Lambert's Problem

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1 Introduction

Every orbit problem so far has been an **initial-value problem**: given $(\mathbf{r}, \mathbf{v})$ now, find the state later, and Kepler's equation did the work. This note turns the question around. Given **two positions** and the **time** between them, find the orbit. That is a **boundary-value problem**, and it is the basis of every targeted transfer, intercept and rendezvous in the chase note.

> Lambert's problem, three ways
>
> **Given** $\mathbf{r}_1$, $\mathbf{r}_2$ and the time of flight Δt:
> 1. find the conic through both points that takes exactly Δt to get from $\mathbf{r}_1$ to $\mathbf{r}_2$; **or**
> 2. find the velocity $\mathbf{v}_1$ at $\mathbf{r}_1$ that will take the body to $\mathbf{r}_2$ in time Δt; **or**
> 3. find the velocity $\mathbf{v}_2$ with which the body arrives at $\mathbf{r}_2$, having left $\mathbf{r}_1$ a time Δt earlier.

The three are **the same problem**. Knowing the conic gives $\mathbf{v}_1$ and $\mathbf{v}_2$ by evaluating the velocity at the two points (Section 5). Knowing $\mathbf{v}_1$ (or $\mathbf{v}_2$) gives a state vector, hence the orbit (the perifocal note), hence the other end.

We work in the orbital plane, with **prograde** motion. The **transfer angle** $\Delta\theta \in (0, 2\pi)$ is measured from $\mathbf{r}_1$ to $\mathbf{r}_2$ in the direction of motion: **short way** if $\Delta\theta < \pi$, **long way** if $\Delta\theta > \pi$. In three dimensions the plane is fixed by $\mathbf{r}_1$ and $\mathbf{r}_2$ (unless they are collinear with the focus), and the direction of motion then decides which of $\Delta\theta$ and $2\pi - \Delta\theta$ is the transfer angle.

What "prograde" means, and why it must be said. Two points and a focus do not, by themselves, say which way round the focus the body goes. Through the same $\mathbf{r}_1$ and $\mathbf{r}_2$ there are transfers that run counter-clockwise, sweeping $\Delta\theta$, and transfers that run clockwise, sweeping $2\pi - \Delta\theta$. They are different orbits with different times of flight, so the sense of motion is part of the data. Choose a unit normal $\hat{\mathbf{w}}$ to the plane. The motion is

- **prograde** if its angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ points along $+\hat{\mathbf{w}}$: counter-clockwise when seen from the tip of $\hat{\mathbf{w}}$;
- **retrograde** if $\mathbf{h}$ points along $-\hat{\mathbf{w}}$: clockwise.

The normal is fixed by convention. For an Earth orbit $\hat{\mathbf{w}}$ is on the side of the north pole $\hat{\mathbf{K}}$, so prograde means $\mathbf{h} \cdot \hat{\mathbf{K}} > 0$, an inclination below $90°$, moving eastward with the Earth's rotation. Almost every spacecraft flies that way, because the launch site's eastward speed comes free. For a transfer between planets $\hat{\mathbf{w}}$ is ecliptic north, and prograde is the sense in which the planets go round the Sun.

Once the sense is chosen, the transfer angle follows from the two position vectors alone:

$$\cos\Delta\theta_0 = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}, \qquad \Delta\theta = \begin{cases} \Delta\theta_0 & \text{if } (\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{w}} \geq 0, \\ 2\pi - \Delta\theta_0 & \text{otherwise,} \end{cases} \tag{1}$$

with $\Delta\theta_0 = \arccos(\cdot) \in [0, \pi]$; a retrograde transfer uses the other choice. Nothing new is needed for retrograde motion. With $\Delta\theta$ the counter-clockwise angle from $\mathbf{r}_1$ to $\mathbf{r}_2$ of Equation 1, a retrograde transfer sweeps $2\pi - \Delta\theta$ clockwise. Reflect the plane in the line of $\mathbf{r}_1$: the clockwise sweep becomes a counter-clockwise one, so the retrograde transfer is the prograde transfer through $2\pi - \Delta\theta$ from $\mathbf{r}_1$ to the mirror image of $\mathbf{r}_2$. So the note treats only prograde motion, and must

treat **both** the short way and the long way. In Figure 1, panel (b) reflected in the horizontal axis is the retrograde transfer between the two points of panel (a).

Does a transfer exist? For any given $(\mathbf{r}_1, \mathbf{r}_2, \Delta t)$ the answer is the following. The sections below derive it, except for one step: that the clock is monotone in p, which is checked numerically rather than proved (Section 4.5).

Existence and uniqueness

Let $r_1, r_2 > 0$ and $\Delta\theta \neq 0, \pi$ (the case $\Delta\theta = \pi$ is special; Section 7.3).

- **No extra revolution** ($N = 0$): for **every** $\Delta t > 0$ there is **exactly one** conic. It is an ellipse if $\Delta t > \Delta t_{\text{par}}$, a parabola if $\Delta t = \Delta t_{\text{par}}$, a hyperbola if $\Delta t < \Delta t_{\text{par}}$, where Δt_{par} is the flight time on the parabola through the two points (Equation 10).
- $N \geq 1$ **extra revolutions**: a solution needs $\Delta t > \Delta t_{\text{min}}(N)$, and then there are **two** (both ellipses). Together with $N = 0$, a long enough Δt gives $2N_{\text{max}} + 1$ solutions, where N_{max} is the largest N with $\Delta t_{\text{min}}(N) < \Delta t$.

Figure 1 shows all four cases for one pair of points and one time of flight.

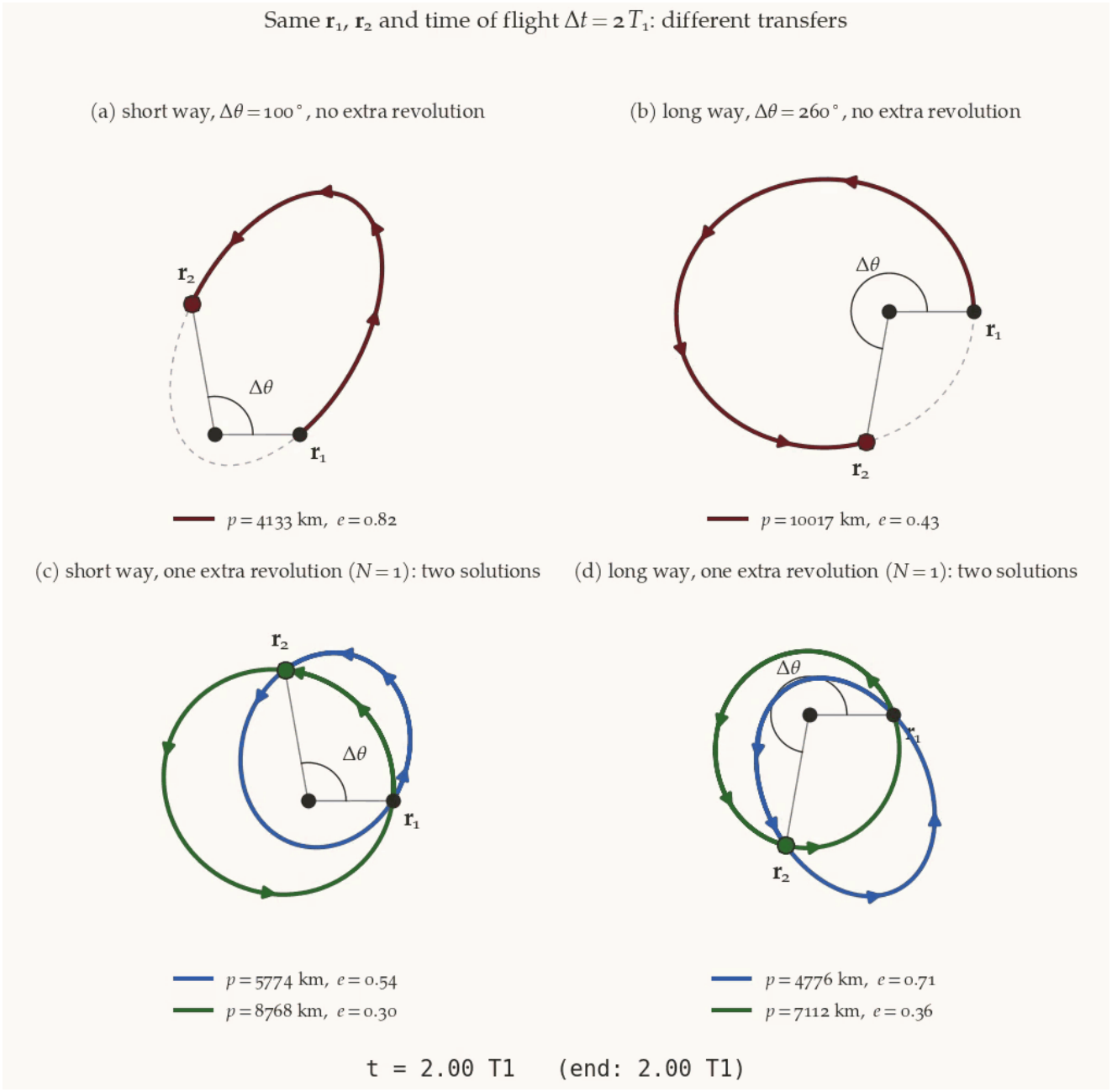


Figure 1: The same two points $\mathbf{r}_1$, $\mathbf{r}_2$ ($r_2/r_1 = 11/7$) and the same time of flight $\Delta t = 2T_1$, where T_1 is the period of the circular orbit of radius r_1. (**a**) Short way, $\Delta\theta = 100°$: one solution. (**b**) Long way, $\Delta\theta = 260°$: one solution. (**c**), (**d**) The same two ways with one extra revolution: each has **two** solutions (blue, green). The craft moves at its true Keplerian speed, slowed down. The central body is at the dot at the origin. *Scripts: anim-lam-types.py; the solutions are found with _lambertp.py.*

