Lunar Trajectories: Phasing Loops and Low-Thrust Spirals

Waiting for the Moon in a loop, and climbing to it on a spiral: two ways to the Moon that trade time for something else

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Where we are going

The flight to the Moon of the patched-conics note (Example 2a) is **one burn and a coast**: at the perigee of a 300 km parking orbit the vehicle gains 3.106 km/s, and about three days later it reaches the Moon's sphere of influence. It has two demands. The burn must be made at the moment when the Moon is 123.07° ahead of the perigee, and the whole 3.106 km/s must be delivered in minutes. This note looks at what to do when one of the two demands cannot be met, and it gives a different answer for each.

- If the **timing** cannot be met, because the launch has happened and the Moon is somewhere else, the vehicle can wait. It waits on a highly eccentric Earth orbit, a **phasing loop**, flying it k times until the Moon is in the right place, and makes the trans-lunar burn at the perigee of the last loop. We shall find that this costs **no extra** Δv: it costs time. It is the technique of the missions whose launchers could not make a direct injection.
- If **short burns** cannot be made, because the engine is an ion thruster that pushes with about two ten-thousandths of a newton per kilogram, there are no burns at all. The vehicle thrusts for months and **spirals** outward. The Δv is *larger* than for the impulsive flight, but the propellant is smaller, because the exhaust speed is five to ten times higher. Here the price is again time, now measured in months. This part is also an introduction to low-thrust flight in general, which the course has met so far only in a few formulas.

> What you should be able to answer at the end
>
> By the end you should be able to answer: (1) Given where the Moon is when the vehicle is injected, how long must it wait for the trans-lunar burn, and what are the period and the apogee of k equal loops that fill the wait? (2) Why does the loop cost nothing in Δv, in the impulsive model, and what in a real mission does it cost? (3) Why does thrusting *along* the velocity make a circular orbit *slower*, and what is the Δv, the time and the number of revolutions of a spiral between two circles? (4) How circular does the spiral stay, and where does the Earth-only model stop? (5) How do I_{sp}, power and trip time trade against each other for an electric spacecraft?

Background. The patched-conics note (its Example 2a is the transfer that the loops deliver) and the Earth–Moon system of the lunar-trajectories note; the phasing condition on a single orbit from the manoeuvres note; the rocket equation; and, for the spiral, the energy of a circular orbit. The finite-burn side of loops ("why split a burn into several perigee burns") is

in the Chobotov note, and Edelbaum's plane-change formula for spirals is derived in its low-thrust section.

Data and conventions. Distances in km, speeds in km/s, times in hours or days. $\mu_\oplus = 398\,600.4418$ km^3/s^2, $\mu_{\rm Moon} = 4902.8$ km^3/s^2, $R_\oplus = 6378.137$ km, $R_{\rm Moon} = 1737.4$ km. The Moon moves on a circle of radius $r_{\rm Moon} = 384\,400$ km at $n_{\rm Moon} = \sqrt{(\mu_\oplus + \mu_{\rm Moon})/r_{\rm Moon}^3} = 2.6653\times10^{-6}$ rad/s, which is 13.194° per day and a period of 27.28 d (the value of every note of the course; the sidereal month, 27.32 d, is the real Moon's). Its sphere of influence has the radius $r_{\rm SOI} = 66\,183$ km. The parking orbit is 300 km high, $r_0 = 6678.137$ km. Everything is **planar**, in the plane of the Moon's orbit.

1 Why the timing matters

1.1 The Moon has to be at ϕ_0

Measure polar angles from the **perigee direction** of the trans-lunar ellipse (the direction from the Earth to the burn point), counter-clockwise, the way both the vehicle and the Moon move. The ellipse is the Hohmann transfer to the Moon's orbit, with $a_{\rm H} = (r_0 + r_{\rm Moon})/2 = 195\,539.1$ km, so the vehicle's apogee is at 180° and it takes $T_{\rm H}/2 = 119.52$ h to get there. During that time the Moon turns $n_{\rm Moon}T_{\rm H}/2 = 65.71°$, so for the Moon to meet the vehicle at the apogee it must be $180° - 65.71° = 114.29°$ ahead of the perigee at the burn. This is the *textbook timing*. To hit the Moon's sphere of influence at the point that gives a 100 km perilune, Example 2a adjusts it to

$$\phi_0 = 123.07°, \tag{1}$$

and we take that value from there; the SOI is entered 78.86 h after the burn, and the burn is $\Delta v_{\rm TLI} = v_{\rm TLI} - v_{c0} = 10.8322 - 7.7258 = 3.1064$ km/s. The angle ϕ_0 is *the* requirement of a lunar transfer: the Moon's angle $\phi(t)$ from the perigee direction must equal ϕ_0 at the moment of the burn. In a time t it grows as

$$\phi(t) = \phi_a + n_{\rm Moon}\, t, \tag{2}$$

where ϕ_a is its value at $t = 0$. It moves 13.19° per day: a burn made a day too early or too late finds the Moon 13° out, and the perilune moves by thousands of kilometres (Section 2.4).

1.2 Two dials: where and when

A trans-lunar burn has two things to get right: **where** in its orbit the vehicle makes it, and **when**. From a *circular* parking orbit the first is free: the burn point goes round the orbit in 90.5 minutes, in which the Moon moves only 0.83°, so the vehicle simply waits for the point that is ϕ_0 behind the Moon. The timing is then solved within one orbit, and no loop is needed. This is how a vehicle with a restartable upper stage and a parking orbit does it.

A loop is needed when **the place is fixed and the time is free**. This happens in four ways.

- The launcher injects the vehicle directly into an *elliptic* orbit, and the perigee point is wherever the ascent puts it. A parking orbit that does not exist cannot be waited in.
- The launch time of day is fixed by the *plane*: the plane that will contain the Moon's position at arrival has to be reached by an ascent from a rotating Earth, once a day (see the lunar note). The perigee direction in that plane then follows from the ascent.
- The date of the launch is fixed by other things (lighting on the Moon, the availability of the range), and not by the Moon's phase.
- The vehicle needs days in orbit, to be checked, commissioned or corrected, before it is committed to the Moon.

In each case the question is the same: *injected where and when it was, the Moon at the angle ϕ_a, how long must the vehicle wait, and in what kind of orbit?*

2 The phasing loop

2.1 The idea and its condition

Instead of the trans-lunar ellipse, the first burn, at the perigee, puts the vehicle on a smaller ellipse with the **same perigee** and a period T_{ph}: the phasing ellipse. After k revolutions the vehicle is back at the perigee, where the second burn finishes the job and puts it on the trans-lunar ellipse. The two burns are at the same point and, as we shall see, in the same direction.

symbol	meaning
ϕ_a, ϕ_0	angle of the Moon from the perigee direction at the first burn; the angle it must have at the trans-lunar burn (Equation 1)
n_{Moon}, T_{Moon}	the Moon's mean motion and period, 27.28 d

symbol	meaning
W, m, k	the wait between the two burns; whole months added to it; number of loops
T_{ph}, a_{ph}, e_{ph}, $r_{a,ph}$, v_{ph}	period, semi-major axis, eccentricity, apogee radius and perigee speed of the phasing ellipse
r_0, v_{c0}	the perigee radius, equal to the parking-orbit radius, and its circular speed
a_{H}, v_{TLI}	the trans-lunar (Hohmann) ellipse of Example 2a and its perigee speed
Δv_1, Δv_2	the first burn and the trans-lunar burn

The vehicle spends $W = k\,T_{ph}$ on the loops. The Moon must advance from ϕ_a to ϕ_0 in that time, and it may advance by a whole number of months on top, so by Equation 2

$$\boxed{W = \frac{\phi_0 - \phi_a + 2\pi m}{n_{\mathrm{Moon}}}, \qquad T_{ph} = \frac{W}{k}, \qquad m = 0, 1, \ldots} \tag{3}$$

The angle $\phi_0 - \phi_a$ is taken between 0 and 2π, the shortest wait; $m = 1$ adds 27.28 d, which nobody does unless forced to. The period fixes the size of the ellipse by Kepler's third law, and the perigee is r_0 whatever the period, so

$$a_{ph} = \left(\frac{\mu_\oplus T_{ph}^2}{4\pi^2} \right)^{1/3}, \qquad r_{a,ph} = 2a_{ph} - r_0, \qquad v_{ph} = \sqrt{\mu_\oplus \Big(\frac{2}{r_0} - \frac{1}{a_{ph}} \Big)}. \tag{4}$$

The loop must be longer than the parking orbit ($r_{a,ph} > r_0$) and it must stay clear of the Moon, wherever the Moon is. If the apogee is to stay at least $1.5\,r_{\mathrm{SOI}}$ inside the Moon's orbit, the largest apogee is

$$r_{a,\max} = r_{\mathrm{Moon}} - 1.5\,r_{\mathrm{SOI}} = 285\,126 \text{ km}, \tag{5}$$

which is a loop period of at most 6.42 d. Waits longer than that are shared among more loops.

