e = 2$ and $\tan(f/2) = \tan 45^\circ = 1$,

$$\tanh\frac{E}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{f}{2} = \sqrt{\tfrac{1}{3}} = \frac{1}{\sqrt{3}}, \qquad E = 2\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx \boxed{1.316958}.$$

For this value, $\cosh E = 2$ and $\sinh E = \sqrt{3}$ exactly.

19.11 Time from periapsis to $f = 90^\circ$

$$M = e\sinh E - E = 2\sqrt{3} - 1.316958 \approx \boxed{2.147144}, \qquad n = \sqrt{\frac{\mu_M}{A^3}} \approx \boxed{6.30429 \times 10^{-4}\ \text{s}^{-1}},$$

$$t = \frac{M}{n} \approx \frac{2.147144}{6.30429 \times 10^{-4}} \approx \boxed{3405.85\ \text{s}} = \boxed{56.76\ \text{min}}.$$

The spacecraft therefore requires less than one hour to move from periapsis to $f = 90^\circ$.

19.12 Radius at $f = 90^\circ$

From the conic equation with $\cos f = 0$, $r = p = \boxed{14276.0\ \text{km}}$. From the eccentric-anomaly form, with $e = 2$ and $\cosh E = 2$,

$$r = A(e\cosh E - 1) = 4758.67(4 - 1) = 4758.67(3) = 14276.0\ \text{km}.$$

The two calculations agree.

19.13 How much does Mars move in these 56.76 minutes?

$$\omega_M = \frac{V_M}{r_M} = \frac{24.13}{2.2794 \times 10^8} \approx 1.0586 \times 10^{-7}\ \text{rad/s},$$

$$\Delta\theta_M = \omega_M t \approx 3.605 \times 10^{-4}\ \text{rad} = \left(3.605 \times 10^{-4}\right)\frac{180}{\pi} \approx \boxed{0.0207^\circ},$$

a very small angle. The corresponding change in the Mars velocity vector is

$$|\Delta\mathbf{V}_M| \approx V_M\Delta\theta_M \approx 24.13\left(3.605 \times 10^{-4}\right) \approx \boxed{0.0087\ \text{km/s} = 8.7\ \text{m/s}},$$

which compared with $v_\infty = 3.00$ km/s is only $0.0087/3.00 \times 100 \approx 0.29\%$. Treating $\mathbf{V}_M$ as unchanged is therefore a very good first approximation. Over the symmetric interval $f = -90^\circ$ to $f = +90^\circ$ the elapsed time is about $2(56.76) = 113.52$ min and Mars moves through only about $\boxed{0.0413^\circ}$ — a quantitative justification for using the same $\mathbf{V}_M$ in the incoming and outgoing transformations.

19.14 State at $f = 90^\circ$

The perifocal position is $\mathbf{r} = r\begin{bmatrix}0 & 1\end{bmatrix}^\mathsf{T}$, i.e.

$$\boxed{\mathbf{r} = \begin{bmatrix}0\\14276.0\end{bmatrix}\ \text{km.}}$$

With $\sin f = 1$, $\cos f = 0$ and $\mu_M/h = 42828/24726.8 \approx 1.73205$ km/s,

$$\mathbf{v} = \frac{\mu_M}{h}\begin{bmatrix}-1\\2\end{bmatrix} \approx \boxed{\begin{bmatrix}-1.73205\\3.46410\end{bmatrix}\ \text{km/s}}, \qquad v = \sqrt{1.73205^2 + 3.46410^2} \approx \boxed{3.87298\ \text{km/s}}.$$

The spacecraft is already slowing down after periapsis, but has not yet returned to the asymptotic speed of 3 km/s.

19.15 Energy check at $f = 90^\circ$

With $v^2 \approx 15$ and $\mu_M/r = 42828/14276 = 3.0$ km^2/s^2,

$$\varepsilon = \frac{v^2}{2} - \frac{\mu_M}{r} = 7.5 - 3.0 = \boxed{4.5\ \text{km}^2/\text{s}^2} = \frac{v_\infty^2}{2}.$$

19.16 Asymptote true anomaly and turning angle

$1 + e\cos f_\infty = 0$ with $e = 2$ gives $\cos f_\infty = -\frac{1}{2}$, so $\boxed{f_\infty = 120^\circ}$ and

$$\delta = 2f_\infty - 180^\circ = 240^\circ - 180^\circ = \boxed{60^\circ}.$$

19.17 Turning angle directly from the miss distance

Writing $q = \Delta v_\infty^2/\mu_M$ so that $e = \sqrt{1+q^2}$, the relation $\sin(\delta/2) = 1/e$ becomes $\sin(\delta/2) = 1/\sqrt{1+q^2}$. A right triangle with opposite side 1 and adjacent side q then gives $\tan(\delta/2) = 1/q$, i.e.