Explore it yourself. The explorer below solves the problem for any r_1, r_2, transfer angle $\Delta\theta$ and time of flight Δt, with up to N extra revolutions. It draws **every** solution: one with no extra revolution, and two for each $N \geq 1$ whose minimum time (Table 3) is below Δt. Each craft flies its arc at its true Keplerian speed, and all of them reach $\mathbf{r}_2$ together at $t = \Delta t$. The arrows are $\mathbf{v}_1$ (solid) and $\mathbf{v}_2$ (open head), to a common scale. Tick *retrograde* to add the clockwise transfers between the same two points (dashed). It uses the p-iteration of Section 6 with $\mu_\oplus$, and opens on the geometry of Figure 1. Things to try:

- Lower Δt slowly at $\Delta\theta = 100°$. The $N = 1$ pair merges and disappears at $1.766\,T_1$, and below Δt_{par} the $N = 0$ transfer becomes a hyperbola.
- Set $N = 2$ and raise Δt past $3.013\,T_1 = 4.88$ h: a second pair appears, and

five transfers now arrive together.
- Set $r_1 = 6678$ km, $r_2 = 42\,164$ km, $\Delta\theta = 175°$, $\Delta t = 4$ h and $N = 0$: Example 1 ($p = 11\,598$ km, $e = 0.744$).

Explorer: lambert-explorer.js; its solver is checked against the examples of this note, Table 3 and the Python solver by lambert-explorer-test.js, run with node. T_1 is the period of the circular orbit of radius r_1, γ the flight-path angle.

The plan, with every step done in full:

1. **The unknowns** (Section 2). Two points and a focus leave a one-parameter family of conics; the time of flight selects one member.
2. **Labelling the family by** p (Section 3). Given p, the eccentricity and true anomaly follow in closed form. We find which p are allowed, and why.
3. **The clock** (Section 4). Kepler's equation for each conic type, the parabola, and the monotonicity that gives existence and uniqueness; then extra revolutions.
4. **Velocities** (Section 5) and the **p-iteration algorithm** (Section 6), with two worked examples.
5. **Where p-iteration struggles** (Section 7), which motivates universal variables.
6. **Velocities without elements** (Section 8): the Lagrange coefficients.

Table 1: Notation

Symbol	Meaning
$\mathbf{r}_1$, $\mathbf{r}_2$, r_1, r_2	the two position vectors and their lengths
$\Delta\theta$, Δt	transfer angle (in the direction of motion) and time of flight
$c = \lvert\mathbf{r}_2 - \mathbf{r}_1\rvert$, $s = \frac{1}{2}(r_1 + r_2 + c)$	chord and semi-perimeter
$k = r_1 r_2(1 - \cos\Delta\theta)$, $\ell = r_1 + r_2$, $m = r_1 r_2(1 + \cos\Delta\theta)$	geometry constants used for the p window
p, e, a	semi-latus rectum, eccentricity, semi-major axis of the trial conic
f_1, $f_2 = f_1 + \Delta\theta$	true anomalies at the two points
$A = p/r_1 - 1$, $B = p/r_2 - 1$	shorthand for $e\cos f_1$ and $e\cos f_2$
N	number of complete revolutions added to the transfer
F, G, $\dot F$, $\dot G$	the Lagrange coefficients (the same symbols are used in the universal-variables note)

Throughout, $\mu = 398\,600.4418$ km^3/s^2 and $R_\oplus = 6378.137$ km.

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

History. The problem is named after Johann Heinrich Lambert, who in *Insigniores orbitae cometarum proprietates* (1761) showed that the time on an elliptic arc depends only on the semi-major axis, the sum $r_1 + r_2$ and the chord c, not on the eccentricity. Euler had proved the parabolic case about 1740, and Lambert reinvented it without knowing. Lagrange gave the first analytic expression for the time of flight in 1778. In 1801 Gauss used a method for this problem to find the orbit of the planet Ceres, which had been lost, from a 3° arc observed over 41 days; astronomers found it again where he had predicted. He set the method out in the *Theoria motus* (1809) (Gauss 1809), which made it the central problem of orbit determination.

History: the modern solvers. The mid-1960s saw intensive work. Lancaster and Blanchard (1969) (Lancaster and Blanchard 1969) reduced the problem to iterations that each need one inverse trigonometric or hyperbolic function; Battin and Vaughan (1984) (Battin and Vaughan 1984) made it more robust; Gooding (1990) (Gooding 1990) and Izzo (2015) (Izzo 2015) followed. Richard Battin, at the MIT Instrumentation Laboratory, spent years on the problem for Apollo guidance (Battin 1964). Today it is solved thousands of times per second in mission design: every point of a launch-window "porkchop" plot is one Lambert solution.

2 The problem and its unknowns

2.1 Counting

Let the conic be

$$r = \frac{p}{1 + e\cos f},$$

with the usual unknowns p and e. A planar orbit is fixed by three elements (p, e, ω) plus one epoch. Measure polar angles θ from a fixed axis in the plane: $\mathbf{r}_1$ is at θ_1 and $\mathbf{r}_2$ at $\theta_2 = \theta_1 + \Delta\theta$. Let f_1 be the true anomaly of $\mathbf{r}_1$, so that the argument of periapsis is $\omega = \theta_1 - f_1$. The second point is at true anomaly $f_2 = f_1 + \Delta\theta$. The orbit equation, written at the two points, gives

$$r_1 = \frac{p}{1 + e\cos f_1}, \qquad r_2 = \frac{p}{1 + e\cos(f_1 + \Delta\theta)}. \tag{2}$$

- **Unknowns:** p, e, f_1.
- **Equations:** two.

So the geometry alone leaves **one free parameter**: a one-parameter family of conics through the two points (Figure 2). The time of flight is the third equation that pins down one member.

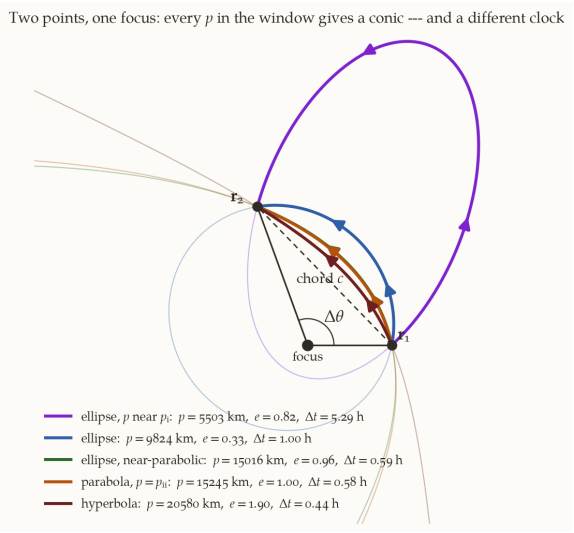


Figure 2: A family of conics through the same two points, with the direction of motion marked. Thin lines: the whole conic. Thick lines: the arc from $\mathbf{r}_1$ to $\mathbf{r}_2$ in the direction of motion. Each member has a different semi-latus rectum p and a different time of flight; the arrows run counter-clockwise, in the direction in which the true anomaly grows. *Script: fig-lam-family.py.*

> Important
>
> Geometry gives a one-parameter family. **Time selects one member.** Every Lambert method is a choice of how to label the family and how to compute the clock.

Why label by p? The semi-latus rectum $p = h^2/\mu$ is finite for **every** conic (the semi-major axis a is not: it is infinite for the parabola and negative for the hyperbola), and it is a positive number that varies continuously across all three types. Given p, the rest of the conic follows in closed form, as the next section shows.

3 Labelling the family by p

3.1 e and f_1 in closed form

Write $A = p/r_1 - 1$ and $B = p/r_2 - 1$. Then Equation 2 says

$$e\cos f_1 = A, \qquad e\cos(f_1 + \Delta\theta) = B.$$

Expand the second with the angle-addition formula, $e\cos f_1\cos\Delta\theta - e\sin f_1\sin\Delta\theta = B$, and substitute the first, $e\cos f_1 = A$:

$$\boxed{e\cos f_1 = A, \qquad e\sin f_1 = \frac{A\cos\Delta\theta - B}{\sin\Delta\theta}} \tag{3}$$

Given a trial p, this delivers $e = \sqrt{(e\cos f_1)^2 + (e\sin f_1)^2}$ and $f_1 = \operatorname{atan2}(e\sin f_1, e\cos f_1)$ **in closed form**. Nothing is iterated yet. The same steps give the other end:

$$e\cos f_2 = B, \qquad e\sin f_2 = e\sin f_1\cos\Delta\theta + e\cos f_1\sin\Delta\theta = \frac{A - B\cos\Delta\theta}{\sin\Delta\theta}, \tag{4}$$

since $A\cos^2\Delta\theta - B\cos\Delta\theta + A\sin^2\Delta\theta = A - B\cos\Delta\theta$.

> Note
>
> Note the division by $\sin\Delta\theta$. It is the symptom of a real difficulty at $\Delta\theta = \pi$ (Section 7.3).