2.2 The loop costs no Δv

At the perigee the vehicle's velocity is perpendicular to the radius, on the circle, on the phasing ellipse and on the trans-lunar ellipse alike. Both burns are therefore along the velocity, and each changes only the speed:

$$\Delta v_1 = v_{ph} - v_{c0}, \qquad \Delta v_2 = v_{\mathrm{TLI}} - v_{ph}, \qquad \Longrightarrow \qquad \boxed{\Delta v_1 + \Delta v_2 = v_{\mathrm{TLI}} - v_{c0} = 3.1064 \text{ km/s}} \tag{6}$$

whatever v_{ph}, that is whatever the period and the number of loops. The intermediate speed cancels. **In the impulsive model a phasing loop is free.** It splits the one burn into two, the first one large and the second small (the longer the loop, the smaller the second burn), and it pays in time and nothing else. The same argument covers any number of perigee burns, which is why a sequence of apogee-raising burns, as flown by several missions, costs the same Δv as the direct injection.

2.3 Worked example: three loops

Example 1. Three phasing loops before the trans-lunar burn.

Problem. A spacecraft is injected at the perigee of a phasing ellipse, 300 km above the Earth ($r_0 = 6678.137$ km), at the moment when the Moon is $\phi_a = 35°$ ahead of the perigee direction. It will make the trans-lunar burn of Example 2a, which needs the Moon $\phi_0 = 123.07°$ ahead, at the perigee of the third loop. Find (1) the waiting time and the period, (2) the phasing ellipse, (3) the two burns, (4) how close the vehicle comes to the Moon during the loops, (5) what the Moon and the Earth's oblateness do to the loops.

Data. r_0, $\mu_\oplus$, n_{Moon}, r_{SOI} as above; $v_{c0} = 7.7258$ km/s; the trans-lunar ellipse of Example 2a, $a_{\mathrm{H}} = 195\,539.1$ km, with $v_{\mathrm{TLI}} = 10.8322$ km/s at perigee; $\phi_0 = 123.07°$; $k = 3$.

Plan. Equation 3 gives the wait and so the period; Equation 4 the ellipse; Equation 6 the burns (and a check); then a numerical integration of the vehicle with the Earth and the Moon both pulling answers (4) and (5).

Step 1: the wait. The Moon must advance by $\phi_0 - \phi_a = 123.07° - 35° = 88.07°$, at 13.1943° per day:

$$W = \frac{88.07°}{13.1943°/\mathrm{d}} = 6.675 \text{ d} = 160.19 \text{ h}, \qquad T_{ph} = \frac{W}{3} = 2.2249 \text{ d} = 53.40 \text{ h}.$$

Step 2: the phasing ellipse. From Equation 4,

History: three Earth-orbit sequences. *LADEE* (NASA, 2013) was launched on 7 September 2013 by a Minotaur V that could not give the direct trans-lunar injection. It was put in a very elliptic Earth orbit, with an apogee of about 50 Earth radii (319 000 km, beyond the 285 000 km cap used below, which is a conservative choice of this planar model), and flew **three phasing loops** (one account counts three and a half) before lunar orbit insertion on 6 October

$$a_{ph} = \left(\frac{398\,600.4 \times (2.2249 \times 86\,400)^2}{4\pi^2} \right)^{1/3} = 71\,990.9 \text{ km},$$
$$r_{a,ph} = 2(71\,990.9) - 6678.1 = 137\,304 \text{ km},$$

an eccentricity $e_{ph} = (r_{a,ph} - r_0)/(r_{a,ph} + r_0) = 0.90724$ and a perigee speed $v_{ph} = \sqrt{398\,600.4\,(2/6678.137 - 1/71\,990.9)} = 10.6695$ km/s. The apogee, 137 000 km, is well below the cap of Equation 5.

Step 3: the burns.

$$\Delta v_1 = 10.6695 - 7.7258 = 2.9437 \text{ km/s}, \qquad \Delta v_2 = 10.8322 - 10.6695 = 0.1627 \text{ km/s},$$

and their sum is 3.1064 km/s, the direct injection of Example 2a, as Equation 6 says. After the second burn the vehicle is on the ellipse of Example 2a at the moment the Moon is at ϕ_0 (the Moon has turned $3\,n_{\text{Moon}} T_{ph} = 88.07°$, so $\phi = 123.07°$), and it reaches the sphere of influence 78.86 h later.

Step 4: the Moon during the loops. The loops are the same size in every revolution; the apogee, 137 300 km on the side opposite to the perigee, is the point nearest the Moon's orbit, and the Moon is at 35° to 123° while the apogee is at 180°. Integrating the planar motion with the Earth and the Moon, the smallest distance to the Moon's centre during the three loops is 351 724 km, 5.3 sphere-of-influence radii. For the longest loop of the table below, $k = 1$ with an apogee of 292 816 km, it is still 376 357 km.

Step 5: what the idealisation leaves out. In the same integration the Moon's pull raises the perigee: it returns at 6719.6, 6749.1 and 6729.6 km at the three passes (41, 71 and 51 km up, against 6678.137 km with the Earth alone), each return is late by 78, 76 and 40 s, and the perigee direction, in the planar model, moves by 0.008°, −0.011° and −0.040° at the three returns. The Earth's oblateness (see the gravity note) acts on the line of apsides *and* on the plane. For $p_{ph} = 12\,737$ km and $i = 28.5°$ the apse turns at $\dot{\omega} = \frac{3}{4} n_{ph} J_2 (R_\oplus / p_{ph})^2 (5\cos^2 i - 1) = +0.0943°$ per day, and the node regresses at $\dot{\Omega} = -\frac{3}{2} n_{ph} J_2 (R_\oplus / p_{ph})^2 \cos i = -0.0579°$ per day: over the wait, +0.63° and −0.39°. The two combine in the direction of the perigee, which is the burn point, and it turns *in space* by 0.29° to 0.34°, depending on where in the plane it lies. The Moon must make up the difference, so the wait is longer by 0.53 to 0.62 h (0.33% to 0.39%), and the condition of Equation 3 has to be corrected by that. The plane itself tilts: the orbit normal turns by 0.18° (about $|\sin i\,\Delta\Omega|$), which is 1240 km out of the plane of the design at the Moon's distance. The period correction is cheap: a prograde burn of 0.54 m/s at the perigee lengthens the loop by 10 minutes (the perigee is the

cheap place to change a period, as in the manoeuvres note); the out-of-plane correction is not, and is one of the reasons for keeping the loops few and short.

The loop count is a free choice. The table gives the answer for the same wait, $W = 6.675$ d, with other values of k.

Table 2: The same wait filled by k equal loops. The sum is the same for every row.

k	T_{ph} [d]	$r_{a,ph}$ [km]	Δv_1 [km/s]	Δv_2 [km/s]	sum [km/s]
1	6.675	292 816	3.0776	0.0288	3.1064
2	3.337	181 991	3.0050	0.1014	3.1064
3	2.225	137 304	2.9437	0.1627	3.1064
4	1.669	112 176	2.8887	0.2177	3.1064
5	1.335	95 747	2.8379	0.2685	3.1064

More loops make smaller ellipses, safer from the Moon and from its perturbation, and each passes through the radiation belts and the atmosphere's reach at the perigee once more; $k = 1$ is nearly at the Moon's orbit. The designer's choice is the middle.

