Capture at Mars: an Elliptical Mars-Centred Orbit

A worked companion to the hyperbolic-flyby problem — showing $\varepsilon < 0$ and $e < 1$

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1 Why this problem is different from the flyby problem

In the flyby problem the spacecraft arrives from far outside Mars's gravitational reach, so its Mars-centred energy is evaluated in the limit $r \to \infty$, where $\varepsilon = v_\infty^2/2 > 0$ **always**. Within a pure Mars-only two-body model, energy is conserved, so a spacecraft that starts outside can never end up bound: *unpowered capture on a patched conic is impossible.*

To obtain a bound Mars-centred orbit the state must be evaluated at a **finite** radius r, with the Mars-relative speed below the local escape speed,

$$v < v_{\rm esc}(r) = \sqrt{\frac{2\mu_M}{r}} \qquad \Longleftrightarrow \qquad \varepsilon = \frac{v^2}{2} - \frac{\mu_M}{r} < 0,$$

and something must have put it there: a Mars-orbit-insertion (MOI) burn, aerobraking, or three-body ballistic capture. The example below uses an MOI burn, and gives the heliocentric velocity **before** and **after** it at the *same* point, so that the same arithmetic yields a hyperbola in one case and an ellipse in the other.

2 Problem statement

A spacecraft approaches Mars on a slow, nearly velocity-matched trajectory. Use the same heliocentric frame as in the flyby problem: at the encounter epoch $+x$ points radially outward from the Sun through Mars, and $+y$ lies along the instantaneous direction of motion of Mars.

Symbol	Value	Meaning
$\mathbf{r}$	$\begin{bmatrix}-1.0 \times 10^5 & 0\end{bmatrix}^{\mathsf{T}}$ km	position of the spacecraft **relative to Mars** at the epoch
$\mathbf{V}_M$	$\begin{bmatrix}0 & 24.13\end{bmatrix}^{\mathsf{T}}$ km/s	heliocentric velocity of Mars
$\mathbf{V}_0$	$\begin{bmatrix}0.585 & 23.35\end{bmatrix}^{\mathsf{T}}$ km/s	heliocentric velocity of the spacecraft **before** the MOI burn
$\mathbf{V}_1$	$\begin{bmatrix}0.36 & 23.65\end{bmatrix}^{\mathsf{T}}$ km/s	heliocentric velocity of the spacecraft **after** the MOI burn
μ_M	4.2828×10^4 km^3/s^2	gravitational parameter of Mars
R_M	3396 km	radius of Mars
r_{SOI}	5.77×10^5 km	radius of Mars's sphere of influence

The burn is instantaneous and occurs at the position $\mathbf{r}$ given above, so the position is the same before and after.

Tasks.

1. Transform both heliocentric velocities into the Mars-centred frame and compare each Mars-relative speed with the local escape speed $v_{\mathrm{esc}}(r)$.
2. For each case compute the specific energy $\varepsilon = v^2/2 - \mu_M/r$ and the specific angular momentum $h = xv_y - yv_x$, and hence the eccentricity $e = \sqrt{1 + 2\varepsilon h^2/\mu_M^2}$.
3. Show that the pre-burn state is hyperbolic ($\varepsilon > 0$, $e > 1$) and the post-burn state is elliptical ($\varepsilon < 0$, $e < 1$): the spacecraft has been **captured**.
4. For the captured orbit find a, p, r_p, r_a, the periapsis altitude, the orbital period, and the true anomaly at the epoch.
5. Check that the capture orbit lies entirely inside the sphere of influence, so that the Mars-only two-body model is self-consistent.
6. Find the burn magnitude $|\Delta\mathbf{V}|$, and comment on what happened to the *heliocentric* speed.

3 Solution

3.1 Mars-relative velocities

Subtracting the velocity of Mars, $\mathbf{v} = \mathbf{V} - \mathbf{V}_M$, gives

$$\mathbf{v}_0 = \begin{bmatrix} 0.585 \\ 23.35 - 24.13 \end{bmatrix} = \begin{bmatrix} 0.585 \\ -0.780 \end{bmatrix} \text{ km/s}, \qquad \mathbf{v}_1 = \begin{bmatrix} 0.36 \\ 23.65 - 24.13 \end{bmatrix} = \begin{bmatrix} 0.36 \\ -0.48 \end{bmatrix} \text{ km/s}.$$

Both are multiples of the same unit vector $\hat{\mathbf{u}} = \begin{bmatrix} 0.6 & -0.8 \end{bmatrix}^\mathsf{T}$, so the burn is purely retrograde relative to Mars:

$$v_0 = |\mathbf{v}_0| = 0.975 \text{ km/s}, \qquad v_1 = |\mathbf{v}_1| = \sqrt{0.36^2 + 0.48^2} = \sqrt{0.36} = 0.600 \text{ km/s}.$$

Note that both heliocentric speeds, $|\mathbf{V}_0| = 23.357$ km/s and $|\mathbf{V}_1| = 23.653$ km/s, are *below* the Mars speed of 24.13 km/s: the spacecraft is being overtaken by Mars, which is exactly the slow-approach geometry that makes capture affordable.