3.2 The eccentricity as a function of p

Square and add the two parts of Equation 3, and multiply by $\sin^2\Delta\theta$:

$$e^2\sin^2\Delta\theta = A^2 + B^2 - 2AB\cos\Delta\theta. \tag{5}$$

This is a **quadratic in** p. Substitute $A = p/r_1 - 1$, $B = p/r_2 - 1$ and expand:

$$A^2+B^2-2AB\cos\Delta\theta = p^2\Big(\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\cos\Delta\theta}{r_1r_2}\Big)-2p\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)(1-\cos\Delta\theta)+2(1-\cos\Delta\theta).$$

Multiply through by $r_1^2r_2^2$. The bracket becomes c^2 (the square of the chord, $c^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\Delta\theta$), and the other coefficients become k and ℓ:

$$r_1^2r_2^2\,e^2\sin^2\Delta\theta = c^2p^2 - 2k\ell\,p + 2k\,r_1r_2.$$

Now compute $e^2 - 1$. Note $r_1^2r_2^2\sin^2\Delta\theta = km$, because $k\,m = r_1^2r_2^2(1-\cos^2\Delta\theta)$. So

$$r_1^2r_2^2\sin^2\Delta\theta\,(e^2-1) = c^2p^2 - 2k\ell\,p + 2k\,r_1r_2 - km = c^2p^2 - 2k\ell\,p + k^2,$$

where the constant term collapses because $2r_1r_2 - m = r_1r_2(1-\cos\Delta\theta) = k$. **The conic through the two points is parabolic** ($e = 1$) **at the roots of**

$$c^2p^2 - 2k\ell\,p + k^2 = 0.$$

Its roots are $p = k(\ell \pm \sqrt{\ell^2 - c^2})/c^2$. Since $\ell^2 - c^2 = 2r_1r_2(1+\cos\Delta\theta) = 2m$, we have $c^2 = \ell^2 - 2m = (\ell - \sqrt{2m})(\ell + \sqrt{2m})$, and so

$$p_{\mathrm{i}} = \frac{k}{\ell + \sqrt{2m}}, \qquad p_{\mathrm{ii}} = \frac{k}{\ell - \sqrt{2m}}. \tag{6}$$

These are the two **parabolas** through the points. Since the quadratic has leading coefficient c^2 and roots p_i, p_ii, it factorises, and we get the compact result

$$\boxed{e^2 = 1 + \frac{c^2\,(p - p_\mathrm{i})(p - p_\mathrm{ii})}{(r_1 r_2 \sin\Delta\theta)^2}} \tag{7}$$

Reading it off:

- **Ellipse** ($e < 1$) for $p_\mathrm{i} < p < p_\mathrm{ii}$; **hyperbola** ($e > 1$) outside that interval. The type of the conic is known from p alone, with no iteration.
- e^2 depends on $\Delta\theta$ only through $\cos\Delta\theta$ and $\sin^2\Delta\theta$, so $\Delta\theta$ **and** $2\pi - \Delta\theta$ **share the same** $e(p)$**,** p_i **and** p_ii (Figure 3). They differ only in which p are admissible.
- **The minimum eccentricity.** The quadratic is smallest at the midpoint of its roots, $p_e = \frac{1}{2}(p_\mathrm{i}+p_\mathrm{ii}) = k\ell/c^2$. There $p_\mathrm{ii} - p_\mathrm{i} = 2k\sqrt{2m}/c^2$, so $c^2(p_\mathrm{ii}-p_\mathrm{i})^2/4 = 2k^2m/c^2$; dividing by $r_1^2 r_2^2 \sin^2\Delta\theta = km$,

  $$e^2_\mathrm{min} = 1 - \frac{2k}{c^2} = \frac{c^2 - 2k}{c^2} = \frac{(r_2 - r_1)^2}{c^2}, \qquad e_\mathrm{min} = \frac{|r_2 - r_1|}{c}.$$

  No conic through the two points has a smaller eccentricity. (For $r_1 = r_2$ the minimum is the circle, as it should be.)

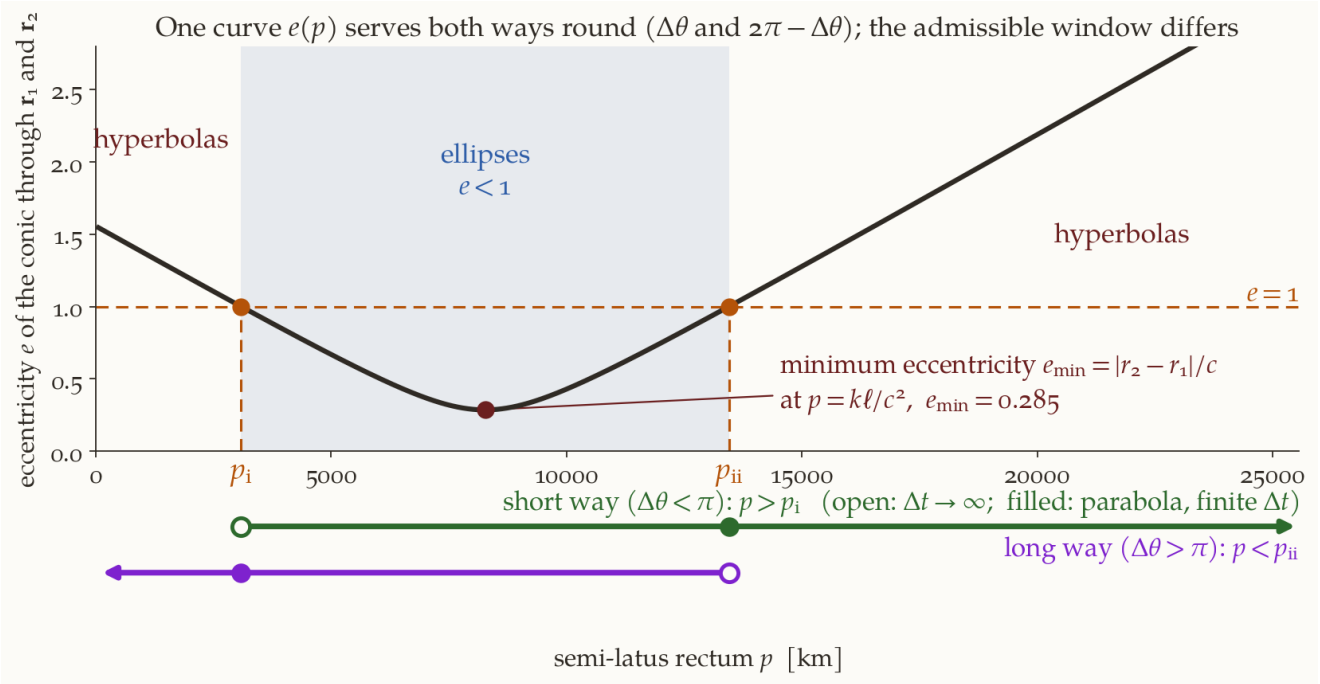


Figure 3: The eccentricity of the conic through $\mathbf{r}_1$ and $\mathbf{r}_2$ as a function of p, for $r_1 = 7000$ km, $r_2 = 11\,000$ km. The same curve serves the short way ($\Delta\theta = 100°$) and the long way ($\Delta\theta = 260°$). $e = 1$ at the two parabolic values p_i and p_ii; the minimum is $e_\mathrm{min} = |r_2 - r_1|/c$. The bars show which p are admissible (next section); open circles mark ends where the time of flight diverges, filled circles the admissible parabola. *Script: fig-lam-window.py.*

3.3 Which values of p are allowed?

Equation 3 returns a real e and an angle f_1 for **every** $p > 0$, so a conic of parameter p through both points always exists. But we need an **arc of it, run in the direction of motion from $\mathbf{r}_1$ to $\mathbf{r}_2$, that sweeps the true anomaly** $\Delta\theta$. Call p *admissible* if there is one.

- **Ellipse** ($e < 1$): every true anomaly occurs, so the arc from f_1 to $f_1 + \Delta\theta$ exists. Always admissible.
- **Parabola or hyperbola** ($e \geq 1$): the true anomaly is confined to $|f| < f_\infty = \cos^{-1}(-1/e)$ (with $f_\infty = \pi$ for the parabola). Admissible if and only if **both** f_1 and $f_2 = f_1 + \Delta\theta$ lie in this interval (with f_1 taken from atan2, in $(-\pi, \pi]$). Otherwise the arc would have to cross an asymptote.

Where can admissibility change? On the hyperbolic range ($e > 1$) neither f_1 nor f_2 can ever reach $\pm f_\infty$ for a finite p: that would need $1 + e\cos f_k = 0$, but $1 + e\cos f_k = p/r_k > 0$. So within each connected hyperbolic interval of p (that is, $p < p_{\mathrm{i}}$ and $p > p_{\mathrm{ii}}$) the admissibility is constant, and one test value settles each interval.

- $p \to \infty$**.** $e \to \infty$ and $f_\infty \to \pi/2$: the hyperbola degenerates into a straight line run infinitely fast. A straight line joins $\mathbf{r}_1$ to $\mathbf{r}_2$ along the chord, sweeping an angle $< \pi$ as seen from the focus. So $p > p_{\mathrm{ii}}$ is admissible for the **short way** and not for the long way.
- $p \to 0$**.** Then $A = B = -1$, and Equation 5 gives $e^2 \sin^2\Delta\theta = 2(1 - \cos\Delta\theta)$, that is $e = 1/|\cos(\Delta\theta/2)|$. Both points have $1 + e\cos f_k = p/r_k \to 0$, so they sit near the two asymptotes, $f_1 \approx -f_\infty$ and $f_2 \approx +f_\infty$, and the arc between them sweeps $2f_\infty = 2\cos^{-1}(-|\cos\frac{\Delta\theta}{2}|)$. For $\Delta\theta > \pi$ this is $\Delta\theta$ (since $\cos(\Delta\theta/2) < 0$ there), exactly what we need. For $\Delta\theta < \pi$ it is $2\pi - \Delta\theta \neq \Delta\theta$. So $p < p_{\mathrm{i}}$ is admissible for the **long way** and not for the short way.