The Moon is 60° **behind** the perigee direction at the first burn ($\phi_a = -60°$). How long is the shortest wait, and which values of k keep the apogee below the cap of Equation 5?

What you should find

$\phi_0 - \phi_a = 183.07°$, so $W = 183.07°/13.1943°/\text{d} = 13.875$ d. One loop would be 13.9 d long, with an apogee of 481 000 km, beyond the Moon's orbit; $k = 2$ ($T_{ph} = 6.94$ d) has $r_{a,ph} = 300\,600$ km, above the cap; $k = 3$ ($T_{ph} = 4.625$ d) has 227 800 km and $k = 4$ (3.47 d) has 186 900 km. $k = 3$ is the least number of loops that fits.

2.4 What the loops buy: every day is a launch day

Figure 1 shows the example to scale, and the clock on which it works: the Moon's angle $\phi(t)$ is a straight line in time, the vehicle comes back to the perigee at 0, T_{ph}, $2T_{ph}$ and $3T_{ph}$, and the trans-lunar burn is made at the one of those times at which the line is at ϕ_0.

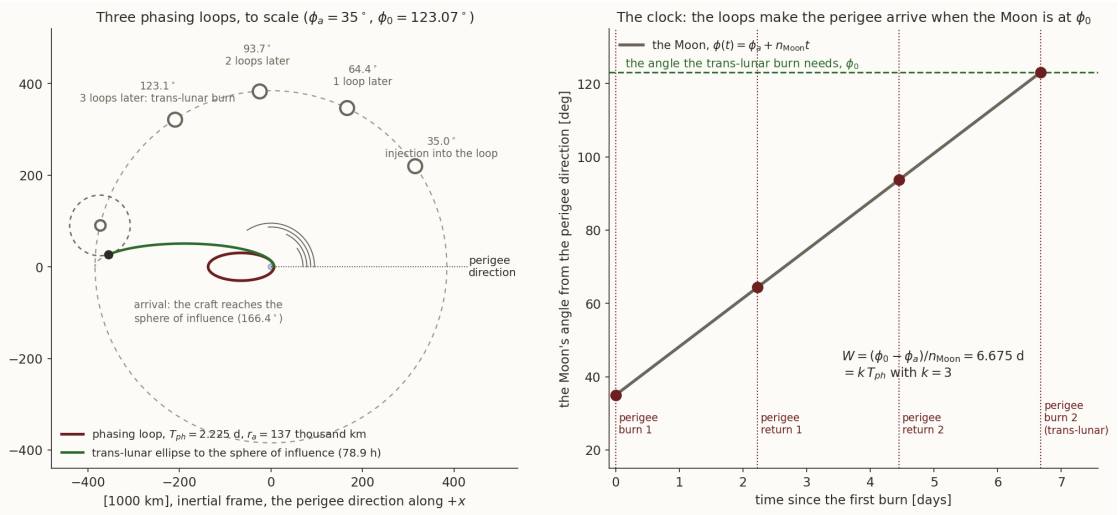


Figure 1: The three-loop example. **Left:** to scale, in the inertial frame with the perigee direction along $+x$: the phasing loop (maroon), the trans-lunar ellipse up to the sphere of influence (green), and the Moon (rings) at the four perigee passes and at the arrival. **Right:** the clock: the Moon's angle $\phi(t)$ from the perigee direction (Equation 2) against the time since the first burn. The loops are chosen so that the third return to the perigee is the moment when $\phi = \phi_0$. *Script: fig-lpl-phasing.py.*

How much launch-time freedom does this give? Without a loop, and with the burn point fixed, the burn has to be made when $\phi = \phi_0$ to within a fraction of a degree. With the patched-conic solution of Example 2a, the perilune altitude is 100 km at ϕ_0, 1094 km one degree later and an impact (-634 km) one degree earlier: 730 to 990 km per degree of ϕ_0, and a correction of that much is expensive. One degree of the Moon's motion is 1.82 h, so the Moon spends an hour or two in the usable range, once a month. With k loops of at most 6.42 d (Equation 5), any wait up to $6.42\,k$ days can be filled. Figure 2 shows that $k = 4$ covers waits up to 25.7 d and $k = 5$ covers 32.1 d, longer than a month: **five loops can serve any phase of the Moon.** The only cost of a longer wait is more orbits to fly (and, in a real mission, more time for things to go wrong).

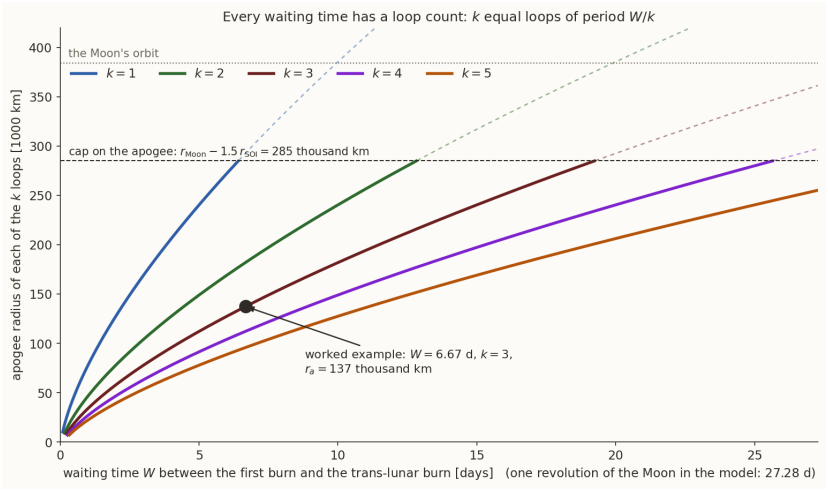


Figure 2: Waiting time against the apogee of each of k equal loops (period W/k, Equation 4). The solid parts are below the cap of Equation 5, 285 000 km. A given wait can be filled by several k; the example of the text is the dot. The curves for $k = 4$ and $k = 5$ reach the end of the month (27.28 d) below the cap. *Script: fig-lpl-phasing.py.*

Figure 3 shows the difference in motion. Both panels start at the same instant, with the Moon 35° ahead. On the left the whole injection is made at once: the vehicle reaches the apogee A at 119.5 h, and the Moon, which is 65.7° further on, is 79.3° short of it. On the right the vehicle flies three loops, and the second burn is small.

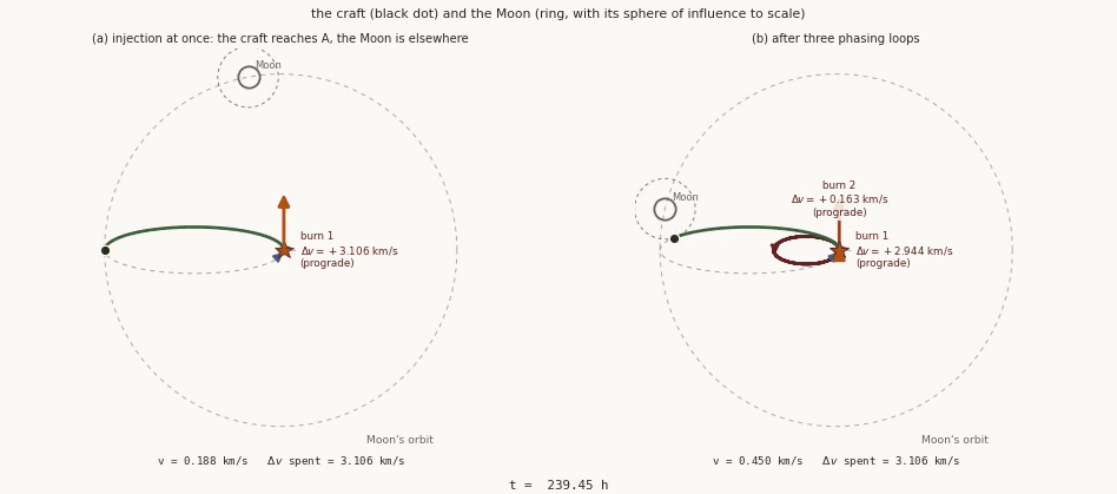


Figure 3: Direct injection against three phasing loops, on one clock (the craft is the black dot; the Moon is the ring, with its sphere of influence to scale). The Moon is 35° ahead of the perigee at the first burn and must be 123.07° ahead at the trans-lunar burn. (**a**) Injection at once: 3.106 km/s, arrival at A after 119.5 h, with the Moon 79.3° short of it. (**b**) Burn 1 (2.944 km/s), three loops of 2.225 d, burn 2 (0.163 km/s) and 78.86 h to the sphere of influence, which the craft reaches as the Moon does. The craft moves at its true speed, only slowed down, and the clock stops during a burn; the long loops are fast near the perigee, so the dot jumps there. *Script: anim-lpl.py, with the engine _manim.py.*

2.5 What it costs in a real mission

The impulsive model said "nothing". Four things say otherwise.

