Deriving the Gauss Variational Equations

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November 21, 2026

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1 Where we are going

A satellite is never on a perfect Kepler orbit. The Earth's bulge, the air, the Sun's light and its own engine all push on it a little. The *Gauss variational equations* say how fast each orbital element changes under such a push: which direction of force moves which element, by how much, and where on the orbit. They are the tool behind every result about slow orbit change in this course, from the drift of a sun-synchronous orbit's plane to the cost of a small correction burn.

You need first the two-body note (the integrals $\mathbf{h}$, $\mathbf{e}$ and ε, the radial and transverse velocity, the eccentric anomaly), the perifocal note (the triad $\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{w}}$ and the 3–1–3 rotation) and the orbital-elements note (the six elements and the orbits where they break down). The cw note helps with the frame. The gravity note supplies the J_2 force used in the worked check, and uses this note's results in turn. Symbols follow the notation page.

When you finish, you should be able to answer the following.

1. Why does a perturbing force change the elements only through their gradients with respect to the velocity?
2. Which force component changes a, which changes the plane, and which turns the apse line?
3. Why is a plane change cheapest far from the planet, and an inclination change cheapest at a node?
4. Why do ω, Ω and M have equations that blow up at $e = 0$ or $i = 0$, and what replaces them?
5. How does the J_2 regression of the node, and the rotation of the apse line, come out of these equations by averaging over one orbit?

The problem is the perturbed two-body equation, with the perturbing acceleration $\mathbf{a}_d$ resolved in the RSW frame of Section 3:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\,\mathbf{r} + \mathbf{a}_d, \qquad \mathbf{a}_d = a_R\,\hat{\mathbf{R}} + a_S\,\hat{\mathbf{S}} + a_W\,\hat{\mathbf{W}}. \tag{1}$$

The result has the structure below. The 6×3 matrix $\mathbf{B}$ depends only on the current elements, and only the mean anomaly carries a Keplerian drift; Section 10 lists $\mathbf{B}$ in full.

$$\frac{d}{dt}\begin{pmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ M \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix} + \mathbf{B}(a, e, i, \omega, M)\begin{pmatrix} a_R \\ a_S \\ a_W \end{pmatrix}.$$

History. The idea of letting the constants of an unperturbed solution vary goes back to Euler's 1748 paper on the mutual perturbations of Jupiter and Saturn. Lagrange used the method of variation of parameters in his mémoires of 1778 and gave the planetary equations, written through derivatives of a disturbing potential with respect to the elements, their final form in 1808–1810. The form written with the force components instead, which also covers drag and thrust, is usually attributed to Gauss. An elementary derivation in the spirit of this note is J. A. Burns, *Am. J. Phys.* **44**, 944–949 (1976).

The Gauss form needs only the force, so it covers drag, radiation pressure and thrust as readily as gravity. Lagrange's planetary equations need a disturbing potential; when one exists, writing the force as its gradient turns the Gauss form into Lagrange's (Section 12.2).

The derivation has three steps.

1. Show that a perturbation changes any Keplerian integral only through that integral's velocity gradient (Section 2.2).
2. For each element, differentiate the state function whose velocity gradient is simplest:

state function	gives	force components that act
angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$	h, p, i, Ω	a_S for h and p; a_W for i and Ω
energy $\varepsilon = v^2/2 - \mu/r$	a	the power $\mathbf{v} \cdot \mathbf{a}_d$
the pair $(e\cos f,\ e\sin f)$	e and f	a_R and a_S
the difference $u - f$	ω	all three
Kepler's equation $M(f, e)$	M	a_R and a_S

3. Cross-check the results against the eccentricity vector, a J_2 orbit average and a numerical test (Section 14).

2 Osculating elements and the fundamental rule

A perturbing force changes a Keplerian integral only through that integral's gradient with respect to velocity. This one rule generates every Gauss equation.

2.1 Variation of parameters

Without the perturbation, the motion is a conic fixed by six constants, the element vector α:

$$\mathbf{r} = \phi(t;\alpha), \qquad \mathbf{v} = \psi(t;\alpha) \equiv \frac{\partial \phi}{\partial t}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\mu}{\phi^3}\,\phi.$$

With the perturbation on, keep the same functional form but let the constants vary:

$$\mathbf{r}(t) = \phi(t;\alpha(t)) \quad \Longrightarrow \quad \dot{\mathbf{r}} = \psi + \frac{\partial \phi}{\partial \alpha}\,\dot{\alpha}.$$

Six unknown functions against three equations of motion leave three conditions free. The *osculating* choice demands that the velocity also keep its Keplerian form, which is Lagrange's constraint:

$$\frac{\partial \phi}{\partial \alpha}\,\dot{\alpha} = \mathbf{0} \quad \Longleftrightarrow \quad \dot{\mathbf{r}} = \psi(t;\alpha(t)).$$

Differentiate the velocity once more and substitute the equation of motion. The Keplerian acceleration cancels on both sides:

$$\ddot{\mathbf{r}} = \underbrace{\frac{\partial \psi}{\partial t}}_{-\mu \mathbf{r}/r^3} + \frac{\partial \psi}{\partial \alpha}\,\dot{\alpha} \quad \Longrightarrow \quad \frac{\partial \psi}{\partial \alpha}\,\dot{\alpha} = \mathbf{a}_d .$$

Stacking the two conditions gives a 6×6 linear system whose matrix is the Jacobian of the state with respect to the elements. Its inverse is the Jacobian of the elements with respect to the state, and the zero block kills the position gradient:

$$\begin{bmatrix} \partial \phi / \partial \alpha \\ \partial \psi / \partial \alpha \end{bmatrix} \dot{\alpha} = \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_d \end{bmatrix} \quad \Longrightarrow \quad \boxed{\dot{\alpha} = \frac{\partial \alpha}{\partial \mathbf{r}}\,\mathbf{0} + \frac{\partial \alpha}{\partial \mathbf{v}}\,\mathbf{a}_d = \frac{\partial \alpha}{\partial \mathbf{v}}\,\mathbf{a}_d} \tag{2}$$

The inversion needs the Jacobian to be non-singular. It is singular exactly where an element is undefined, at $e = 0$ (no periapsis) and at $i = 0$ (no node), and that is where the equations below will blow up (Section 12.3).