Table 2: The admissible window.

	admissible p	ellipse	hyperbola
short way, $\Delta\theta < \pi$	$p_{\mathrm{i}} < p < \infty$	$p_{\mathrm{i}} < p < p_{\mathrm{ii}}$	$p > p_{\mathrm{ii}}$
long way, $\Delta\theta > \pi$	$0 < p < p_{\mathrm{ii}}$	$p_{\mathrm{i}} < p < p_{\mathrm{ii}}$	$p < p_{\mathrm{i}}$

The parabolas. Exactly one of the two parabolas is admissible for each way, and it is the one that joins the hyperbolic and elliptic ranges inside the window: p_{ii} for the short way, p_{i} for the long way. The other one is only approached from the elliptic side, by ellipses whose arc runs round a very distant apoapsis, so **the time of flight diverges there** ($\Delta t \to \infty$ as $p \to p_{\mathrm{i}}$ for the short way, and as $p \to p_{\mathrm{ii}}$ for the long way). For the admissible parabola the unwrapped f_2 stays below π and the time is finite; it is computed in Section 4.3.

(lambert-verify.py checks Table 2 against the direct test, with the unwrapped anomalies, on 15 440 random cases.)

4 The clock

With (p, e, f_1, f_2) known, the time of flight is **Kepler's equation, twice**, with a different form for each conic.

4.1 Ellipse ($e < 1$)

Write the same radius in two ways, $r = a(1 - e\cos E) = p/(1 + e\cos f)$, with $p = a(1 - e^2)$. Then $1 - e\cos E = (1 - e^2)/(1 + e\cos f)$, so $e\cos E = (e\cos f + e^2)/(1 + e\cos f)$ and

$$\cos E = \frac{e + \cos f}{1 + e\cos f}.$$

The half-angle relation follows from $\tan^2(E/2) = (1 - \cos E)/(1 + \cos E)$:

$$\tan^2\frac{E}{2}=\frac{(1+e\cos f)-(e+\cos f)}{(1+e\cos f)+(e+\cos f)}=\frac{(1-e)(1-\cos f)}{(1+e)(1+\cos f)}=\frac{1-e}{1+e}\tan^2\frac{f}{2}.$$

E and f lie in the same half-plane (both grow together from periapsis), so $\tan\frac{E}{2}=\sqrt{(1-e)/(1+e)}\,\tan\frac{f}{2}$. For computation use the form that is safe at $f=\pi$ and for every f, $E=2\,\mathrm{atan2}(\sqrt{1-e}\,\sin\frac{f}{2},\ \sqrt{1+e}\,\cos\frac{f}{2})$. Kepler's equation $M=E-e\sin E$ and $M=n\,(t-t_p)$ with $n=\sqrt{\mu/a^3}$ give

$$\begin{aligned}\Delta t&=\frac{\Delta E-e\,[\,\sin(E_1+\Delta E)-\sin E_1\,]}{n},\\ \Delta E&=[\,(E_2-E_1)\bmod 2\pi\,]+2\pi N.\end{aligned}\tag{8}$$

For $N=0$ take $E_2-E_1\in(0,2\pi)$, no complete revolution. Each extra revolution adds 2π to ΔE, and since sines are periodic that adds exactly one period $T=2\pi/n$ to the time: $\Delta t_N=\Delta t_0+N\,T(p)$.

4.2 Hyperbola ($e>1$)

The same steps with $a=p/(1-e^2)<0$ and $r=|a|(e\cosh H-1)=p/(1+e\cos f)$ give $\cosh H=(e+\cos f)/(1+e\cos f)$ and

$$\tanh^2\frac{H}{2}=\frac{\cosh H-1}{\cosh H+1}=\frac{e-1}{e+1}\tan^2\frac{f}{2}.$$

With the hyperbolic Kepler equation $M_h=e\sinh H-H$ and $n_h=\sqrt{\mu/(-a)^3}$:

$$\begin{aligned}\tanh\frac{H}{2}&=\sqrt{\frac{e-1}{e+1}}\tan\frac{f}{2},\qquad M_h=e\sinh H-H,\\ \Delta t&=\frac{M_{h,2}-M_{h,1}}{n_h}.\end{aligned}\tag{9}$$

Both anomalies must satisfy $|f|<f_\infty$, as in Section 3.3. There are no complete revolutions: $N=0$.

4.3 Parabola ($e=1$)

For $e=1$, a is infinite and neither form applies. Integrate the angular-momentum equation $dt=r^2\,df/h$ directly, with $r=p/(1+\cos f)=\frac{p}{2}\sec^2\frac{f}{2}$ and $h=\sqrt{\mu p}$. With $D=\tan\frac{f}{2}$, $df=2\,dD/(1+D^2)$ and $\sec^2\frac{f}{2}=1+D^2$:

$$r^2\,df=\frac{p^2}{4}(1+D^2)^2\,\frac{2\,dD}{1+D^2}=\frac{p^2}{2}(1+D^2)\,dD,$$

and integrating, with $p^2/\sqrt{\mu p}=\sqrt{p^3/\mu}$ (**Barker's equation**):

$$\Delta t=\frac{1}{2}\sqrt{\frac{p^3}{\mu}}\Big[\Big(D_2+\frac{D_2^3}{3}\Big)-\Big(D_1+\frac{D_1^3}{3}\Big)\Big],\qquad D_k=\tan\frac{f_k}{2}.$$

For the parabola through the two points the same time has a closed form in the geometry alone, **Euler's parabolic theorem**:

$$\Delta t_{\mathrm{par}} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} \mp (s-c)^{3/2}\right], \tag{10}$$

with the **minus** sign for the short way and the **plus** sign for the long way. (lambert-verify.py checks that Equation 10 equals Barker's equation at the admissible parabola, for six geometries and both ways.) This is the threshold of the existence result in Section 1: a given Δt is an ellipse for $\Delta t > \Delta t_{\mathrm{par}}$ and a hyperbola for $\Delta t < \Delta t_{\mathrm{par}}$. For Example 1 below, $\Delta t_{\mathrm{par}} = 2.238$ h.

4.4 Lambert's theorem

Euler's parabolic theorem says that the time on the parabola depends only on s and c. Lambert saw that the same is true of every conic: **the time of flight depends only on the semi-major axis a, the sum $r_1 + r_2$ and the chord c**, not on the eccentricity or on where the arc sits on its orbit. (Here $c = |\mathbf{r}_2 - \mathbf{r}_1|$ and $s = (r_1 + r_2 + c)/2$ is the semi-perimeter of the triangle formed by the focus and the two points.) Lagrange put it in the form

$$n\,\Delta t = (\alpha - \sin\alpha) - (\beta - \sin\beta), \qquad \sin^2\frac{\alpha}{2} = \frac{s}{2a}, \quad \sin^2\frac{\beta}{2} = \frac{s-c}{2a}, \tag{11}$$

for an elliptic arc with $n = \sqrt{\mu/a^3}$. The right-hand side contains only a, s and c. Two branch rules pick the right arc of each inverse sine: $\beta \to -\beta$ for the long way ($\Delta\theta > \pi$), and $\alpha \to 2\pi - \alpha$ when the flight time exceeds that of the minimum-energy ellipse below. For Example 1, with $c = 48\,820.2$ km and $s = 48\,831.2$ km, $\alpha = 2.64678$, $\beta = 0.02907$, and Equation 11 returns 14 400.0 s.

The minimum-energy ellipse. Equation 11 needs $s/2a \le 1$, so no ellipse through the two points has $a < s/2$. The ellipse with $a_m = s/2$ has $\alpha = \pi$; it is the transfer of lowest energy $-\mu/(2a)$, and its time of flight separates the two α branches. For Example 1, $a_m = 24\,415.6$ km. The theorem is why Lambert's problem is often solved in a (or in a variable built from it) rather than in p; this note uses p because e and f_1 then follow in closed form.

4.5 The clock is monotone in p

Put the pieces together: for a given $\Delta\theta$ and a trial p in the admissible window, Equation 3 gives (e, f_1) and the matching clock gives $\Delta t(p)$. For zero revolutions this is a **monotone** function of p across the **whole** window (Figure 4):

- **Short way**: Δt **falls** as p grows, from $+\infty$ at $p \to p_{\mathrm{i}}^{+}$ (ellipses running round a distant apoapsis), through the parabolic value Δt_{par} at p_{ii}, down to 0 as $p \to \infty$ (numerically $\Delta t \propto p^{-1/2}$ there).
- **Long way**: Δt **rises** as p grows, from 0 at $p \to 0$ (a fast dive past the focus), through Δt_{par} at p_{i}, up to $+\infty$ as $p \to p_{\mathrm{ii}}^{-}$.

So each $\Delta t \in (0, \infty)$ is reached **exactly once**: this is the existence and uniqueness statement of Section 1. We have not proved the monotonicity analytically; lambert-verify.py confirms it on a scan of 116 random geometries (both ways, over the whole window, ellipse and hyperbola), and checks both limits.