Finite burns. A real engine takes minutes, and a burn that is long compared with the time near the perigee loses Δv to gravity (the Chobotov note works this out). The cure is the same technique: **split the burn into several made at successive perigees**, which is what a phasing loop does. In the experiment of that note (the table in its section on several perigee burns), an engine of thrust-to-weight $T/W_0 = 0.03$ that raises a 278 km orbit to a geosynchronous apogee loses 499.8 m/s in one burn (1.9 h) and 21.8 m/s in eight (23.8 h). So for a stage of modest thrust-to-weight ratio the loops *save* Δv against one long burn, and that is a main reason why Earth-orbit sequences are flown at all: the real exchange is time for Δv, the opposite of the impulsive model's "time for nothing".

Perturbations. The Moon's pull changed the perigee by tens of kilometres in three loops (Step 5), the Earth's oblateness turns the line of apsides and tilts the plane (0.18°, 1240 km at the Moon, in the example), and the Sun does as much as the Moon at these apogees. The perigee is raised or lowered, and a lowered perigee can bring the vehicle into the atmosphere. The loops need tracking and small corrections, in the plane and out of it.

Exposure. Each perigee passage crosses the radiation belts, and a loop of 2.2

d has two crossings per period.

Moon's neighbourhood. The cap of Equation 5 keeps the apogee away from the Moon; a $k = 1$ loop of 6.7 d reaches 293 000 km and the geometry is no longer simple.

The answer to the first question of the note is therefore: *the wait is Equation 3, the loops are the shortest that keep the apogee under the cap, and the cost in Δv is zero on paper and, if anything, negative in practice.*

3 Low thrust: an introduction

3.1 What "low thrust" means

What matters about an engine is not its thrust F but the **acceleration** $a_T = F/m$ it gives, compared with gravity's, $g = \mu_\oplus / r^2$. Their ratio is

$$\kappa = \frac{a_T}{g} = \frac{a_T\, r^2}{\mu_\oplus}. \tag{7}$$

(The letter ε stays for the specific energy, as everywhere else in the course; κ is used for this ratio only.)

A chemical stage has κ between 0.03 and 10 at the start of a burn: it burns for minutes, and the burn is nearly an impulse when $\kappa \gtrsim 0.5$. For κ from 0.01 to 0.1 the burn is long enough to lose a lot to gravity, and one splits it into several, as the loops above do. An ion or Hall thruster has $\kappa \approx 10^{-5}$ to 10^{-3}: it can only be on for months, at an acceleration that moves the vehicle by a negligible amount in one orbit, and the orbit changes slowly and continuously.

The reason for building such engines is the exhaust speed, $c = g_0 I_{sp}$, $g_0 = 9.80665 \times 10^{-3}$ km/s^2. A chemical stage has $I_{sp} \approx 300$ s, $c \approx 3$ km/s; an electric thruster has 1500 to 3000 s, $c \approx 15$ to 30 km/s, and by the rocket equation the propellant for a given Δv is

$$\frac{m_p}{m_0} = 1 - e^{-\Delta v / c} \tag{8}$$

so 10 times the exhaust speed needs far less than a tenth of the propellant when the Δv is of order c. The price is that the thrust is small, for reasons of power (Section 3.5), and the transfer takes a long time.

History: SMART-1. ESA's SMART-1 was launched in September 2003 and used a Snecma PPS-1350-G Hall thruster, of

3.2 The spiral and its speed

Suppose a vehicle on a circular orbit of radius r thrusts along its velocity with the acceleration a_T, small enough that the orbit stays almost circular (this will be shown in Section 3.3). The energy of the orbit is $\varepsilon = -\mu_\oplus/(2r)$, and it changes at the rate at which the thrust does work,

$$\frac{d\varepsilon}{dt} = \mathbf{a}_d \cdot \mathbf{v} = a_T v. \tag{9}$$

On a circular orbit $v^2 = \mu_\oplus/r$, so $\varepsilon = -v^2/2$ and $d\varepsilon/dt = -v\,\dot{v}$. Setting this equal to $a_T v$,

$$\boxed{\frac{dv_c}{dt} = -a_T} \tag{10}$$

Thrusting forward makes the circular speed fall at exactly the rate of the thrust acceleration. The paradox is only apparent: the thrust does positive work, so the *total* energy rises, but in a circular orbit the kinetic energy is $-\varepsilon$, so a rise of the total energy by $\delta\varepsilon$ means a *fall* of the kinetic energy by $\delta\varepsilon$ and a rise of the potential energy by twice that. The vehicle slows down because it climbs.

The consequences are immediate. The speed falls linearly in time and the radius follows from the circular-speed relation $r = \mu_\oplus/v_c^2$:

$$v_c(t) = v_{c1} - a_T t, \qquad r(t) = \frac{\mu_\oplus}{(v_{c1} - a_T t)^2}. \tag{11}$$

A spiral from a circle of radius r_1 to a circle of radius $r_2 > r_1$ therefore costs

$$\boxed{\Delta v = v_{c1} - v_{c2}}, \qquad t = \frac{\Delta v}{a_T} \quad (\text{constant } a_T). \tag{12}$$

A spiral between coplanar circles costs the difference of the circular speeds, not the sum of two impulses. For the number of revolutions, each takes $2\pi r/v_c$, so the revolution rate is $v_c/(2\pi r) = v_c^3/(2\pi\mu_\oplus)$ (using $r = \mu_\oplus/v_c^2$), and with $dt = -dv_c/a_T$

$$N = \int_0^t \frac{v_c^3}{2\pi\mu_\oplus}\,dt = \frac{1}{2\pi\mu_\oplus a_T}\int_{v_{c2}}^{v_{c1}} v_c^3\,dv_c = \frac{v_{c1}^4 - v_{c2}^4}{8\pi\mu_\oplus a_T} = \frac{1}{8\pi\kappa_1}\Big[1 - \Big(\frac{r_1}{r_2}\Big)^2\Big], \tag{13}$$

with $\kappa_1 = a_T r_1^2/\mu_\oplus$ from Equation 7. The shape of the spiral depends on κ_1 alone: Figure 4 draws one with $\kappa_1 = 0.003$ (about 130 times SMART-1's), where 12.9 turns take the radius from r_1 to $6r_1$, and shows the speed on its straight line.