The three osculating conditions are a *gauge choice*, not a law of physics. They make the elements describe the conic tangent to the true path: the orbit the body would fly if the perturbation vanished at that instant. With this choice the perturbation acts as if it changed the velocity but never the position. Other gauges are legitimate, but give non-osculating elements.

2.2 The fundamental rule

The same result holds for any smooth state function $g(\mathbf{r}, \mathbf{v})$. The chain rule along the true path splits its rate into a Keplerian part and a perturbation part:

$$\dot{g} = \underbrace{\frac{\partial g}{\partial \mathbf{r}} \cdot \mathbf{v} - \frac{\mu}{r^3} \frac{\partial g}{\partial \mathbf{v}} \cdot \mathbf{r}}_{\mathcal{K} g} + \underbrace{\frac{\partial g}{\partial \mathbf{v}} \cdot \mathbf{a}_d}_{\mathcal{D} g} . \tag{3}$$

For a Keplerian integral (a, e, i, Ω, ω, the angular momentum, the energy) the Keplerian part vanishes, so the rate is just $\mathcal{D} g$. The fast angles keep a known Keplerian part:

$$\mathcal{K} f = \mathcal{K} u = \frac{h}{r^2}, \qquad \mathcal{K} M = n.$$

The operator $\mathcal{D}$ is linear and obeys the product and chain rules. Its action on the basic quantities is

$$\mathcal{D}\mathbf{r} = \mathbf{0}, \qquad \mathcal{D}\mathbf{v} = \mathbf{a}_d, \qquad \mathcal{D}\, G(\mathbf{r}) = 0 \ \text{ for any function of position alone.}$$

Physically, $\mathcal{D} g\, dt$ is the change in g caused by the velocity increment $\mathbf{a}_d\, dt$ applied at a fixed position. The Gauss equations are therefore the first-order impulsive-manoeuvre equations of the impulsive-model note divided by dt, and a small impulse changes any element by

$$\Delta \alpha \simeq \frac{\partial \alpha}{\partial \mathbf{v}} \cdot \Delta \mathbf{v}. \tag{4}$$

Nothing here assumes the force is conservative or independent of velocity, so drag and thrust are covered. The rest of the derivation evaluates $\mathcal{D}$ on well-chosen state functions.

3 Frames, notation and identities

Forces are resolved in the **RSW** frame, which rides with the body: $\hat{\mathbf{R}}$ along the radius, $\hat{\mathbf{W}}$ along the angular momentum, and $\hat{\mathbf{S}}$ completing the triad:

$$\hat{\mathbf{R}} = \frac{\mathbf{r}}{r}, \qquad \hat{\mathbf{W}} = \frac{\mathbf{h}}{h} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|}, \qquad \hat{\mathbf{S}} = \hat{\mathbf{W}}\times\hat{\mathbf{R}}.$$

This is the frame of the cw note ($\hat{\mathbf{R}} = \hat{\mathbf{i}}$, $\hat{\mathbf{S}} = \hat{\mathbf{j}}$, $\hat{\mathbf{W}} = \hat{\mathbf{k}}$, there attached to a target) and the polar basis $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{h}}$ of the two-body note; it is also called RTN or the Hill frame. Note that $\hat{\mathbf{S}}$ is *transverse*, perpendicular to the radius. It lies along the velocity only on a circular orbit, or at an apsis; elsewhere the velocity is tilted from it by the flight-path angle γ (Figure 1, left, and the TNW frame of Section 11).

The triad is right-handed, so cross products cycle $R \to S \to W$:

$$\hat{\mathbf{R}}\times\hat{\mathbf{S}} = \hat{\mathbf{W}}, \qquad \hat{\mathbf{S}}\times\hat{\mathbf{W}} = \hat{\mathbf{R}}, \qquad \hat{\mathbf{W}}\times\hat{\mathbf{R}} = \hat{\mathbf{S}}.$$