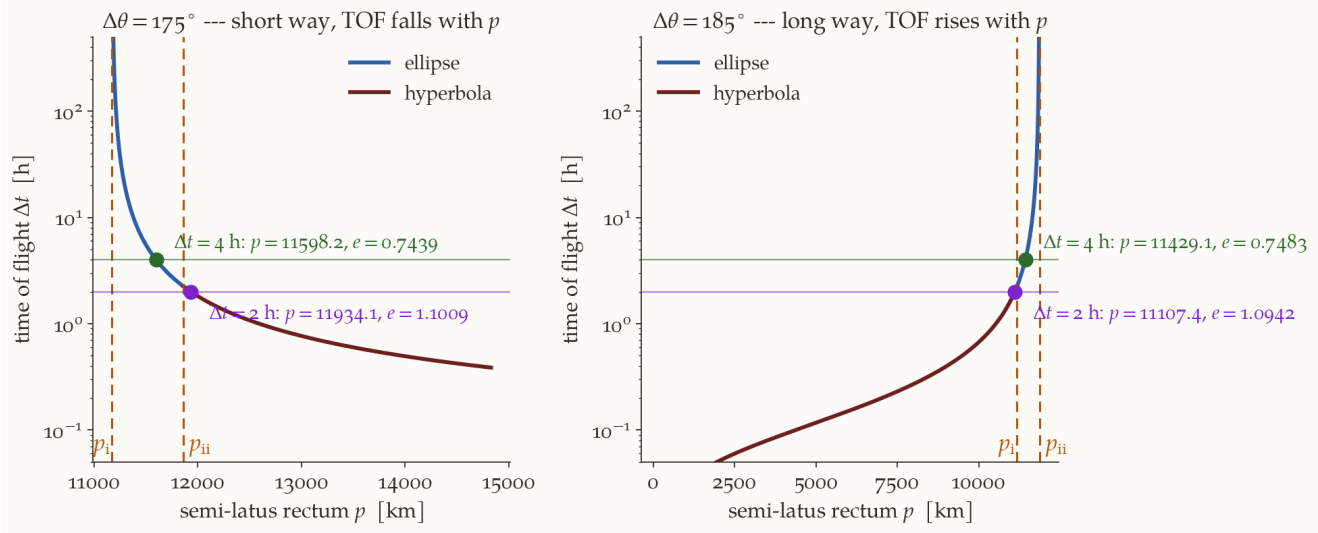


Figure 4: Time of flight against p for the LEO → GEO geometry ($r_1 = 6678.137$ km, $r_2 = 42\,164$ km), at $\Delta\theta = 175°$ (short way, left) and 185° (long way, right). Short way: the clock falls as p grows; long way: it rises. In both cases it is **monotone** across the whole window, so each Δt has exactly one p. *Script: fig-lam-tof.py.*

4.6 More than one revolution

Allow N complete revolutions. Only ellipses can complete a revolution, so $N \geq 1$ lives entirely in $p_i < p < p_{ii}$, and by Equation 8

$$\Delta t_N(p) = \Delta t_0(p) + N\,T(p), \qquad T(p) = 2\pi\sqrt{\frac{a^3}{\mu}}, \quad a = \frac{p}{1-e^2}.$$

At **both** ends of the window the period diverges ($e \to 1$ gives $a \to \infty$), so $\Delta t_N \to \infty$ at both ends, and a continuous function that is infinite at both ends has a **minimum** $\Delta t_{\min}(N)$ in between. The clock is therefore **no longer monotone**: for Δt below the minimum there is no solution, and above it there are **two**, one on each side of the minimum. That there is only *one* minimum (the curve is unimodal) is, like the monotonicity of Section 4.5, checked numerically rather than proved. Call the two solutions the **low-p** and **high-p** branches; in the examples below the low-p one is the more eccentric. Bisection alone can no longer be trusted: it must be told which side to search.

For the geometry of Figure 1 ($r_2/r_1 = 11/7$), the minimum times are

Table 3: Smallest **elliptic** time of flight for each number of extra revolutions. For $N = 0$ it is the parabolic time Δt_{par} (faster transfers are hyperbolas, down to $\Delta t \to 0$); for $N \geq 1$ it is the minimum of $\Delta t_N(p)$. T_1 is the period of the circular orbit of radius r_1.

	$N = 0$	$N = 1$	$N = 2$
short way, $\Delta\theta = 100°$	$0.248\,T_1$	$1.766\,T_1$	$3.013\,T_1$
long way, $\Delta\theta = 260°$	$0.271\,T_1$	$1.790\,T_1$	$3.037\,T_1$

At $\Delta t = 2T_1$ both $N = 0$ and $N = 1$ exist (one and two solutions per way, as in Figure 1) and $N = 2$ does not, since $2T_1 < 3.013\,T_1$. In Figure 1 (c) both $N = 1$ orbits sweep $100° + 360° = 460°$ of true anomaly in the same time; the green one

($e = 0.30$, the high-p branch) is close to the least eccentric conic through the two points ($e = 0.285$), the blue one ($e = 0.54$, low p) a more eccentric ellipse that dives closer to the focus and is flown faster there.

5 Velocities

Once p is found, angular momentum $h = \sqrt{\mu p}$ and the orbit equation give the radial and transverse speeds at any true anomaly. (This is the result of the perifocal note, derived there in full; here is the short version.) From $h = r^2\dot{f}$, $v_\theta = r\dot{f} = h/r$. Differentiating $r = p/(1 + e\cos f)$ in time, $\dot{r} = \dfrac{p\,e\sin f}{(1+e\cos f)^2}\dot{f} = \dfrac{r^2\dot{f}}{p}\,e\sin f = \dfrac{h}{p}\,e\sin f$. So

$$v_\theta = \frac{h}{r} = \sqrt{\frac{\mu}{p}}\,(1 + e\cos f), \qquad v_r = \dot{r} = \sqrt{\frac{\mu}{p}}\,e\sin f. \tag{12}$$

At the two ends, $1 + e\cos f_k = p/r_k$, and Equation 3 and Equation 4 give $e\sin f_k$, so **the velocities need neither e nor f separately**:

$$\begin{aligned} v_{r1} &= \sqrt{\frac{\mu}{p}}\,\frac{A\cos\Delta\theta - B}{\sin\Delta\theta}, & v_{\theta 1} &= \frac{\sqrt{\mu p}}{r_1}, \\ v_{r2} &= \sqrt{\frac{\mu}{p}}\,\frac{A - B\cos\Delta\theta}{\sin\Delta\theta}, & v_{\theta 2} &= \frac{\sqrt{\mu p}}{r_2}. \end{aligned} \tag{13}$$

Rotate into the inertial frame with the unit vectors at each end, $\hat{\mathbf{u}}_r(\theta) = (\cos\theta, \sin\theta)$ and $\hat{\mathbf{u}}_\theta(\theta) = (-\sin\theta, \cos\theta)$:

$$\mathbf{v}_k = v_{r,k}\,\hat{\mathbf{u}}_r(\theta_k) + v_{\theta,k}\,\hat{\mathbf{u}}_\theta(\theta_k), \qquad k = 1, 2.$$

6 The p-iteration algorithm

The algorithm

Input r_1, r_2, $\Delta\theta$, Δt (and N).

1. Compute p_{i}, p_{ii} from Equation 6; the admissible window is read from Table 2.
2. For a trial p: get e, f_1 from Equation 3 (closed form).
3. Branch on e: time of flight from Equation 8 ($e < 1$) or Equation 9 ($e > 1$).
4. Update p to drive $\Delta t(p) - \Delta t$ to zero: **bisection** (safe, because for $N = 0$ the clock is monotone) or **secant** (fast).
5. Recover $\mathbf{v}_1$, $\mathbf{v}_2$ from Equation 13 (or element-free, from Equation 19 in Section 8).

- **The secant update** is $p_{k+1} = p_k - (\Delta t_k - \Delta t)\dfrac{p_k - p_{k-1}}{\Delta t_k - \Delta t_{k-1}}$. It converges superlinearly (order ≈ 1.62), where bisection only gains one binary digit per step.
- If a secant step leaves the window, **fall back to bisection** on the current bracket. A bracket always exists for $N = 0$ (Section 4.5): $(p_{\mathrm{i}}, p_{\mathrm{ii}})$ if an ellipse is expected, with the long way the mirror image.

- **A bracket for the short-way hyperbola.** Its window, $p > p_{\mathrm{ii}}$, has no upper end. Start at $p = 2p_{\mathrm{ii}}$ and double p until $\Delta t(p) < \Delta t$; then bisect. (The long-way hyperbola lives in the bounded window $0 < p < p_{\mathrm{i}}$.)
- **For $N \geq 1$.** First find the minimiser p^* of $\Delta t_N(p)$ on $(p_{\mathrm{i}}, p_{\mathrm{ii}})$, by golden-section search or Brent's method. If $\Delta t < \Delta t_N(p^*)$ there is no solution with N revolutions. Otherwise bisect separately on (p_{i}, p^*) and on (p^*, p_{ii}): one solution on each side.
- Bisection from the window of Example 1 below (690 km wide) down to 10^{-6} km takes about 30 steps; the secant method takes seven updates.

6.1 Example 1 — LEO → GEO in four hours

Given $r_1 = 6678.137$ km, $r_2 = 42\,164$ km, $\Delta\theta = 175°$, $\Delta t = 4$ h $= 14\,400$ s. Here $\cos 175° = -0.996195$ and $1 - \cos\Delta\theta = 1.996195$.