$$\boxed{\tan\frac{\delta}{2} = \frac{\mu_M}{\Delta v_\infty^2}.}$$

Here $\mu_M/(\Delta v_\infty^2) \approx 1/\sqrt{3}$, so $\delta/2 = 30^\circ$ and again $\boxed{\delta = 60^\circ}$. The physical reading is direct: **smaller $\Delta \implies$ larger turning angle.**

19.18 Outgoing Mars-relative velocity

With $R(60^\circ) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ applied to $\mathbf{v}_{\infty 1} = \begin{bmatrix} 1.80 & -2.40 \end{bmatrix}^{\mathsf{T}}$,

$$v_{\infty 2,x} = \tfrac{1}{2}(1.80) - \tfrac{\sqrt{3}}{2}(-2.40) = 0.90 + 2.07846 = 2.97846 \text{ km/s},$$

$$v_{\infty 2,y} = \tfrac{\sqrt{3}}{2}(1.80) + \tfrac{1}{2}(-2.40) = 1.55885 - 1.20 = 0.35885 \text{ km/s},$$

$$\boxed{\mathbf{v}_{\infty 2} = \begin{bmatrix} 2.97846 \\ 0.35885 \end{bmatrix} \text{ km/s}, \qquad |\mathbf{v}_{\infty 2}| = 3.00 \text{ km/s}.}$$

19.19 Transform back to the heliocentric frame

$$\mathbf{V}_2 = \mathbf{V}_M + \mathbf{v}_{\infty 2} = \begin{bmatrix} 0 \\ 24.13 \end{bmatrix} + \begin{bmatrix} 2.97846 \\ 0.35885 \end{bmatrix} = \boxed{\begin{bmatrix} 2.97846 \\ 24.48885 \end{bmatrix} \text{ km/s.}}$$

$$|\mathbf{V}_1| = \sqrt{1.80^2 + 21.73^2} \approx 21.80 \text{ km/s}, \qquad |\mathbf{V}_2| = \sqrt{2.97846^2 + 24.48885^2} \approx 24.67 \text{ km/s},$$

so that in the Sun frame $21.80 \to 24.67$ km/s while in the Mars frame $3.00 \to 3.00$ km/s. The speed gain is a frame-dependent consequence of rotating $\mathbf{v}_\infty$ and then adding the moving planet's heliocentric velocity.

19.20 Optional: orientation of the perifocal hyperbola

$$\alpha_1 = \operatorname{atan2}(-2.40,\, 1.80) \approx -53.130^\circ, \qquad \alpha_2 = \alpha_1 + 60^\circ \approx 6.870^\circ,$$

$$\alpha_p = \frac{-53.130^\circ + 6.870^\circ}{2} = -23.130^\circ, \qquad \omega = \alpha_p - 90^\circ = \boxed{-113.130^\circ}.$$

The complete Mars-centred hyperbola can therefore be drawn in the encounter frame by rotating the perifocal coordinates through ω.

19.21 Suggested teaching sequence

Begin only with the two heliocentric velocities and ask students to transform into the Mars frame. Use the sign of ε to establish that the orbit is hyperbolic, then introduce Δ geometrically and derive $h = \Delta v_\infty$. Determine e, p, r_p and the periapsis altitude, and derive the gravitational-focusing relation between Δ and r_p. Introduce $a < 0$ and $A = -a > 0$, then the hyperbolic eccentric anomaly E from the hyperbola geometry, the relation between f and E, and the hyperbolic Kepler equation — derived rather than merely quoted. Calculate the time from $f = 0$ to $f = 90^\circ$ and use it to justify treating $\mathbf{V}_M$ as nearly constant. Present the Newton algorithm for $e \sinh E - E = M$ and calculate $\mathbf{r}(t)$, $\mathbf{v}(t)$. Finally determine the asymptotic angle and the 60° turning angle, rotate the incoming $\mathbf{v}_\infty$ to obtain the outgoing one, transform back to the Sun frame, and only then emphasize that the spacecraft has experienced a gravity assist. The conceptual flow is

$$\boxed{\mathbf{V}_1 \to \mathbf{v}_{\infty 1} \to (\varepsilon, h) \to (\Delta, e, a, r_p) \to E(t), f(t), \mathbf{r}(t), \mathbf{v}(t) \to \delta \to \mathbf{v}_{\infty 2} \to \mathbf{V}_2,}$$

which makes the flyby a natural application of hyperbolic two-body motion rather than an isolated set of gravity-assist formulas.