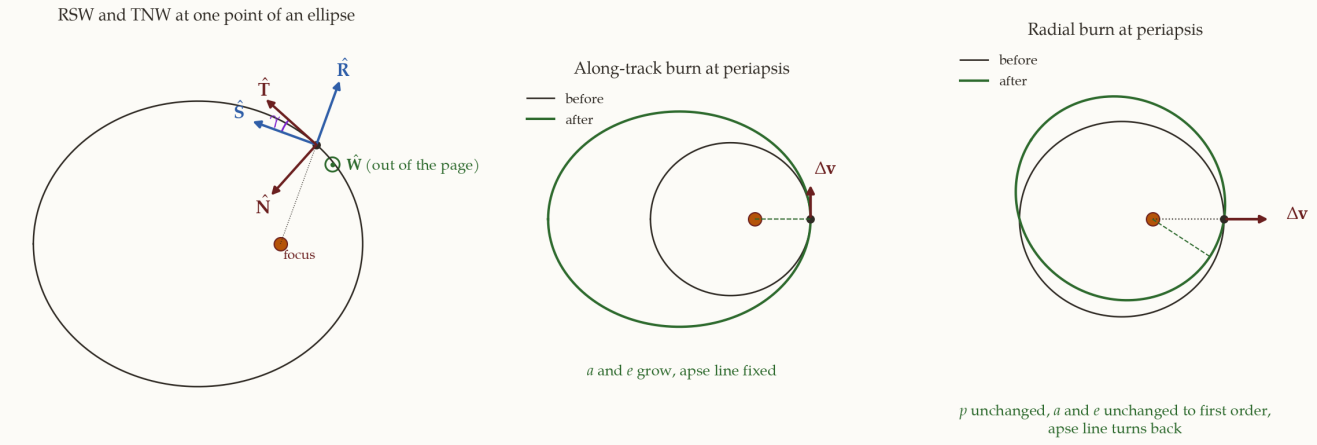


Figure 1: **Left:** the RSW frame (blue) and the TNW frame (maroon) at one point of an ellipse; $\hat{\mathbf{W}}$ points out of the page, and $\hat{\mathbf{T}}$ is tilted from $\hat{\mathbf{S}}$ by the flight-path angle γ. **Centre:** an along-track impulse at periapsis, to scale: a and e grow and the apse line stays put. **Right:** a radial impulse at periapsis: p is unchanged, a and e are unchanged to first order, and the apse line turns backwards, leaving the body just past its new periapsis (Section 12). *Script: fig-gve-frames.py*.

3.1 Notation

The elements are a, e, i, Ω, ω and M, with true anomaly f and eccentric anomaly E. The derived quantities are

$u = \omega + f$		argument of latitude
$p = a\,(1-e^2) = h^2/\mu$		semi-latus rectum
$n = \sqrt{\mu/a^3}$		mean motion
$b = a\sqrt{1-e^2},$	$h = n\,a\,b$	semi-minor axis

3.2 Unit vectors

With $\hat{\mathbf{K}}$ the inertial polar axis, $\hat{\mathbf{N}}$ the ascending-node direction and $\hat{\mathbf{M}}$ its in-plane companion 90° ahead (the names of the perifocal note), the inertial components are

$$\hat{\mathbf{N}} = (\cos\Omega,\ \sin\Omega,\ 0), \qquad \hat{\mathbf{M}} = \hat{\mathbf{W}} \times \hat{\mathbf{N}} = (-\cos i \sin\Omega,\ \cos i \cos\Omega,\ \sin i),$$
$$\hat{\mathbf{W}} = (\sin i \sin\Omega,\ -\sin i \cos\Omega,\ \cos i).$$

The radial and transverse directions sit at angle u from the node, and the perifocal directions $\hat{\mathbf{p}}$ (to periapsis) and $\hat{\mathbf{q}}$ sit at angle ω:

$$\hat{\mathbf{R}} = \cos u\, \hat{\mathbf{N}} + \sin u\, \hat{\mathbf{M}}, \qquad \hat{\mathbf{S}} = -\sin u\, \hat{\mathbf{N}} + \cos u\, \hat{\mathbf{M}},$$
$$\hat{\mathbf{p}} = \cos\omega\, \hat{\mathbf{N}} + \sin\omega\, \hat{\mathbf{M}} = \cos f\, \hat{\mathbf{R}} - \sin f\, \hat{\mathbf{S}}, \quad \hat{\mathbf{q}} = -\sin\omega\, \hat{\mathbf{N}} + \cos\omega\, \hat{\mathbf{M}} = \sin f\, \hat{\mathbf{R}} + \cos f\, \hat{\mathbf{S}}.$$

Projecting onto the polar axis gives the three components used for i, Ω and u:

$$\hat{\mathbf{K}}\cdot\hat{\mathbf{R}} = \sin i \sin u, \qquad \hat{\mathbf{K}}\cdot\hat{\mathbf{S}} = \sin i \cos u, \qquad \hat{\mathbf{K}}\cdot\hat{\mathbf{W}} = \cos i, \qquad \hat{\mathbf{K}} = \cos i\, \hat{\mathbf{W}} + \sin i\, \hat{\mathbf{M}}.$$

3.3 Orbit geometry and velocity

On the osculating conic, the radius and the velocity components are

$$r = \frac{p}{1 + e\cos f}, \qquad \mathbf{v} = v_R\, \hat{\mathbf{R}} + v_S\, \hat{\mathbf{S}}, \qquad v_R = \frac{\mu}{h}\, e \sin f, \qquad v_S = \frac{h}{r} = \frac{\mu}{h}\,(1 + e\cos f).$$

The energy integral and the eccentric-anomaly relations complete the toolkit:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \qquad r = a\,(1 - e\cos E), \qquad \cos E = \frac{e + \cos f}{1 + e\cos f}, \qquad \sin E = \frac{\sqrt{1 - e^2}\,\sin f}{1 + e\cos f}.$$

3.4 Three identities

Three identities do most of the algebra. (I1) is the orbit equation rearranged, (I2) restates the definition of p, and (I3) follows from the expression for $\cos E$:

$$p - r = r\, e \cos f \qquad \text{(I1)}$$
$$\frac{h}{\mu} = \frac{p}{h} \qquad \text{(I2)}$$
$$\frac{r}{p}\,(e + \cos f) = \cos E \qquad \text{(I3)}$$

4 Angular momentum: h, p and the orbit normal

Only the along-track force changes the size of the angular momentum, and only the cross-track force tilts it.