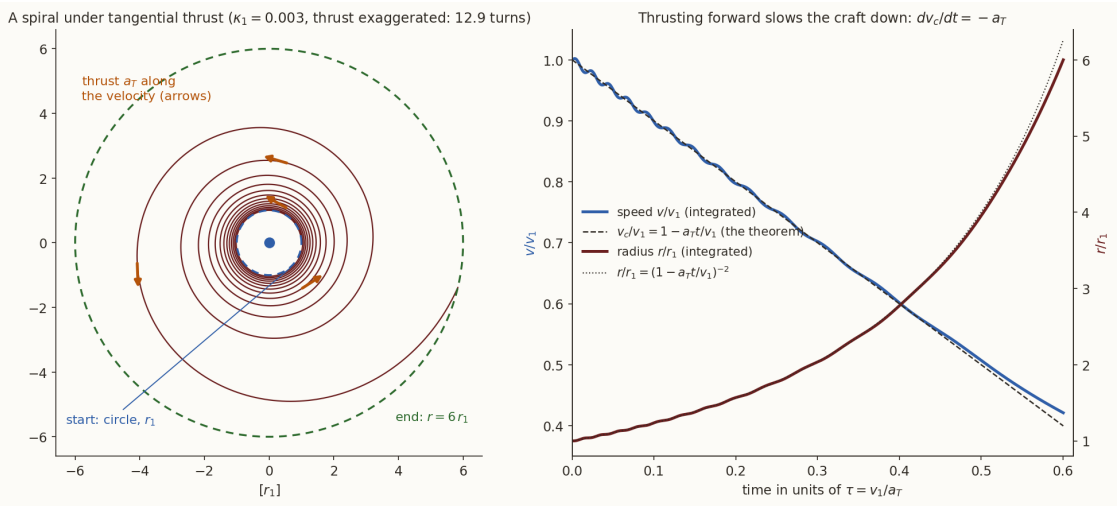


Figure 4: **Left:** a spiral under tangential thrust, integrated numerically with $\kappa_1 = a_T r_1^2/\mu_\oplus = 0.003$; the thrust is **exaggerated** (a real one has 10^{-5} and a thousand turns, which cannot be drawn). **Right:** the speed (blue, left axis) falls on the straight line $v_c = v_{c1} - a_T t$ (dashed) and the radius (maroon, right axis) rises as $(1 - a_T t/v_{c1})^{-2}$ (dotted), Equation 11; the small waves are the departure from a circle. *Script: fig-lpl-lowthrust.py*.

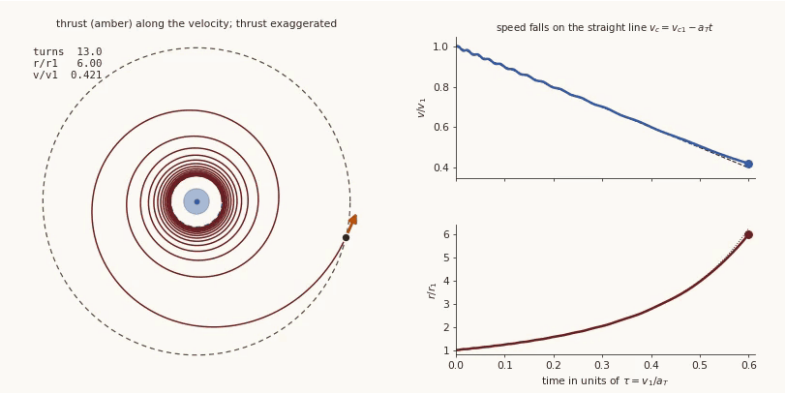


Figure 5: The same spiral as a film: the amber arrow is the thrust, along the velocity; the counter gives the turns, the radius and the speed. The thrust is **exaggerated** ($\kappa_1 = 0.003$), so that twelve turns replace seventeen hundred. *Script: anim-lpl-spiral.py*.

Against the Hohmann transfer. The same two circles, joined impulsively by

a Hohmann transfer, cost $\Delta v_H = \Delta v_1 + \Delta v_2$, and for LEO to the Moon's orbit that is 3.9365 km/s against the spiral's $v_{c1} - v_{c2} = 7.7258 - 1.0183 = 6.7075$ km/s: 70% more.

The reason is the speed at which the energy is bought. Every unit of Δv delivered at the speed v raises the energy by v per unit mass, $d\varepsilon = v\, d(\Delta v)$ (Equation 9), so the Δv for a given energy gain is $\int d\varepsilon / v$, and the higher the speed at the burn the cheaper it is: the Oberth effect of the manoeuvres note. From LEO to a circle at 100 000 km the energy gain is $\frac{1}{2}\mu_\oplus(1/r_1 - 1/r_2) = 27.85$ km^2/s^2 for either route. An impulse along the velocity buys its energy at the *mean* of the speeds before and after it, $\Delta\varepsilon = v\,\Delta v + \frac{1}{2}\Delta v^2$: the first Hohmann burn, from 7.73 to 10.58 km/s, at 9.15 km/s, and the second, from 0.71 to 2.00 km/s, at 1.35 km/s. Together they buy the 27.85 km^2/s^2 for 4.1427 km/s, which is 6.72 km/s of energy for each km/s of Δv. The spiral can only push at the circular speed of the orbit it is on, falling from 7.73 to 2.00 km/s, an average of 4.86 km/s (27.85/5.7293), and the same energy then costs 5.7293 km/s. The spiral pays for having no good place to burn.

3.3 How circular does the spiral stay?

The derivation assumed a circular orbit throughout. The orbit is perturbed from a circle at every instant, and we have to show by how much. The eccentricity vector is $\mathbf{e} = (\mathbf{v}\times\mathbf{h})/\mu_\oplus - \hat{\mathbf{r}}$, with $\mathbf{h} = \mathbf{r}\times\mathbf{v}$, and with an extra acceleration $\mathbf{a}_d$ the equations of motion are $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = -\mu_\oplus\mathbf{r}/r^3 + \mathbf{a}_d$, so $\dot{\mathbf{h}} = \mathbf{r}\times\mathbf{a}_d$. Differentiating, the two-body terms cancel (this is why $\mathbf{e}$ is constant without $\mathbf{a}_d$) and what is left is

$$\dot{\mathbf{e}} = \frac{1}{\mu_\oplus}\Big[\mathbf{a}_d\times\mathbf{h} + \mathbf{v}\times(\mathbf{r}\times\mathbf{a}_d)\Big] = \frac{1}{\mu_\oplus}\Big[2(\mathbf{v}\cdot\mathbf{a}_d)\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{a}_d)\,\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{a}_d\Big], \tag{14}$$

by the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ applied to both terms. For a tangential acceleration, $\mathbf{a}_d = a_T\hat{\mathbf{v}}$, on a circle ($\mathbf{r}\cdot\mathbf{v} = 0$, $\mathbf{r}\cdot\mathbf{a}_d = 0$, $\mathbf{v}\cdot\mathbf{a}_d = va_T$, $rv^2 = \mu_\oplus$):

$$\dot{\mathbf{e}} = \frac{2va_T\,r}{\mu_\oplus}\,\hat{\mathbf{r}} = \frac{2a_T}{v}\,\hat{\mathbf{r}}. \tag{15}$$

(Its component along the periapsis direction, $2a_T\cos f/v$, is the circular limit of the scalar equation $\dot{e} = (2/v)(e + \cos f)a_T$ of the Gauss equations in tangential–normal form.) The vector $\hat{\mathbf{r}} = (\cos\theta, \sin\theta)$ turns with the vehicle, θ being its polar angle, at the mean motion $n = v/r$, so $\dot{\mathbf{e}}\,dt = (2a_T/vn)\,\hat{\mathbf{r}}\,d\theta$ with $d\theta = n\,dt$, and $2a_T/(vn) = 2a_Tr^2/\mu_\oplus = 2\kappa$, the ratio of Equation 7. Over a short stretch of the spiral κ is constant, and integrating from $\mathbf{e} = 0$ at $\theta = 0$,

$$\mathbf{e} = 2\kappa\,(\sin\theta,\ 1-\cos\theta), \qquad |\mathbf{e}| = 4\kappa\left|\sin\frac{\theta}{2}\right|. \tag{16}$$