Window. With $k = 5.62082 \times 10^8$ km^2, $\ell = 48\,842.137$ km and $m = 1.07149 \times 10^6$ km^2,

$$p_{\mathrm{i}} = 11\,173.263 \text{ km}, \qquad p_{\mathrm{ii}} = 11\,863.724 \text{ km}.$$

The parabola at p_{ii} takes 2.238 h $<$ 4 h (Euler's theorem, Equation 10, gives the same), so the answer is an **ellipse**. **Secant iteration** from $p = 11\,300$ and 11 800 km converges in seven updates; every iterate stays inside the window:

step	p [km]	Δt [h]	e
0	11300	19.207	0.847
1	11800	2.507	0.918
2	11755.29	2.736	0.866
3	11508.72	5.364	0.727
4	11636.70	3.598	0.764
5	11607.55	3.894	0.748
6	11597.17	4.012	0.7435
7	11598.19	3.9997	0.7440
8	**11598.16**	**4.0000**	**0.74395**

The converged arc. $p = 11\,598.163$ km gives, from Equation 3,

$$A = 0.736736, \quad B = -0.724927, \quad e\sin f_1 = -0.103326,$$

$$e = 0.74395, \qquad f_1 = -7.984°, \qquad f_2 = 167.016°.$$

Clock check. $a = 25\,973.20$ km, $n = 1.508276 \times 10^{-4}$ rad/s, $E_1 = -0.053465$, $E_2 = 2.564252$, so $M_1 = -0.013709$, $M_2 = 2.158208$ and

$$\frac{M_2 - M_1}{n} = \frac{2.171917}{1.508276 \times 10^{-4}} = 14\,400.0 \text{ s} \checkmark$$

Velocities from Equation 12, $(v_r,\ v_\theta)$ in km/s:

$$\mathbf{v}_1 = (-0.60573,\ 10.18142), \qquad \mathbf{v}_2 = (0.97986,\ 1.61258),$$

with flight-path angles $-3.40°$ at departure and $31.28°$ at arrival. Departing from the circular LEO speed 7.7258 km/s and arriving into the circular GEO speed 3.0747 km/s:

$\Delta v_1 = 2.52926, \quad \Delta v_2 = 1.76006, \quad \Delta v_{\text{total}} = 4.28932$ km/s
Hohmann: 3.89255 km/s in 5.275 h

Saving 1.3 hours of the Hohmann time of flight costs 0.40 km/s.

6.2 Example 2 — the same trip in two hours

Now $\Delta t = 2$ h < 2.238 h, so the answer must be a **hyperbola**, $p > p_{\text{ii}}$. Secant from $p = 11\,900$ and $12\,200$ km:

p [km]	Δt [h]	e
11900	2.1088	1.051
12200	1.4243	1.538
11947.69	1.9597	1.121
11928.68	2.0166	1.093
11934.21	1.9997	1.101
11934.10	**2.0000**	**1.10087**

The converged hyperbola has $f_1 = -44.363°$ and $f_2 = 130.637°$, both inside $f_\infty = 155.281°$. $H_1 = -0.179153$, $H_2 = 1.037685$, so $M_{h,1} = -0.019128$, $M_{h,2} = 0.321009$. With $a = -56\,316.04$ km and $n_h = 4.724118 \times 10^{-5}$ s^{-1}:

$$\frac{M_{h,2} - M_{h,1}}{n_h} = \frac{0.340137}{4.724118 \times 10^{-5}} = 7200 \text{ s} \checkmark$$

The velocities are $\mathbf{v}_1 = (-4.44847,\ 10.32782)$ and $\mathbf{v}_2 = (4.82797,\ 1.63577)$ km/s, and

$\Delta v_1 = 5.15360, \quad \Delta v_2 = 5.03783, \quad \Delta v_{\text{total}} = 10.19143$ km/s

Halving the flight time of Example 1 more than doubles the bill, from 4.289 to 10.191 km/s. Even the last 14 minutes below the parabola (2.238 h → 2 h) add 1.443 km/s to the parabola's 8.748 km/s, 16% more.

(lambert-verify.py reproduces both iteration tables and every number above.)

7 Where p-iteration struggles

7.1 Difficulty 1: three clocks for three conics

The time equation is **different** for $e < 1$, $e = 1$ and $e > 1$, and we do not know which applies until p is known.

- Starting guesses on one side of p_{ii} may converge to the other, so the code must switch formulas mid-iteration.
- Near $e = 1$ each formula is a ratio of vanishing quantities: $a \to \infty$, $n \to 0$, $\Delta M \to 0$.

Where the digits go. Look at the elliptic clock when $e \to 1$ at fixed p. Then $a = p/(1 - e^2) \to \infty$, and the anomalies shrink, $E \sim \sqrt{(1-e)/(1+e)}\, 2 \tan \frac{f}{2} \to 0$. The mean anomaly $M = E - e \sin E$ is a difference of two nearly equal numbers:

$$M = (1 - e)\, E + e\, (E - \sin E) \approx (1 - e)\, E + \frac{E^3}{6},$$

which is of order $(1-e)^{3/2}$ while each of E and $e\sin E$ is of order $(1-e)^{1/2}$. So **a relative cancellation of order** $1-e$: the relative error of Δt in double precision grows like $\varepsilon_{\text{mach}}/(1-e)$. Measured against 50-digit arithmetic, just below p_{ii} in the geometry of Example 1:

$1-e$	1.6×10^{-5}	1.6×10^{-8}	1.6×10^{-11}	1.6×10^{-13}
relative error in Δt	2.2×10^{-12}	1.6×10^{-9}	2.8×10^{-6}	3.0×10^{-4}

Digits leak away exactly where the physics is perfectly smooth: nothing special happens to the *trajectory* at $e=1$ (Equation 10 is smooth across it).

7.2 Difficulty 2: the window moves

The admissible interval (p_{i},∞) or $(0,p_{\text{ii}})$ depends on r_1, r_2 and $\Delta\theta$. A trial p outside it gives a conic that **does not join the two points** in the direction of motion (a hyperbolic arc would have to cross its asymptote). The algorithm must compute the bracket before it can take a single step.

7.3 Difficulty 3: the window pinches at 180°

At $\Delta\theta=\pi$, Equation 2 gives $e\cos f_1 = A$ and $-e\cos f_1 = B$, so $A+B=0$:

$$\frac{p}{r_1}+\frac{p}{r_2}=2 \quad\Longrightarrow\quad \boxed{p=p_H\equiv\frac{2r_1r_2}{r_1+r_2}}$$

the **harmonic mean** of the two radii: the p of the Hohmann ellipse, and of every other conic whose chord passes through the focus. The free parameter becomes $e\sin f_1$, and p can no longer label the family (Figure 5, right).

How fast does the window close? Near π, $k\to 2r_1r_2$, $c\to\ell$ and $\sqrt{2m}=2\sqrt{r_1r_2}\,|\cos\frac{\Delta\theta}{2}|\approx\sqrt{r_1r_2}\,(\pi-\Delta\theta)$, so from Equation 6

$$p_{\text{ii}}-p_{\text{i}}=\frac{2k\sqrt{2m}}{c^2}\approx\frac{4\,(r_1r_2)^{3/2}}{(r_1+r_2)^2}\,(\pi-\Delta\theta)\,:$$

linear in the distance from 180°. For the radii of Example 1 and $\Delta\theta=179.9^\circ$ the window is 13.83 km wide around $p_H=11\,530.08$ km (the formula gives 13.83), and the clock is almost vertical across it: $\Delta t=5$ h needs $p=11\,530.375$ km and $\Delta t=1.8$ h needs $p=11\,539.775$ km, a change of only 9.4 km (Figure 5, left, shows the same effect at 179.8° and below).

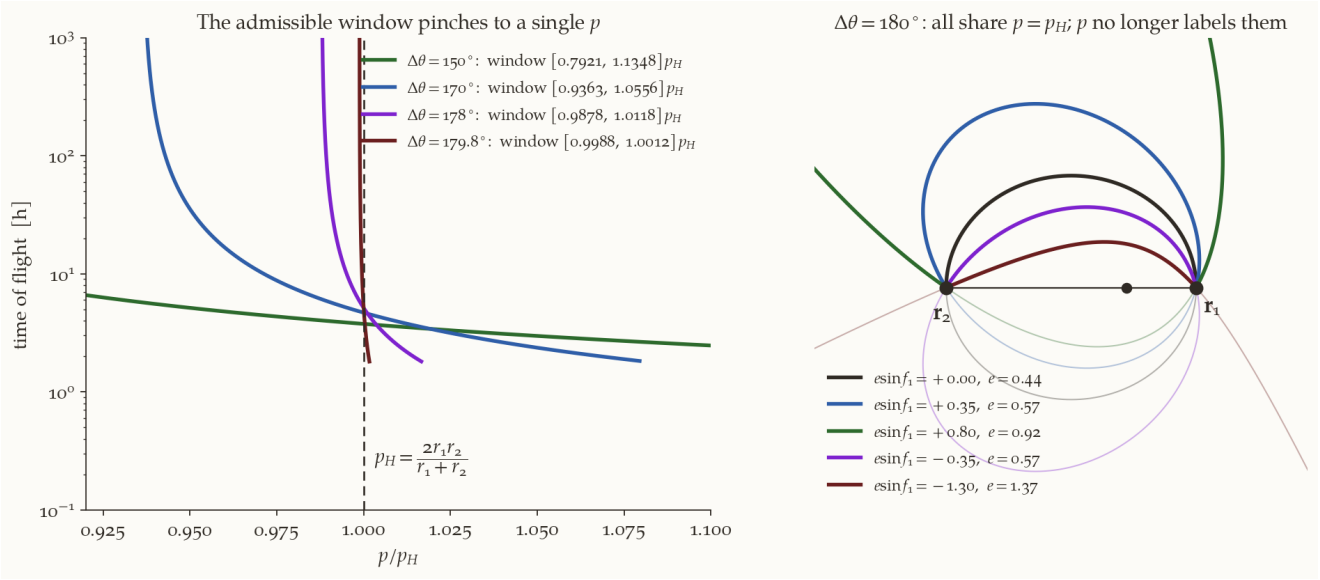


Figure 5: Left: the admissible window shrinks onto $p_H = 2r_1r_2/(r_1 + r_2)$ as $\Delta\theta \to 180°$ (curves for 150°, 170°, 178°, 179.8°), and the clock becomes almost vertical. Right: at exactly 180° every connecting conic has the same p; they differ only in $e\sin f_1$. *Script: fig-lam-180.py.*

> **Important**
>
> Near 180° the division by $\sin\Delta\theta$ in Equation 3 is the symptom. In the plane one can switch to iterating on $e\sin f_1$. In three dimensions the two position vectors no longer define the orbital plane at all: that singularity is **geometric**, and no choice of variable removes it. (A Hohmann transfer is exactly this case; one must say in which plane it is flown.)