Angular momentum is a Keplerian integral, so its rate is $\mathcal{D}$ of it. Position is untouched by $\mathcal{D}$, which leaves the torque of the perturbing force:

$$\dot{\mathbf{h}} = \mathcal{D}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \mathcal{D}\mathbf{v} = \mathbf{r} \times \mathbf{a}_d = r\, \hat{\mathbf{R}} \times (a_R\, \hat{\mathbf{R}} + a_S\, \hat{\mathbf{S}} + a_W\, \hat{\mathbf{W}}) = r a_S\, \hat{\mathbf{W}} - r a_W\, \hat{\mathbf{S}}.$$

Projecting on the normal gives the magnitude; the remainder gives the rotation of the orbit normal:

$$\boxed{\dot{h} = \hat{\mathbf{W}} \cdot \dot{\mathbf{h}} = r a_S} \qquad \dot{\hat{\mathbf{W}}} = \frac{\dot{\mathbf{h}} - \dot{h}\,\hat{\mathbf{W}}}{h} = -\frac{r a_W}{h}\,\hat{\mathbf{S}}. \tag{5}$$

The semi-latus rectum follows from its definition:

$$p = \frac{h^2}{\mu} \quad \Longrightarrow \quad \boxed{\dot{p} = \frac{2h}{\mu}\,\dot{h} = 2r\sqrt{\frac{p}{\mu}}\,a_S} \tag{6}$$

A radial force exerts no torque, so it cannot change h or p. It reshapes the orbit only by trading a against e at fixed p. The normal's rotation feeds the inclination and node equations of Section 7.

5 Energy and the semi-major axis

The semi-major axis changes only through the power the perturbation delivers, so only the velocity-aligned part of the force matters.

The specific energy is a Keplerian integral whose potential term depends on position alone. Applying $\mathcal{D}$ gives the work–energy theorem:

$$\varepsilon = \tfrac{1}{2}\,\mathbf{v}\cdot\mathbf{v} - \frac{\mu}{r} \quad \Longrightarrow \quad \dot{\varepsilon} = \mathcal{D}\varepsilon = \mathbf{v}\cdot\mathcal{D}\mathbf{v} = \mathbf{v}\cdot\mathbf{a}_d.$$

In RSW components, using the velocity components of the osculating conic,

$$\dot{\varepsilon} = v_R a_R + v_S a_S = \frac{\mu}{h}\left[e\sin f\, a_R + (1 + e\cos f)\, a_S\right] = \frac{\mu}{h}\left[e\sin f\, a_R + \frac{p}{r}\,a_S\right].$$

The energy fixes the semi-major axis, so

$$a = -\frac{\mu}{2\varepsilon} \quad \Longrightarrow \quad \dot{a} = \frac{\mu}{2\varepsilon^2}\,\dot{\varepsilon} = \frac{2a^2}{\mu}\,\mathbf{v}\cdot\mathbf{a}_d,$$

$$\boxed{\dot{a} = \frac{2a^2}{h}\left(e\sin f\, a_R + \frac{p}{r}\,a_S\right) = \frac{2}{n\sqrt{1-e^2}}\left(e\sin f\, a_R + \frac{p}{r}\,a_S\right)} \tag{7}$$

The second form uses $h = nab$. On a circular orbit the radial term vanishes and only the along-track force changes a: for $e = 0$, $\dot{a} = (2/n)\,a_S$.

The mean motion follows from a, and its change is what makes the mean anomaly drift later:

$$n = \sqrt{\mu/a^3} \quad \Longrightarrow \quad \dot{n} = -\frac{3n}{2a}\,\dot{a}.$$

6 Eccentricity and true anomaly

Treating $e\cos f$ and $e\sin f$ as the Cartesian coordinates of one point makes the eccentricity and true-anomaly equations fall out together.

6.1 The two state functions

Both coordinates are functions of the state. The first is the orbit equation solved for $e\cos f$; the second comes from the radial velocity:

$$e\cos f = \frac{h^2}{\mu r} - 1, \qquad e\sin f = \frac{h\,v_R}{\mu}, \qquad v_R = \hat{\mathbf{R}}\cdot\mathbf{v}.$$

The radius and the radial unit vector depend on position alone, and h was handled in Section 4:

$$\mathcal{D}r = 0, \qquad \mathcal{D}v_R = \hat{\mathbf{R}}\cdot\mathbf{a}_d = a_R, \qquad \mathcal{D}h = r a_S,$$

$$\mathcal{D}(e\cos f) = \frac{2h}{\mu r}\,\mathcal{D}h = \frac{2h}{\mu}\,a_S, \qquad \mathcal{D}(e\sin f) = \frac{v_R\,\mathcal{D}h + h\,\mathcal{D}v_R}{\mu} = \frac{h a_R + r v_R a_S}{\mu}.$$

6.2 Polar decomposition

Rotating the variation of the pair by $-f$ separates its radial part (the change in e) from its angular part (the change in f):

$$\mathcal{D}e = \cos f\,\mathcal{D}(e\cos f) + \sin f\,\mathcal{D}(e\sin f), \qquad e\,\mathcal{D}f = \cos f\,\mathcal{D}(e\sin f) - \sin f\,\mathcal{D}(e\cos f).$$

6.3 Eccentricity

Substitute the two variations, the radial velocity from the osculating conic, and (I2):

$$\dot e = \frac{2h\cos f}{\mu}\,a_S + \frac{\sin f\,(h a_R + r v_R a_S)}{\mu} = \frac{1}{h}\Big[p\sin f\,a_R + (2p\cos f + re\sin^2 f)a_S\Big].$$