3.2 The escape-speed test at $r = 10^5$ km

With $r = |\mathbf{r}| = 1.0 \times 10^5$ km,

$$v_{\text{esc}}(r) = \sqrt{\frac{2\mu_M}{r}} = \sqrt{\frac{2(42828)}{10^5}} = \sqrt{0.85656} = \boxed{0.9255 \text{ km/s}}.$$

Therefore $v_0 = 0.975 > v_{\text{esc}}$ (unbound) while $v_1 = 0.600 < v_{\text{esc}}$ (bound). Everything that follows is just this statement expressed through ε and e.

3.3 Specific energy

$$\varepsilon_0 = \frac{v_0^2}{2} - \frac{\mu_M}{r} = \frac{0.975^2}{2} - \frac{42828}{10^5} = 0.475313 - 0.428280 = \boxed{+0.047032 \text{ km}^2/\text{s}^2},$$

$$\varepsilon_1 = \frac{v_1^2}{2} - \frac{\mu_M}{r} = \frac{0.600^2}{2} - \frac{42828}{10^5} = 0.180000 - 0.428280 = \boxed{-0.248280 \text{ km}^2/\text{s}^2}.$$

The post-burn energy is **negative**: the kinetic term $v^2/2$ no longer covers the potential well depth μ_M/r at this radius. The corresponding semi-major axes are

$$a_0 = -\frac{\mu_M}{2\varepsilon_0} = -\frac{42828}{0.094065} = -455302 \text{ km} < 0 \quad \text{(hyperbola)},$$

$$a_1 = -\frac{\mu_M}{2\varepsilon_1} = +\frac{42828}{0.496560} = \boxed{86249 \text{ km} > 0} \quad \text{(ellipse)}.$$

3.4 Specific angular momentum

With $\mathbf{r} = \begin{bmatrix}-10^5 & 0\end{bmatrix}^\mathsf{T}$ and $h = xv_y - yv_x$,

$$h_0 = (-10^5)(-0.780)-0 = 7.8000\times10^4 \text{ km}^2/\text{s}, \qquad h_1 = (-10^5)(-0.480)-0 = \boxed{4.8000 \times 10^4 \text{ km}^2/\text{s}}.$$

Both are positive, so the motion is counter-clockwise in this frame; the retro burn removed angular momentum in the same proportion as speed, because it did not change the velocity direction.

3.5 Eccentricity

Using $e = \sqrt{1 + 2\varepsilon h^2/\mu_M^2}$ with $\mu_M^2 = 1.834238 \times 10^9$ km^6/s^4:

$$e_0 = \sqrt{1 + \frac{2(0.047032)(7.8 \times 10^4)^2}{1.834238 \times 10^9}} = \sqrt{1 + 0.312006} = \boxed{1.1454 > 1} \quad \text{(hyperbola)},$$

$$e_1 = \sqrt{1 - \frac{2(0.248280)(4.8 \times 10^4)^2}{1.834238 \times 10^9}} = \sqrt{1 - 0.623677} = \sqrt{0.376323} = \boxed{0.6134 < 1} \quad \text{(ellipse)}.$$

The sign of ε and the value of e carry the same information, since $2\varepsilon h^2/\mu_M^2 = e^2-1$: $\varepsilon < 0 \Leftrightarrow e < 1$. The pre-burn orbit would have escaped with $v_\infty = \sqrt{2\varepsilon_0} = 0.3067$ km/s; the post-burn orbit has no v_∞ at all.

3.6 Elements of the capture orbit

$$p = \frac{h_1^2}{\mu_M} = \frac{(4.8 \times 10^4)^2}{42828} = 53797 \text{ km},$$

$$r_p = a_1(1-e_1) = 86249(0.386594) = \boxed{33343 \text{ km}}, \qquad r_a = a_1(1+e_1) = 86249(1.613406) = \boxed{139155 \text{ km}},$$

$$h_p = r_p - R_M = 33343 - 3396 = 29947 \text{ km (periapsis altitude)}.$$

The periapsis and apoapsis speeds follow from $h = rv$ at the apsides:

$$v_p = \frac{h_1}{r_p} = \frac{48000}{33343} = 1.4396 \text{ km/s}, \qquad v_a = \frac{h_1}{r_a} = \frac{48000}{139155} = 0.3449 \text{ km/s}.$$

The period is

$$T = 2\pi\sqrt{\frac{a_1^3}{\mu_M}} = 2\pi\sqrt{\frac{(86249)^3}{42828}} = 7.690 \times 10^5 \text{ s} = \boxed{8.90 \text{ days}}.$$

The true anomaly at the epoch comes from the conic equation, $1 + e\cos f = p/r$:

$$\cos f = \frac{1}{e_1}\left(\frac{p}{r} - 1\right) = \frac{0.537966 - 1}{0.613406} = -0.75323 \quad\Longrightarrow\quad f = 138.87^\circ \text{ or } 221.13^\circ.$$

The radial velocity is $v_r = \hat{\mathbf{r}} \cdot \mathbf{v}_1 = (-1)(0.36) = -0.36$ km/s < 0, so the spacecraft is falling inward and the correct branch is $\boxed{f = 221.13^\circ}$, i.e. -138.87°: periapsis lies ahead, about 139° of true anomaly away.