7.4 Difficulty 4: more than one revolution

As shown in Section 4.6, the clock for $N \geq 1$ is not monotone: it has a minimum, and each Δt above it has **two** solutions. Bisection alone can no longer be trusted.

7.5 What we would like instead

Difficulties 1 and 2 are artefacts of the **variable**, not of the physics. We want:

- **one** iteration variable, defined for every conic;
- **one** time-of-flight equation, smooth straight through $e = 1$;
- an iteration range that does not have to be computed from the geometry first.

That is exactly what the **universal-variable** formulation provides: a single anomaly-like variable whose sign tells you the conic, and a single Kepler-like equation that reduces to the elliptic, parabolic and hyperbolic forms as special cases. It is the subject of the universal-variables note.

> **Note**
>
> Difficulties 3 and 4 do **not** go away with universal variables. The 180° case is geometric, and multiple revolutions always admit two solutions. A good formulation handles the avoidable problems so the unavoidable ones stand out.

8 Velocities without elements: the Lagrange coefficients

8.1 The idea: stay in the plane of $\mathbf{r}_1$ and $\mathbf{v}_1$

Section 5 found $\mathbf{v}_1$ by first computing e and f_1 (Equation 13 already hides them). We can skip that detour. The motion is planar, and $\mathbf{r}_1$, $\mathbf{v}_1$ are not parallel (unless the orbit is a straight line through the focus). So they form a **basis for the orbital plane**, and every later position and velocity is a combination of them:

$$\boxed{\mathbf{r}_2 = F\,\mathbf{r}_1 + G\,\mathbf{v}_1, \qquad \mathbf{v}_2 = \dot{F}\,\mathbf{r}_1 + \dot{G}\,\mathbf{v}_1} \tag{14}$$

- F, G, $\dot{F}$, $\dot{G}$ are scalars: the **Lagrange coefficients**.
- The second equation is the time derivative of the first ($\mathbf{r}_1$, $\mathbf{v}_1$ are fixed vectors; only the coefficients change as the end point moves), hence the dots.
- Units: F and $\dot{G}$ are dimensionless, G is a time, $\dot{F}$ is an inverse time.

The payoff. Once F and G are known, the first equation solves for the unknown velocity: $\mathbf{v}_1 = (\mathbf{r}_2 - F\,\mathbf{r}_1)/G$.

The goal. Express F, G, $\dot{G}$ using **only** r_1, r_2, $\Delta\theta$ **and** p.

8.2 Step 1: components in the frame at departure

Use the radial and transverse unit vectors **at** $\mathbf{r}_1$: $\hat{\mathbf{u}}_r$ along $\mathbf{r}_1$, and $\hat{\mathbf{u}}_\theta$ perpendicular to it, in the direction of motion. The end point sits at angle $\Delta\theta$, where the local unit vectors are

$$\begin{aligned}\hat{\mathbf{u}}_{r2} &= \cos\Delta\theta\,\hat{\mathbf{u}}_r + \sin\Delta\theta\,\hat{\mathbf{u}}_\theta,\\ \hat{\mathbf{u}}_{\theta 2} &= -\sin\Delta\theta\,\hat{\mathbf{u}}_r + \cos\Delta\theta\,\hat{\mathbf{u}}_\theta.\end{aligned}$$

vector	$\hat{\mathbf{u}}_r$ component	$\hat{\mathbf{u}}_\theta$ component
$\mathbf{r}_1$	r_1	0
$\mathbf{v}_1$	v_{r1}	$v_{\theta 1}$
$\mathbf{r}_2$	$r_2\cos\Delta\theta$	$r_2\sin\Delta\theta$
$\mathbf{v}_2 = v_{r2}\hat{\mathbf{u}}_{r2} + v_{\theta 2}\hat{\mathbf{u}}_{\theta 2}$	$v_{r2}\cos\Delta\theta - v_{\theta 2}\sin\Delta\theta$	$v_{r2}\sin\Delta\theta + v_{\theta 2}\cos\Delta\theta$

The one tool we need. For in-plane vectors the cross product is along the normal, $\mathbf{a}\times\mathbf{b} = (\mathbf{a}\wedge\mathbf{b})\,\hat{\mathbf{w}}$, with

$$\mathbf{a}\wedge\mathbf{b} \equiv a_r b_\theta - a_\theta b_r.$$

Two facts: $\mathbf{a}\wedge\mathbf{a} = 0$, and $\mathbf{r}_1\wedge\mathbf{v}_1 = r_1 v_{\theta 1} = h = \sqrt{\mu p}$ (likewise $r_2 v_{\theta 2} = h$).

8.3 Step 2: isolate G and F

Wedging with a vector deletes its own term. Wedge $\mathbf{r}_1$ into Equation 14 from the left:

$$\begin{aligned}\mathbf{r}_1\wedge\mathbf{r}_2 &= F\underbrace{\mathbf{r}_1\wedge\mathbf{r}_1}_{0} + G\underbrace{\mathbf{r}_1\wedge\mathbf{v}_1}_{h}\\ \Longrightarrow\qquad G &= \frac{\mathbf{r}_1\wedge\mathbf{r}_2}{h} = \frac{r_1\cdot r_2\sin\Delta\theta - 0\cdot r_2\cos\Delta\theta}{h},\end{aligned}$$

$$\boxed{G = \frac{r_1 r_2 \sin\Delta\theta}{\sqrt{\mu p}}} \tag{15}$$

Now wedge Equation 14 with $\mathbf{v}_1$ from the right:

$$\begin{aligned}
\mathbf{r}_2 \wedge \mathbf{v}_1 &= F\,\underbrace{\mathbf{r}_1 \wedge \mathbf{v}_1}_{h} + G\,\underbrace{\mathbf{v}_1 \wedge \mathbf{v}_1}_{0} \\
\Longrightarrow \qquad F &= \frac{\mathbf{r}_2 \wedge \mathbf{v}_1}{h} = \frac{r_2\cos\Delta\theta\, v_{\theta 1} - r_2 \sin\Delta\theta\, v_{r1}}{r_1 v_{\theta 1}}, \\
F &= \frac{r_2}{r_1}\Big[\cos\Delta\theta - \frac{v_{r1}}{v_{\theta 1}}\sin\Delta\theta\Big].
\end{aligned}$$

G is finished. F still contains the **unknown** velocity ratio $v_{r1}/v_{\theta 1}$; Step 3 removes it.

8.4 Step 3: eliminate the velocity ratio from F

From Equation 12 and $1 + e\cos f_1 = p/r_1$, and from Equation 3:

$$\frac{v_{r1}}{v_{\theta 1}} = \frac{e\sin f_1}{1 + e\cos f_1} = \frac{r_1}{p}\, e\sin f_1, \qquad e\sin f_1 = \frac{A\cos\Delta\theta - B}{\sin\Delta\theta}.$$

Multiply the two, so the $\sin\Delta\theta$ cancels, and substitute $A = p/r_1 - 1$, $B = p/r_2 - 1$:

$$\begin{aligned}
\frac{v_{r1}}{v_{\theta 1}}\sin\Delta\theta &= \frac{r_1}{p}\Big[\Big(\frac{p}{r_1} - 1\Big)\cos\Delta\theta - \Big(\frac{p}{r_2} - 1\Big)\Big] \\
&= \cos\Delta\theta - \frac{r_1}{p}\cos\Delta\theta - \frac{r_1}{r_2} + \frac{r_1}{p} \\
&= \cos\Delta\theta - \frac{r_1}{r_2} + \frac{r_1}{p}(1 - \cos\Delta\theta).
\end{aligned}$$

Put this into the bracket of F. The two $\cos\Delta\theta$ terms cancel:

$$F = \frac{r_2}{r_1}\Big[\cos\Delta\theta - \cos\Delta\theta + \frac{r_1}{r_2} - \frac{r_1}{p}(1 - \cos\Delta\theta)\Big],$$

so that

$$\boxed{F = 1 - \frac{r_2}{p}(1 - \cos\Delta\theta)} \tag{16}$$

> **Important**
>
> e and f_1 entered through $e\sin f_1$ and have **cancelled completely**.