Identity (I1) puts the along-track coefficient into its standard form:

$$2p\cos f + re\sin^2 f = 2p\cos f + re - \underbrace{re\cos f}_{p-r}\cos f = (p+r)\cos f + re,$$

$$\boxed{\dot e = \frac{1}{h}\Big\{p\sin f\,a_R + [(p+r)\cos f + re]a_S\Big\} = \sqrt{\frac{p}{\mu}}\Big[\sin f\,a_R + (\cos f + \cos E)\,a_S\Big]} \tag{8}$$

The second form uses (I3). Unlike the angle rates below, $\dot e$ stays finite as $e \to 0$. It still depends on the true anomaly, which is undefined there, so near-circular orbits are better followed with the pair $(e\cos\varpi,\ e\sin\varpi)$ of Section 12.4.

6.4 True anomaly

The angular part uses the same substitutions, and (I1) collapses the along-track coefficient:

$$e\,\mathcal{D}f = \frac{\cos f\,(h a_R + r v_R a_S)}{\mu} - \frac{2h\sin f}{\mu}\,a_S = \frac{1}{h}\Big[p\cos f\,a_R - (\underbrace{2p - re\cos f}_{p+r})\sin f\,a_S\Big].$$

Adding the Keplerian part gives the full rate:

$$\boxed{\dot{f} = \frac{h}{r^2} + \frac{1}{he}\left[p\cos f\, a_R - (p+r)\sin f\, a_S\right]} \tag{9}$$

The perturbative part is singular at $e = 0$, where periapsis, and with it f, is undefined.

7 Orientation of the orbital plane: i and Ω

Only the cross-track force moves the orbital plane. It does so by rotating the whole conic about the current radius vector.

7.1 Inclination

The inclination is the angle between the orbit normal and the polar axis. Differentiate its cosine and use the normal's rotation from Equation 5:

$$\cos i = \hat{\mathbf{K}}\cdot\hat{\mathbf{W}} \quad\Longrightarrow\quad -\sin i\,\dot{i} = \hat{\mathbf{K}}\cdot\dot{\hat{\mathbf{W}}} = -\frac{ra_W}{h}\,\hat{\mathbf{K}}\cdot\hat{\mathbf{S}} = -\frac{ra_W}{h}\,\sin i\cos u,$$

$$\boxed{\dot{i} = \frac{r\cos u}{h}\,a_W} \tag{10}$$

7.2 Node

The node is the azimuth of the normal's equatorial projection. With the components of the normal from Section 3.2,

$$h_x = h\sin i\sin\Omega,\quad h_y = -h\sin i\cos\Omega \quad\Longrightarrow\quad \dot{\Omega} = \frac{h_x\dot{h}_y - h_y\dot{h}_x}{h_x^2 + h_y^2} = \frac{\hat{\mathbf{K}}\cdot(\mathbf{h}\times\dot{\mathbf{h}})}{h^2\sin^2 i}.$$

The cross product needs only the torque found earlier:

$$\mathbf{h}\times\dot{\mathbf{h}} = h\,\hat{\mathbf{W}}\times(ra_S\,\hat{\mathbf{W}} - ra_W\,\hat{\mathbf{S}}) = h\,ra_W\,\hat{\mathbf{R}},\qquad \hat{\mathbf{K}}\cdot\hat{\mathbf{R}} = \sin i\sin u,$$

$$\boxed{\dot{\Omega} = \frac{r\sin u}{h\sin i}\,a_W} \tag{11}$$

7.3 Geometric view: rigid rotation about the radius vector

A cross-track impulse leaves the position fixed and tilts the velocity. A small rotation of the entire state about the radial direction does exactly the same thing:

$$\begin{aligned}\text{impulse:}\quad & (\mathbf{r},\ \mathbf{v}) \ \mapsto\ (\mathbf{r},\ \mathbf{v} + a_W\,dt\,\hat{\mathbf{W}}),\\ \text{rotation by } \delta\phi \text{ about } \hat{\mathbf{R}}:\quad & (\mathbf{r},\ \mathbf{v}) \ \mapsto\ (\mathbf{r},\ \mathbf{v} + \delta\phi\,\hat{\mathbf{R}}\times\mathbf{v}) = (\mathbf{r},\ \mathbf{v} + \delta\phi\,v_S\,\hat{\mathbf{W}}).\end{aligned}$$

The two agree to first order for the angle below. The Kepler problem is rotation-invariant, so the new conic is the old one turned rigidly: a, e and f are untouched, and the orientation spins with angular velocity

$$\delta\phi = \frac{a_W\,dt}{v_S} = \frac{r a_W}{h}\,dt \qquad \Longrightarrow \qquad \Omega_\perp = \frac{r a_W}{h}\,\hat{\mathbf{R}}.$$

Now resolve this angular velocity along the axes of the 3–1–3 sequence of the perifocal note: Ω about $\hat{\mathbf{K}}$, i about the node line $\hat{\mathbf{N}}$, ω about the normal $\hat{\mathbf{W}}$. Each Euler-angle rate spins the triad about the axis of its own rotation *at that instant*, so the angular velocity of the orbit's orientation is the sum $\dot{\Omega}\,\hat{\mathbf{K}} + \dot{i}\,\hat{\mathbf{N}} + \dot{\omega}\,\hat{\mathbf{W}}$. With the unit-vector relations of Section 3.2,

$$\Omega_\perp = \frac{r a_W}{h}\left[\cos u\,\hat{\mathbf{N}} + \sin u\,\frac{\hat{\mathbf{K}} - \cos i\,\hat{\mathbf{W}}}{\sin i}\right] \equiv \dot{\Omega}\,\hat{\mathbf{K}} + \dot{i}\,\hat{\mathbf{N}} + \dot{\omega}_\perp\,\hat{\mathbf{W}}.$$