For constant κ the eccentricity vector goes round a circle of radius 2κ that touches the origin, oscillating between 0 and 4κ in every revolution. But along the spiral κ **grows as** r^2, and the circle does not simply grow with it. Write $\hat{\mathbf{r}} = d\hat{\mathbf{s}}/d\theta$ with $\hat{\mathbf{s}} = (\sin\theta, -\cos\theta)$ (the vector $\hat{\mathbf{r}}$ turned 90° backwards), and integrate by parts:

$$\mathbf{e} = \int_0^\theta 2\kappa\,\hat{\mathbf{r}}\,d\theta = 2\kappa(r)\,\hat{\mathbf{s}} + 2\kappa_1\,\hat{\mathbf{j}} + O(\kappa^2), \qquad \hat{\mathbf{j}} = (0,1), \tag{17}$$

where κ_1 is the initial value, and the integral that is left, $-\int 2\,(d\kappa/d\theta)\,\hat{\mathbf{s}}\,d\theta$, is of order κ^2 because $d\kappa/d\theta = 4\kappa^2$ (the radius grows by $4\pi\kappa$ of itself in a revolution, and $\kappa \propto r^2$). Two terms remain. The **forced** one, $2\kappa(r)\,\hat{\mathbf{s}}$, has the length 2κ *at the present radius* and keeps 90° behind the vehicle. The **free** one, $2\kappa_1\hat{\mathbf{j}}$, is a constant: the remnant of the abrupt start, which cancels the forced term at $\theta = 0$ where $\mathbf{e} = 0$, and which is small, $2\kappa_1 = 4.7\times10^{-5}$ in the example below. So

$$\big|\,|\mathbf{e}| - 2\kappa(r)\,\big| \le 2\kappa_1, \qquad |\mathbf{e}| \le 4\kappa(r),$$

the second because $\kappa(r) \ge \kappa_1$, with equality only at the start. After the first revolutions, when $\kappa(r) \gg \kappa_1$, $e \approx 2\kappa(r)$. (If the initial velocity is adjusted to cancel the remnant, which takes a small radial velocity, $|\mathbf{e}| = 2\kappa(r)$ from the start.)

What the steady eccentricity is. The forced vector points 90° behind the vehicle, so the periapsis of the osculating orbit is 90° behind it, and the vehicle is at the true anomaly $f = 90°$. On an orbit of small eccentricity the radial speed there is $v_r = (\mu_\oplus/h)\,e\sin f \approx e\,v$. With $e = 2\kappa$,

$$v_r = 2\kappa\,v = \frac{2a_T r^2}{\mu_\oplus}\sqrt{\frac{\mu_\oplus}{r}} = \frac{2a_T\,r^{3/2}}{\sqrt{\mu_\oplus}},$$

which is the climb rate $\dot r$ that the energy equation gives (Exercise 4). The eccentricity 2κ is not a distortion of the path. The path is very nearly a circle at every instant (it climbs $4\pi\kappa r$ in a revolution); the osculating ellipse is slightly eccentric only because the velocity is not quite perpendicular to the radius. The eccentricity *is* the flight-path angle, $\gamma = v_r/v = 2\kappa$. The orbit is almost circular when $\kappa \ll 1$.

A numerical integration at a SMART-1-class acceleration, $a_T = 2.08\times10^{-7}$ km/s^2 ($\kappa_1 = 2.33\times10^{-5}$ at the 300 km orbit), from $r_1 = 6678$ km to 100 000

km, confirms all of it (Figure 6). The integrated time is 318.81 d against 318.80 d from Equation 12 (0.003%), and the number of revolutions is 1702.10 against 1702.09 from Equation 13. The osculating eccentricity stays within $2\kappa_1$ of $2\kappa(r)$ at every sample (the ratio $|e - 2\kappa|/2\kappa_1$ reaches 1.000 at the start and is at most 0.993 after the first revolutions), never exceeds $4\kappa(r)$ (the largest ratio is 0.994), and ends at 0.01042 against $2\kappa = 0.01044$; the radial speed is $2\kappa v$ to within $2\kappa_1 v$. For constant κ ($\kappa_1 = 10^{-4}$, two revolutions) the vector itself follows Equation 16 to 1.3×10^{-6}, the growth of κ, and Equation 17 follows the vector for growing κ to 2.6×10^{-7}.

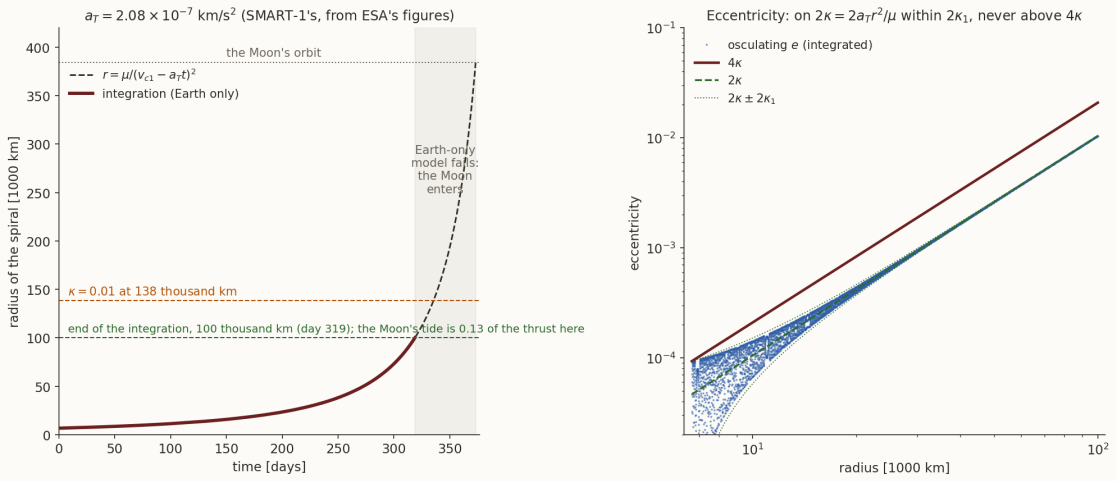


Figure 6: A thrust of 2.08×10^{-7} km/s², the average of SMART-1's from ESA's figures, from the 300 km orbit. **Left:** the integrated radius against time (maroon) and the analytic $r = \mu_\oplus/(v_{c1} - a_T t)^2$ (dashed), continued to the Moon's orbit. The integration stops at 100 000 km (day 319). **Right:** the osculating eccentricity along the spiral (dots): it oscillates between 0 and $4\kappa_1$ in the first revolutions, then settles on $2\kappa(r)$ (green), within $2\kappa_1$ of it throughout (Equation 17); 4κ (maroon) is the bound. *Script: fig-lpl-lowthrust.py, the integration by _lunarlt.py.*

3.4 Where the Earth-only model stops

Two things limit the spiral before the vehicle reaches the Moon, and they are different.

The orbit stops being circular. The acceleration a_T is constant and gravity falls as $1/r^2$, so $\kappa = a_T r^2/\mu_\oplus$ **grows as** r^2: for $a_T = 2.08 \times 10^{-7}$ km/s² it is 2.3×10^{-5} at LEO, 0.0052 at 100 000 km, 0.01 at 138 000 km and 0.077 at the Moon's distance. Since $e \approx 2\kappa$ is the flight-path angle of the climb, the spiral there climbs at $2\kappa = 0.154$ rad, 8.8° above the horizontal, and the orbit is no longer close to a circle: the model fails of its own accord, before the Moon is reached.

The Moon pulls. The Moon's differential acceleration at the vehicle, on the Moon's side, $\mu_{\text{Moon}}[(r_{\text{Moon}} - r)^{-2} - r_{\text{Moon}}^{-2}]$, is 13% of the thrust acceleration

at 100 000 km, 53% at 200 000 km and equal to it at 241 800 km (the lunar *tide* of the patched-conics note; see also the three-body note). Close to the Moon its direct pull equals the thrust within 153 500 km, 2.3 sphere-of-influence radii.

So the analytic spiral is trustworthy to about 100 000 km. That is 85% of the Δv and of the time of the whole spiral to the Moon's distance (5.73 of 6.71 km/s). Beyond it the vehicle is in a three-body problem and the end of the journey needs an integration (Section 3.7).