3.7 Consistency check against the sphere of influence

$$\frac{r_a}{r_{\text{SOI}}} = \frac{139155}{5.77 \times 10^5} = 0.24.$$

The whole capture ellipse stays within a quarter of the sphere of influence, so treating Mars as the only gravitating body is self-consistent — this orbit really is closed and the spacecraft really is trapped.

The same check also exposes the point made at the start: a closed orbit with $r_a = 139155$ km **never reaches the SOI boundary**, so the spacecraft cannot have arrived on it from interplanetary space. The negative energy is itself the proof that a burn (or some non-two-body effect) occurred.

3.8 The burn, and the heliocentric bookkeeping

$$\Delta\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_0 = \begin{bmatrix} 0.36 - 0.585 \\ -0.48 + 0.78 \end{bmatrix} = \begin{bmatrix} -0.225 \\ 0.300 \end{bmatrix} \text{ km/s}, \qquad |\Delta\mathbf{V}| = 0.975 - 0.600 = \boxed{0.375 \text{ km/s}},$$

directed exactly opposite to $\mathbf{v}_0$. Note the frame-dependence, echoing the flyby lesson in reverse: this is a **braking** burn in the Mars frame ($0.975 \to 0.600$ km/s), yet in the heliocentric frame the spacecraft *speeds up*, from 23.357 km/s to 23.653 km/s, because the retro-burn direction relative to Mars happens to point partly along the heliocentric motion. Only the Mars-relative energy decides whether the spacecraft is captured; the heliocentric speed says nothing about it.

4 Summary

Quantity	Before MOI	After MOI
heliocentric $\mathbf{V}$ (km/s)	$\begin{bmatrix}0.585 & 23.35\end{bmatrix}^{\mathsf{T}}$	$\begin{bmatrix}0.36 & 23.65\end{bmatrix}^{\mathsf{T}}$
Mars-relative $\mathbf{v}$ (km/s)	$\begin{bmatrix}0.585 & -0.780\end{bmatrix}^{\mathsf{T}}$	$\begin{bmatrix}0.36 & -0.48\end{bmatrix}^{\mathsf{T}}$
speed v vs. $v_{\text{esc}} = 0.9255$ km/s	$0.975 > v_{\text{esc}}$	$0.600 < v_{\text{esc}}$
ε (km^2/s^2)	$+0.0470$	$\mathbf{-0.2483}$

Quantity	Before MOI	After MOI
h (km^2/s)	7.80×10^4	4.80×10^4
e	1.1454	**0.6134**
a (km)	-455302	$+86249$
conic	hyperbola (escapes, $v_\infty = 0.307$ km/s)	**ellipse — captured**
$r_p,\ r_a$ (km)	—	$33343,\ 139155$
period	—	8.90 days

5 Teaching notes

- **Capture is a statement about a finite radius.** Students who try to evaluate ε "at infinity" for the captured orbit will find no v_∞ exists; the orbit simply does not extend that far. The escape-speed comparison $v \lesseqgtr \sqrt{2\mu_M/r}$ is the most physical way to introduce this.
- **Velocity matching is what makes capture cheap.** Here $|\mathbf{V}_0 - \mathbf{V}_M| = 0.975$ km/s, so only 0.375 km/s of ΔV is needed. In the flyby problem $v_\infty = 3.00$ km/s, and capture at the same radius would demand a burn of $\sqrt{9 + 0.85656} - 0.6 \approx 2.54$ km/s.
- **The threshold at the SOI boundary.** To be even marginally bound at r_{SOI} requires $v < \sqrt{2\mu_M/r_{\text{SOI}}} = 0.385$ km/s — the heliocentric velocities of spacecraft and planet must agree to within about 0.4 km/s. This is the quantitative reason that unpowered capture is a three-body (weak-stability-boundary) phenomenon and not a patched-conic one.
- **A good follow-up exercise.** Ask for the burn that circularises the orbit at r_p: $v_{\text{circ}} = \sqrt{\mu_M/r_p} = 1.1334$ km/s against $v_p = 1.4396$ km/s, so a further 0.306 km/s at periapsis produces a circular 33343 km orbit — the second half of a standard two-burn insertion.