Matching coefficients reproduces both boxed results and adds a third. The cross-track force also moves ω, because the node from which ω is measured slides along the orbit:

$$\dot{\Omega} = \frac{r\sin u}{h\sin i}\,a_W, \qquad \dot{i} = \frac{r\cos u}{h}\,a_W, \qquad \dot{\omega}_\perp = -\frac{r\sin u\cos i}{h\sin i}\,a_W = -\cos i\,\dot{\Omega}.$$

Per unit cross-track impulse, inclination responds most at the nodes and the node responds most 90° from them. Both effects grow with r, which is why plane changes are done far out when the geometry allows.

A spacecraft is on a circular orbit of radius 7000 km ($\mu_\oplus = 398\,600.4418\ \mathrm{km}^3/\mathrm{s}^2$). It makes a 1 m/s burn along $\hat{\mathbf{W}}$. (a) If the burn is at the ascending node ($u = 0$), how much does the inclination change? (b) If the orbit has $i = 51.6°$ and the burn is at $u = 90°$, how much do Ω and ω change? Use Equation 4 with the rates above, multiplied by dt.

What you should find

On a circle $r/h = r/\sqrt{\mu r} = \sqrt{r/\mu} = 0.1325$ s/km, which is $1/v_c$ with $v_c = 7.546$ km/s. (a) $\Delta i = (r/h)\cos 0\,\Delta v = 10^{-3}/7.546 = 1.325\times10^{-4}$ rad $= 0.00759°$; Ω does not change, because $\sin u = 0$. (b) $\Delta i = 0$, and $\Delta\Omega = (r/h)\,\Delta v/\sin 51.6° = 0.00969°$, while $\Delta\omega = -\cos 51.6°\,\Delta\Omega = -0.00602°$. The same impulse that tilts the plane at a node swings it about the pole 90° later.

8 Argument of periapsis: ω

The argument of periapsis moves for two reasons. In-plane forces turn the apse line within the orbital plane, and the cross-track force shifts the node from which ω is measured.

8.1 From the difference $u - f$

The argument of periapsis is the argument of latitude minus the true anomaly. Both angles carry the same Keplerian part h/r^2, so it cancels and only velocity gradients remain:

$$\omega = u - f \qquad \Longrightarrow \qquad \dot{\omega} = (\mathcal{K}u + \mathcal{D}u) - (\mathcal{K}f + \mathcal{D}f) = \mathcal{D}u - \mathcal{D}f.$$

The true-anomaly gradient comes from Section 6. For u, the polar-axis components of $\hat{\mathbf{S}}$ and $\hat{\mathbf{R}}$ form a point at polar angle u, the device already used for e and f:

$$(\hat{\mathbf{K}}\cdot\hat{\mathbf{S}},\ \hat{\mathbf{K}}\cdot\hat{\mathbf{R}}) = \sin i\,(\cos u,\ \sin u) \quad\Longrightarrow\quad \sin^2 i\ \mathcal{D}u = (\hat{\mathbf{K}}\cdot\hat{\mathbf{S}})\,\mathcal{D}(\hat{\mathbf{K}}\cdot\hat{\mathbf{R}}) - (\hat{\mathbf{K}}\cdot\hat{\mathbf{R}})\,\mathcal{D}(\hat{\mathbf{K}}\cdot\hat{\mathbf{S}}).$$

The radial direction depends on position alone, so its term vanishes. The transverse direction inherits the rotation of the normal:

$$\mathcal{D}\hat{\mathbf{R}} = \mathbf{0}, \qquad \mathcal{D}\hat{\mathbf{S}} = \mathcal{D}\hat{\mathbf{W}}\times\hat{\mathbf{R}} = -\frac{ra_W}{h}\,\hat{\mathbf{S}}\times\hat{\mathbf{R}} = \frac{ra_W}{h}\,\hat{\mathbf{W}} \quad\Longrightarrow\quad \mathcal{D}(\hat{\mathbf{K}}\cdot\hat{\mathbf{S}}) = \frac{ra_W}{h}\,\cos i,$$

$$\mathcal{D}u = -\frac{r\sin u\cos i}{h\sin i}\,a_W = -\cos i\,\dot{\Omega}.$$

This is the $\dot{\omega}_\perp$ of the rigid-rotation picture, as it must be. A rotation about $\hat{\mathbf{R}}$ leaves f alone, so all of its change in u is a change in ω. Subtracting $\mathcal{D}f$ gives the full rate:

$$\boxed{\dot{\omega} = \frac{1}{he}\Big[-p\cos f\,a_R + (p+r)\sin f\,a_S\Big] - \frac{r\sin u\cos i}{h\sin i}\,a_W} \tag{12}$$

With $p/h = \sqrt{p/\mu}$ from (I2), this becomes the form quoted in most textbooks:

$$\dot{\omega} = \frac{1}{e}\sqrt{\frac{p}{\mu}}\Big[-\cos f\,a_R + \Big(1+\frac{r}{p}\Big)\sin f\,a_S\Big] - \frac{r\sin u\cos i}{h\sin i}\,a_W.$$