8.5 Step 4: isolate $\dot{G}$

Apply the same trick to the velocity equation in Equation 14. Wedge $\mathbf{r}_1$ in from the left:

$$\begin{aligned} \mathbf{r}_1 \wedge \mathbf{v}_2 &= \dot{F}\,\underbrace{\mathbf{r}_1 \wedge \mathbf{r}_1}_{0} + \dot{G}\,\underbrace{\mathbf{r}_1 \wedge \mathbf{v}_1}_{h} \\ \Longrightarrow \qquad \dot{G} &= \frac{\mathbf{r}_1 \wedge \mathbf{v}_2}{h}. \end{aligned}$$

From the table, $\mathbf{r}_1 \wedge \mathbf{v}_2 = r_1(v_{r2}\sin\Delta\theta + v_{\theta 2}\cos\Delta\theta)$, and this time use $h = r_2 v_{\theta 2}$:

$$\dot{G} = \frac{r_1(v_{r2}\sin\Delta\theta + v_{\theta 2}\cos\Delta\theta)}{r_2 v_{\theta 2}} = \frac{r_1}{r_2}\Big[\cos\Delta\theta + \frac{v_{r2}}{v_{\theta 2}}\sin\Delta\theta\Big].$$

As in Step 2, one unknown remains: the velocity ratio **at the arrival point**,

$$\frac{v_{r2}}{v_{\theta 2}} = \frac{e\sin f_2}{1 + e\cos f_2} = \frac{r_2}{p}\,e\sin f_2, \qquad f_2 = f_1 + \Delta\theta.$$

8.6 Step 5: eliminate the velocity ratio from $\dot{G}$

Equation 4 already gives $e\sin f_2 = (A - B\cos\Delta\theta)/\sin\Delta\theta$. Then, exactly as in Step 3,

$$\begin{aligned} \frac{v_{r2}}{v_{\theta 2}}\sin\Delta\theta &= \frac{r_2}{p}\Big[\Big(\frac{p}{r_1} - 1\Big) - \Big(\frac{p}{r_2} - 1\Big)\cos\Delta\theta\Big] \\ &= \frac{r_2}{r_1} - \cos\Delta\theta - \frac{r_2}{p}(1 - \cos\Delta\theta), \end{aligned}$$

$$\dot{G} = \frac{r_1}{r_2}\Big[\cos\Delta\theta + \frac{r_2}{r_1} - \cos\Delta\theta - \frac{r_2}{p}(1 - \cos\Delta\theta)\Big],$$

so that

$$\boxed{\dot{G} = 1 - \frac{r_1}{p}(1 - \cos\Delta\theta)} \tag{17}$$

Compare with Equation 16: F and $\dot{G}$ are mirror images, with $r_1 \leftrightarrow r_2$.

8.7 Step 6: the fourth coefficient, from angular momentum

Angular momentum is the same at both ends: $\mathbf{r}_2 \times \mathbf{v}_2 = \mathbf{r}_1 \times \mathbf{v}_1 = h\hat{\mathbf{w}}$. Substitute Equation 14 and expand the four terms:

$$\begin{aligned} \mathbf{r}_2 \wedge \mathbf{v}_2 &= (F\mathbf{r}_1 + G\mathbf{v}_1) \wedge (\dot{F}\mathbf{r}_1 + \dot{G}\mathbf{v}_1) \\ &= F\dot{F}\,\underbrace{\mathbf{r}_1 \wedge \mathbf{r}_1}_{0} + F\dot{G}\,\underbrace{\mathbf{r}_1 \wedge \mathbf{v}_1}_{h} + G\dot{F}\,\underbrace{\mathbf{v}_1 \wedge \mathbf{r}_1}_{-h} + G\dot{G}\,\underbrace{\mathbf{v}_1 \wedge \mathbf{v}_1}_{0} \\ &= (F\dot{G} - \dot{F}G)\,h. \end{aligned}$$

Setting this equal to h:

$$\boxed{F\dot{G} - \dot{F}G = 1} \tag{18}$$

So $\dot F = (F\dot G - 1)/G$. With $\kappa = 1-\cos\Delta\theta$, $F\dot G = (1-r_2\kappa/p)(1-r_1\kappa/p)$, so $F\dot G - 1 = r_1 r_2\kappa^2/p^2 - (r_1+r_2)\kappa/p$. Dividing by Equation 15, and using $\kappa/\sin\Delta\theta = \tan\frac{\Delta\theta}{2}$:

$$\dot F = \sqrt{\frac{\mu}{p}}\,\tan\frac{\Delta\theta}{2}\Big[\frac{1-\cos\Delta\theta}{p} - \frac{1}{r_1} - \frac{1}{r_2}\Big].$$

8.8 Both velocities from F, G, $\dot G$

Solve the position equation of Equation 14 for $\mathbf{v}_1$, then substitute it into the velocity equation:

$$\mathbf{v}_1 = \frac{\mathbf{r}_2 - F\,\mathbf{r}_1}{G}, \qquad \mathbf{v}_2 = \dot F\,\mathbf{r}_1 + \dot G\,\frac{\mathbf{r}_2 - F\,\mathbf{r}_1}{G} = \frac{\dot G\,\mathbf{r}_2 - \overbrace{(F\dot G - \dot F G)}^{=1}\,\mathbf{r}_1}{G}.$$

$$\boxed{\mathbf{v}_1 = \frac{\mathbf{r}_2 - F\,\mathbf{r}_1}{G}, \qquad \mathbf{v}_2 = \frac{\dot G\,\mathbf{r}_2 - \mathbf{r}_1}{G}} \tag{19}$$

Thanks to Equation 18, $\dot F$ is not needed at all.

> **Note**
>
> These are vector equations in **any** frame: $\mathbf{r}_1$ and $\mathbf{r}_2$ can be given in inertial components, and $\mathbf{v}_1$, $\mathbf{v}_2$ come out in the same components. No rotation to local radial/transverse axes is needed.

8.9 Check: Example 1 again

$p = 11\,598.163$ km, $\Delta\theta = 175^\circ$, $1-\cos\Delta\theta = 1.996195$:

$$\begin{aligned}
F &= 1 - \frac{42\,164}{11\,598.163}(1.996195) = -6.25697,\\
G &= \frac{6678.137 \times 42\,164\sin 175^\circ}{\sqrt{\mu \times 11\,598.163}} = 360.936\ \text{s},\\
\dot G &= 1 - \frac{6678.137}{11\,598.163}(1.996195) = -0.14939,\\
\dot F &= -1.8076\times10^{-4}\ \text{s}^{-1}, \qquad F\dot G - \dot F G = 1.000000\ \checkmark
\end{aligned}$$

With $\mathbf{r}_1 = (6678.137,\ 0)$ and $\mathbf{r}_2 = (-42\,003.553,\ 3674.835)$ km in inertial components:

$$\mathbf{v}_1 = \frac{\mathbf{r}_2 - F\mathbf{r}_1}{G} = \frac{(-218.631,\ 3674.835)}{360.936} = (-0.60573,\ 10.18142)\ \text{km/s},$$

$$\mathbf{v}_2 = \frac{\dot G\mathbf{r}_2 - \mathbf{r}_1}{G} = \frac{(-403.048,\ -548.999)}{360.936} = (-1.11668,\ -1.52104)\ \text{km/s},$$

identical to Example 1 (the (v_r, v_θ) pair $(0.97986, 1.61258)$ at angle 175° is the inertial $(-1.11668, -1.52104)$) **without computing e, f_1, or a single anomaly**.

8.10 What the Lagrange coefficients buy

- **No elements.** Only r_1, r_2, $\Delta\theta$ and p appear. The periapsis, e and the individual true anomalies are never needed.
- **Near-circular orbits are safe.** As $e \to 0$, $f_1 = \operatorname{atan2}(e\sin f_1, e\cos f_1)$ is numerically meaningless, but $\Delta\theta$ is always well defined.
- **One formula for every conic.** Equation 16 to Equation 19 hold unchanged for ellipses, parabolas and hyperbolas; no branching.
- **The 180° singularity is still there.** $G \propto \sin\Delta\theta$ vanishes, and Equation 19 divides by G. As Section 7.3 said, no formulation removes it.
- **The same coefficients as in the perifocal note.** There, the last section of the perifocal note derived $F = 1 - \frac{r}{p}(1-\cos\Delta f)$ and $G = \frac{rr_0}{\sqrt{\mu p}}\sin\Delta f$ for a step Δf in true anomaly from the perifocal state vector. With $r_0 \to r_1$, $r \to r_2$, $\Delta f \to \Delta\theta$ they are Equation 16 and Equation 15. Two derivations, one result.

> Important
>
> The same coefficients also propagate a state: given $(\mathbf{r}_1, \mathbf{v}_1)$ and a time, Equation 14 gives $(\mathbf{r}_2, \mathbf{v}_2)$ once F, G are expressed in terms of time. Doing that for every conic at once is the job of **universal variables**.

9 Summary

1. Two points and a focus leave a **one-parameter family** of conics; the time of flight selects one.
2. Labelling the family by p gives e and f_1 **in closed form** (Equation 3), and $e^2 = 1 + c^2(p-p_{\mathrm{i}})(p-p_{\mathrm{ii}})/(r_1 r_2 \sin\Delta\theta)^2$ (Equation 7): the conic type, the window, and the minimum eccentricity $|r_2 - r_1|/c$ all follow.
3. The clock is Kepler's equation evaluated twice, elliptic or hyperbolic, with Barker's equation (and Euler's closed form) on the parabola between.
4. For zero revolutions the clock is **monotone** in p over the admissible window, so **every** $\Delta t > 0$ **has exactly one solution**, and bisection always converges; the secant method converges in under ten steps. With $N \geq 1$ extra revolutions there is a minimum time and then **two** solutions.
5. The method suffers from **three clocks** (with a loss of precision $\sim \varepsilon/(1-e)$ near $e = 1$), a **moving window**, a **pinch at** 180° (window width $\propto \pi - \Delta\theta$) and **non-uniqueness for** $N \geq 1$. The first two motivate universal variables.
6. The **Lagrange coefficients** give $\mathbf{v}_1 = (\mathbf{r}_2 - F\mathbf{r}_1)/G$ and $\mathbf{v}_2 = (\dot{G}\mathbf{r}_2 - \mathbf{r}_1)/G$ from r_1, r_2, $\Delta\theta$ and p alone.

Everything in this note can be reproduced with lambert-verify.py: the symbolic checks of the closed forms, the half-angle relations and the Lagrange coefficients; both worked examples; the admissibility, monotonicity and multiple-revolution results; and the loss of precision near $e = 1$.

Further reading

For the classical treatments, Battin (1999) and Bate, Mueller, and White (1971); for relative merits of modern solvers, Gooding (1990), Battin and Vaughan (1984), Lancaster and Blanchard (1969) and Izzo (2015); and for orbit-determination practice, Vallado (2013) and Curtis (2014).

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