3.5 Propellant, thrust and power

Two kinds of time formula are in use, and they mean different things. With a **constant thrust acceleration** a_T, Equation 12 gives $t = \Delta v/a_T$. A real engine has a constant **thrust** F, and the mass falls as the propellant is used, so the acceleration rises. For a constant exhaust speed $c = g_0 I_{sp}$ the mass flow is $\dot m = F/c$, $m(t) = m_0 - Ft/c$, and $\Delta v = \int F\,dt/m = c\ln(m_0/m)$ (the rocket equation, Equation 8), which gives the time

$$t = \frac{c\,m_0}{F}\left(1 - e^{-\Delta v/c}\right). \tag{18}$$

This is shorter than $\Delta v/a_{T,0}$, with $a_{T,0} = F/m_0$ the acceleration at the start, because the acceleration grows. The revolution count of Equation 13 assumed constant a_T as well; with falling mass one integrates $v_c^3/(2\pi\mu_\oplus)$ over the time with $v_c = v_{c1} - c\ln(m_0/m)$.

The thrust is limited by **power**. A thruster that turns an electrical power P into a jet of mass flow $\dot m$ and speed c with an efficiency η has $\eta P = \frac{1}{2}\dot m c^2 = \frac{1}{2}Fc$, so

$$F = \frac{2\eta P}{c} = \frac{2\eta P}{g_0 I_{sp}}. \tag{19}$$

For a given power, **the thrust falls as** $1/I_{sp}$: doubling the exhaust speed halves the thrust. With Equation 18 the time is $t = c^2 m_0(1 - e^{-\Delta v/c})/(2\eta P)$, which rises with c (for $c \gg \Delta v$ it is $c\,m_0\Delta v/(2\eta P)$, proportional to I_{sp}), while the propellant falls. Figure 7 is the trade. It has no optimum, because the model has no mass for the power system. A real design includes the specific mass of the solar arrays (kg per kW), which grows with the power, and a minimum of the total mass or of the trip time appears at a finite I_{sp}; the figure's trend is still the guide.

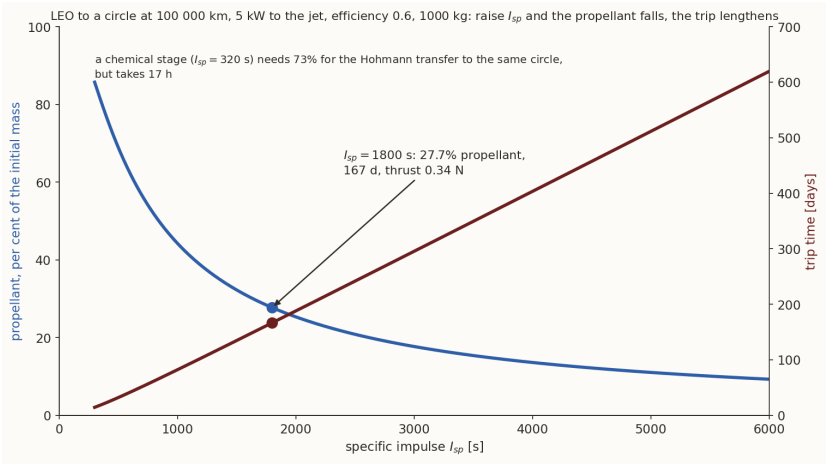


Figure 7: A 1000 kg spacecraft with 5 kW to the jet and efficiency 0.6, spiralling from LEO to a circle at 100 000 km ($\Delta v = 5.73$ km/s), against its specific impulse, with $F = 2\eta P/(g_0 I_{sp})$ and the time of Equation 18. Blue: the propellant as a fraction of the initial mass (left axis). Maroon: the trip time (right axis). The dots are the example of the text. *Script: fig-lpl-lowthrust.py.*

3.6 Worked example: an electric tug

Example 2. LEO to 100 000 km with 5 kW.

Problem. A spacecraft of wet mass $m_0 = 1000$ kg in a 300 km circular orbit has an electric thruster with $P = 5$ kW, an efficiency $\eta = 0.6$ and $I_{sp} = 1800$ s. It spirals to a circle of radius 100 000 km, where the Earth-only model stops. Find (1) the Δv, (2) the thrust and the initial acceleration, (3) the propellant and the time, (4) the comparison with a chemical Hohmann transfer to the same circle with $I_{sp} = 320$ s.

Data. $r_1 = 6678.137$ km, $r_2 = 100\,000$ km, $\mu_\oplus = 398\,600.4$ km^3/s^2, $g_0 = 9.80665 \times 10^{-3}$ km/s^2; $v_{c1} = 7.7258$ km/s, $v_{c2} = \sqrt{\mu_\oplus/r_2} = 1.9965$ km/s.

Plan. Equation 12 for the Δv; Equation 19 for the thrust; Equation 8 and Equation 18 for the propellant and the time; the Hohmann transfer for the comparison.

Step 1: the Δv. $\Delta v = v_{c1} - v_{c2} = 7.7258 - 1.9965 = 5.7293$ km/s.

Step 2: thrust and acceleration. $c = g_0 I_{sp} = 9.80665 \times 10^{-3} \times 1800 = 17.652$ km/s, $F = 2\eta P/c = 2(0.6)(5000)/17\,652$ m/s $= 0.3399$ N, and $a_{T,0} = F/m_0 = 3.399 \times 10^{-4}$ m/s^2 $= 3.40 \times 10^{-7}$ km/s^2, $\kappa_1 = 3.8 \times 10^{-5}$.

Step 3: propellant and time. Equation 8: $m_p/m_0 = 1 - e^{-5.7293/17.652} = 1 - e^{-0.3246} = 0.2772$, so 277 kg of propellant and 723 kg left. Equation 18:

$$t = \frac{c\,m_0}{F}(1 - e^{-\Delta v/c}) = \frac{17\,652 \times 1000}{0.3399} \times 0.2772 = 1.439 \times 10^7 \text{ s} = 166.6 \text{ d}.$$

(With the acceleration frozen at its initial value the time would be $5.7293/3.399 \times 10^{-7} = 195.1$ d: the rising acceleration saves 29 days.)

Step 4: the chemical stage. The Hohmann transfer from 6678.137 km to 100 000 km has $a_t = 53\,339$ km, $\Delta v_1 = 2.8526$ km/s, $\Delta v_2 = 1.2901$ km/s, a total of 4.1427 km/s, in $\pi\sqrt{a_t^3/\mu_\oplus} = 17.03$ h. At $I_{sp} = 320$ s the propellant is $1 - e^{-4.1427/3.1381} = 73.3\%$ of the initial mass.

Table 3: Two ways from LEO to the same 100 000 km circle. The spiral needs 38% more Δv, 2.6 times less propellant (277 kg against 733 kg of 1000 kg), and 235 times the time.

	Δv [km/s]	I_{sp} [s]	propellant	time
Hohmann, chemical	4.1427	320	73.3%	17.0 h
spiral, electric	5.7293	1800	27.7%	166.6 d

What this shows. Δv is no longer the figure of merit: the spiral has the larger one and the smaller propellant, because the exhaust speed is 5.6 times higher. The saving of propellant is bought with *time*, and with the radiation of the belts the vehicle spirals through for months.

Take the engine of Example 2 at $I_{sp} = 3000$ s, with the same 5 kW and $\eta = 0.6$. What are the thrust, the propellant and the time?

What you should find

$F = 2(0.6)(5000)/29\,420 = 0.204$ N, $m_p/m_0 = 1 - e^{-5.7293/29.420} = 17.7\%$, and $t = 295$ d. Against the 1800 s engine: 10 kg less propellant out of every 100 kg, and 128 days more.

3.7 Reaching the Moon

An all-electric journey to the Moon has three parts: the Earth-only spiral, which we can calculate, up to about 100 000 km; the part in which the Moon interferes, which needs a three-body integration and, as SMART-1 showed, lunar resonances and ballistic capture; and the spiral down around the Moon, to which the same theory applies with μ_{Moon} in place of $\mu_\oplus$. For the last, the thrust is *against* the velocity, the speed rises on the way down, and the cost is $\Delta v = v_{c,\text{low}} - v_{c,\text{high}}$: for a 100 km circular lunar orbit, $v_c = 1.6335$ km/s, from the edge of the sphere of influence (0.2722 km/s) it is 1.3613 km/s, and

from 15 350 km, where $\kappa = 0.01$ for this thrust, 1.0684 km/s, in 59 days at 2.08×10^{-7} km/s^2.