The in-plane terms alone give the rotation rate of the apse line within the orbital plane. This combination returns in the mean-anomaly equation:

$$\dot{\omega} + \cos i\,\dot{\Omega} = \frac{1}{he}\Big[-p\cos f\,a_R + (p+r)\sin f\,a_S\Big]. \tag{13}$$

8.2 Cross-check with the eccentricity vector

The eccentricity vector $\mathbf{e}$ points to periapsis with length e, so its rate holds $\dot{e}$ and the apse-line rotation together. It is a Keplerian integral, and $\mathcal{D}$ acts only on its velocities:

$$\mu\,\mathbf{e} = \mathbf{v}\times\mathbf{h} - \mu\,\hat{\mathbf{R}} = (\mathbf{v}\cdot\mathbf{v})\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v} - \mu\,\hat{\mathbf{R}} \quad\Longrightarrow\quad \mu\,\dot{\mathbf{e}} = 2(\mathbf{v}\cdot\mathbf{a}_d)\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{a}_d)\,\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{a}_d.$$

In RSW components, using $rv_S = h$, the radial force drops out of the radial component:

$$\mu\,\dot{\mathbf{e}} = 2ha_S\,\hat{\mathbf{R}} - (ha_R + rv_Ra_S)\,\hat{\mathbf{S}} - rv_Ra_W\,\hat{\mathbf{W}}.$$

On the element side, $\mathbf{e} = e\,\hat{\mathbf{p}}$ and the perifocal triad turns with the Euler-angle rates. Projecting on the triad separates the pieces:

$$\dot{\mathbf{e}} = \dot{e}\,\hat{\mathbf{p}} + e\,\Omega_{PQW}\times\hat{\mathbf{p}}, \qquad \Omega_{PQW} = \dot{\Omega}\,\hat{\mathbf{K}} + \dot{i}\,\hat{\mathbf{N}} + \dot{\omega}\,\hat{\mathbf{W}},$$

$$\hat{\mathbf{p}}\cdot\dot{\mathbf{e}} = \dot{e}, \qquad \hat{\mathbf{q}}\cdot\dot{\mathbf{e}} = e\,(\dot{\omega} + \cos i\,\dot{\Omega}), \qquad \hat{\mathbf{W}}\cdot\dot{\mathbf{e}} = -e\,(\sin i\cos\omega\,\dot{\Omega} - \sin\omega\,\dot{i}).$$

Now project the component form on $\hat{\mathbf{p}} = \cos f\,\hat{\mathbf{R}} - \sin f\,\hat{\mathbf{S}}$ and $\hat{\mathbf{q}} = \sin f\,\hat{\mathbf{R}} + \cos f\,\hat{\mathbf{S}}$. The $\hat{\mathbf{p}}$ projection reproduces the unsimplified $\dot{e}$ of Section 6 term by term. The $\hat{\mathbf{q}}$ projection needs (I1) once:

$$2h\sin f - rv_R\cos f = \frac{\mu\sin f}{h}\,(2p - re\cos f) = \frac{\mu\sin f}{h}\,(p + r),$$

$$e\,(\dot{\omega}+\cos i\,\dot{\Omega}) = \frac{1}{\mu}\Big[-h\cos f\,a_R + \frac{\mu\sin f}{h}\,(p+r)\,a_S\Big] = \frac{1}{h}\Big[-p\cos f\,a_R + (p+r)\sin f\,a_S\Big].$$

That is the apse-line rate Equation 13. The $\hat{\mathbf{W}}$ projection tests the plane equations: inserting $\dot{i}$ and $\dot{\Omega}$ collapses the bracket to $\sin(u-\omega) = \sin f$,

$$-e\,(\sin i\cos\omega\,\dot{\Omega} - \sin\omega\,\dot{i}) = -\frac{e\,ra_W}{h}\,(\sin u\cos\omega - \cos u\sin\omega) = -\frac{re\sin f}{h}\,a_W = -\frac{rv_R}{\mu}\,a_W,$$

which matches the $\hat{\mathbf{W}}$ component above. Three projections, three agreements: the eccentricity vector confirms $\dot{e}$, the in-plane part of $\dot{\omega}$, and the i and Ω equations together.

9 Mean anomaly: M

The mean anomaly keeps its Keplerian drift n and responds only to in-plane forces, because it depends on the state only through f and e.

9.1 M as a function of f and e

Kepler's equation and the eccentric anomaly make M a function of f and e alone. The chain rule splits its rate into the Keplerian drift and two perturbation terms:

$$M = E - e\sin E, \qquad E = E(f,e) \quad\Longrightarrow\quad \dot{M} = \mathcal{K}M + \frac{\partial M}{\partial f}\Big|_e \mathcal{D}f + \frac{\partial M}{\partial e}\Big|_f \mathcal{D}e, \qquad \mathcal{K}M = n.$$

The partial derivatives of E follow from the expressions for $\cos E$ and $\sin E$. Then $dM = (1 - e\cos E)\,dE = (r/a)\,dE$ converts them:

$$\frac{\partial E}{\partial f}\Big|_e = \frac{\sqrt{1-e^2}}{1+e\cos f}, \qquad \frac{\partial E}{\partial e}\Big|_f = -\frac{\sin f}{\sqrt{1-e^2}\,(1+e\cos f)},$$

$$\frac{\partial M}{\partial f}\Big|_e = \frac{r^2}{ab}, \qquad \frac{\partial M}{\partial e}\Big|_f = \frac{r}{a}\,\frac{\partial E}{\partial e}\Big|_f - \sin E = -\sqrt{1-e^2}\,\frac{r\,(p+r)}{p^2}\,\sin f.$$

The first partial gives a quick check: it maps the Keplerian part of f onto exactly the mean motion, $(r^2/ab)(h/r^2) = h/ab = n$.