The table compares three ways of reaching a 100 km circular lunar orbit. The numbers are **not like for like unless read with their labels**: each row says what its Δv buys.

Table 4: Three routes to a 100 km lunar orbit. The last row extends the Earth-only spiral beyond its range, ignores the Moon's gravity until the hand-over at the sphere of influence, and assumes $a_T = 2.08 \times 10^{-7}$ km/s^2 throughout: it is an order of magnitude, not a design.

route	what the Δv buys	Δv [km/s]	I_{sp} [s]	propellant	time
direct (Example 2a)	trans-lunar burn 3.1064 and capture 0.7949, all impulsive	3.9013	320	71.2%	4.1 d
phasing loops (Example 1)	the same burns, split in two at the perigee	3.9013	320	71.2%	6.7 + 4.1 d
all-electric, **idealised**	Earth spiral to r_{Moon} (6.7075) and spiral down from the sphere of influence (1.3613)	8.0688	1800	36.7%	449 d

The phasing loop does not change the propellant of the direct flight but changes when it flies; the electric route halves the propellant and takes over a year. The real SMART-1, which began in a transfer orbit and used the Moon's gravity to end its climb, is reported with a total Δv of 3.5 to 3.9 km/s (the figures differ between sources), about that of the chemical route.

Plane change. The Moon's orbital plane is inclined to the equator by 18.3° to 28.6°, so a spiral from a LEO of a different inclination also has to turn its plane. For low-thrust spirals Edelbaum's formula, derived in the Chobotov note, replaces Equation 12 by $\Delta v = \sqrt{v_{c1}^2 + v_{c2}^2 - 2v_{c1}v_{c2}\cos(\pi\Delta i/2)}$. For the spiral of Example 2, $\Delta i = 10°$ makes the Δv 5.829 km/s instead of 5.729 (an increase of 1.7%), and 28.5° makes it 6.463 km/s (13% more).

Summary

In one paragraph. A lunar transfer needs the Moon at the angle ϕ_0 from the perigee at the moment of the trans-lunar burn. When the vehicle is injected with the Moon at ϕ_a, it may wait in k equal **phasing loops** of period $T_{ph} = (\phi_0 - \phi_a)/(k\, n_{\text{Moon}})$, with the same perigee, and make the trans-lunar burn at the last perigee. Both burns are tangential at that perigee, so in the impulsive model they add up to the direct injection, 3.1064 km/s: **the loop costs time, not** Δv. Five loops under an apogee cap serve every phase of the Moon. In a real mission the loops reduce the gravity loss of a weak engine, and they expose the vehicle to perturbations and to the radiation belts at each perigee. A **low-thrust spiral** between circular orbits obeys $dv_c/dt = -a_T$: the circular speed falls at the thrust acceleration, so $\Delta v = v_{c1} - v_{c2}$, $t = \Delta v/a_T$, and the revolutions are $(v_{c1}^4 - v_{c2}^4)/(8\pi\mu_\oplus a_T)$. The osculating eccentricity settles on $e \approx 2\kappa$ (never above 4κ), $\kappa = a_T r^2/\mu_\oplus$: it is the flight-path angle of the climb, v_r/v, not a distortion of the path; κ grows as r^2, and the Earth-only model holds to about 100 000 km for an acceleration like SMART-1's (85% of the Δv to the Moon). The spiral costs more Δv than the impulsive transfer (38% more to 100 000 km) and much less propellant, because I_{sp} is about six times higher; the price is a trip of months, and for a fixed power the trade between propellant and time has no end.

The numbers of this note are checked by lunar-phasing-lowthrust-verify.py, with the helper module _lunarlt.py; the transfer of Example 2a is reproduced there from _pc_transfer.py. The figure scripts are fig-lpl-phasing.py and fig-lpl-lowthrust.py, and the animations are anim-lpl.py (on the engine _manim.py) and anim-lpl-spiral.py.

What we left out. (i) *The three-dimensional geometry*: the plane of the loops, the Moon's declination and the launch azimuth, which are in the lunar note. (ii) *Finite-burn design* of the loops (gravity loss as a function of the thrust-to-weight ratio): the Chobotov note. (iii) *Low-energy transfers* that arrive at the Moon with a tiny Δv in months, through the Sun's gravity (three-body note). (iv) *Optimal low-thrust control*: the thrust here was along the velocity, with constant magnitude; the general problem, with eclipses and with thrust and coast arcs, is a trajectory optimisation. (v) *Shadow and power*: the thruster cannot fire in the Earth's shadow, and the arrays degrade in the radiation

belts.

Further reading. Chobotov (Chobotov 2002), chapter 6, for the pseudo-velocity derivation of Edelbaum's formula, and Edelbaum's own papers (Edelbaum 1961, 1967); the patched-conics note for the transfer the loops deliver.

Exercises

The numerical answers are checked in lunar-phasing-lowthrust-verify.py; those of exercise 7 are the table of the Chobotov note, checked in complications-verify.py.

1. **The wait.** The Moon is 60° ahead of the perigee at the first burn. How long is the shortest wait, and what are the period and the apogee for $k = 2$ and $k = 3$? Which satisfies the cap of Equation 5? (Answer: $W = 63.07°/13.1943°/\mathrm{d} = 4.78$ d. For $k = 2$: $T_{ph} = 2.39$ d, $r_{a,ph} = 144\,300$ km; for $k = 3$: 1.593 d, 108 600 km. Both are below the cap; $k = 1$ (4.78 d, 233 000 km) is too.)
2. **A free sum.** Show that two perigee burns of any sizes that take the same perigee speed from v_{c0} to v_{TLI} add up to the same Δv, and say why three burns do too. What changes if the second burn is made at the *apogee* of the loop instead? (Answer: $\Delta v_1 + \Delta v_2 = (v_{ph} - v_{c0}) + (v_{\mathrm{TLI}} - v_{ph}) = v_{\mathrm{TLI}} - v_{c0}$ whatever v_{ph} (Equation 6); with three burns the intermediate speeds cancel in pairs, a telescoping sum, as long as every burn is tangential at the same point. A burn at the apogee of the loop is also tangential, but it changes the *perigee* radius, not the apogee, so it cannot raise the apogee to the Moon's distance: the trans-lunar burn has to be at the perigee.)
3. **The climb rate as an eccentricity.** On an orbit of small eccentricity the radial speed at the true anomaly $f = 90°$ is $v_r = (\mu_\oplus/h)\, e \sin f \approx e\, v$. Show that the forced eccentricity $e = 2\kappa$ of Equation 17 then gives exactly the climb rate $\dot{r}$ of exercise 4, and find the radius at which $e = 0.05$ for $a_T = 2.08 \times 10^{-7}$ km/s^2. (Answer: $\kappa = 0.025$, $r = \sqrt{0.025\mu_\oplus/a_T} = 218\,900$ km, where the flight-path angle of the spiral is $2\kappa = 0.05$ rad $= 2.9°$.)
4. **The radial speed.** From Equation 9 and $\varepsilon = -\mu_\oplus/(2r)$ show that $\dot{r} = 2a_T r^{3/2}/\sqrt{\mu_\oplus}$ on a circular orbit, and integrate it to recover Equation 11.
5. **A heavier engine.** Repeat Example 2 with $P = 10$ kW. (Answer: $F = 0.680$ N, the same propellant, $t = 83.3$ d.)
6. **Spiral down.** How long does it take to spiral from 15 350 km to a 100 km lunar orbit with $a_T = 2.08 \times 10^{-7}$ km/s^2? (Answer: 59.4 d.)
7. **When the loops win.** A stage has $T/W_0 = 0.03$. Use the experiment of the Chobotov note to say how much of the Δv a single burn loses

compared with eight, and what the eight cost in time. (Answer: 499.8 and 21.8 m/s, a difference of 478 m/s, for 23.8 h in place of 1.9 h.)

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