9.2 Assembling the rate

Substitute $\mathcal{D}f$ and $\mathcal{D}e$ from Section 6 and collect the coefficients of a_R and a_S. Both collapse after replacing $er\cos f$ by $p - r$, as (I1) allows, and a by $p/(1-e^2)$:

$$a_R: \quad \frac{r^2\cos f}{a} - \frac{er\,(p+r)\sin^2 f}{p} = p\cos f - 2re$$

$$a_S: \quad \frac{r^2}{a} + \frac{er\,(p+r)\cos f + e^2 r^2}{p} = p$$

$$\boxed{\dot{M} = n + \frac{b}{ahe}\Big[(p\cos f - 2re)\,a_R - (p+r)\sin f\,a_S\Big]} \tag{14}$$

Splitting off the apse-line rate Equation 13 gives a more revealing form:

$$\dot{M} = n - \frac{2r}{\sqrt{\mu a}}\,a_R - \sqrt{1-e^2}\,(\dot{\omega} + \cos i\,\dot{\Omega}).$$

Apart from one radial term, M falls back as the apse line advances, scaled by $\sqrt{1-e^2}$. The $1/e$ singularity is inherited entirely from the apse line, so the sum $M+\omega$ stays regular as $e \to 0$. It still carries the $\cos i\,\dot{\Omega}$ term, though, and with it a $1/\sin i$: the sum is regular in e but not at $i = 0$, which is why the mean longitude $\lambda = M + \omega + \Omega$ of Section 12.4 is used instead.

9.3 The mean-anomaly-at-epoch subtlety

The boxed rate is for M itself, which is what a numerical integrator should carry. Many analytical theories use the mean anomaly at epoch instead, and there the varying mean motion leaves an explicit time in the equation:

$$M = M_0 + n\,(t - t_0) \quad\Longrightarrow\quad \dot{M}_0 = (\dot{M} - n) - \dot{n}\,(t - t_0) = (\dot{M} - n) + \frac{3n}{2a}\,(t - t_0)\,\dot{a}.$$

Even a purely periodic $\dot{a}$ then drives terms in M_0 that grow in proportion to $t - t_0$. These mixed secular terms are an artefact of the element choice, not of the physics. The standard remedy integrates the mean motion separately and keeps only the perturbative part in a new element σ:

$$M = \int_{t_0}^{t} n\,dt' + \sigma \quad\Longrightarrow\quad \dot{\sigma} = \dot{M} - n = \frac{b}{ahe}\Big[(p\cos f - 2re)\,a_R - (p+r)\sin f\,a_S\Big].$$

The integral of n is carried alongside a, and σ obeys a Gauss equation with no explicit time. The time of periapsis passage τ, defined by $M = n(t-\tau)$, has the same $\dot{n}(t-\tau)$ problem.

10 Summary: the complete equations in RSW form

Six equations give the rates of the osculating elements from the radial, transverse and cross-track force components. Only M carries a Keplerian term.

$$
\begin{aligned}
\dot{a} &= \frac{2a^2}{h}\Big(e\sin f\, a_R + \frac{p}{r}\, a_S\Big) \\
\dot{e} &= \frac{1}{h}\Big\{p\sin f\, a_R + \big[(p+r)\cos f + re\big]\, a_S\Big\} \\
\dot{i} &= \frac{r\cos u}{h}\, a_W \\
\dot{\Omega} &= \frac{r\sin u}{h\sin i}\, a_W \\
\dot{\omega} &= \frac{1}{he}\Big[-p\cos f\, a_R + (p+r)\sin f\, a_S\Big] - \frac{r\sin u\cos i}{h\sin i}\, a_W \\
\dot{M} &= n + \frac{b}{ahe}\Big[(p\cos f - 2re)\, a_R - (p+r)\sin f\, a_S\Big]
\end{aligned}
\tag{15}
$$

Here $r = p/(1 + e\cos f)$, $u = \omega + f$, $b = a\sqrt{1-e^2}$, $h = \sqrt{\mu p}$ and $n = \sqrt{\mu/a^3}$, all evaluated on the osculating conic. The auxiliary quantities derived along the way are

$$
\begin{aligned}
\dot{h} &= r a_S, \qquad \dot{p} = 2r\sqrt{\frac{p}{\mu}}\, a_S \\
\dot{f} &= \frac{h}{r^2} + \frac{1}{he}\Big[p\cos f\, a_R - (p+r)\sin f\, a_S\Big] \\
\dot{u} &= \frac{h}{r^2} - \frac{r\sin u\cos i}{h\sin i}\, a_W
\end{aligned}
$$

The argument of latitude u stays defined on a circular orbit, so it replaces f or M as the fast angle when e is near zero. (It is still undefined at $i = 0$, where there is no node to measure it from.)

10.1 Matrix form

The same equations as one linear map from the force to the element rates:

$$
\frac{d}{dt}\begin{pmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ M \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix} + \underbrace{\begin{pmatrix} \dfrac{2a^2 e\sin f}{h} & \dfrac{2a^2 p}{hr} & 0 \\[2ex] \dfrac{p\sin f}{h} & \dfrac{(p+r)\cos f + re}{h} & 0 \\[2ex] 0 & 0 & \dfrac{r\cos u}{h} \\[2ex] 0 & 0 & \dfrac{r\sin u}{h\sin i} \\[2ex] -\dfrac{p\cos f}{he} & \dfrac{(p+r)\sin f}{he} & -\dfrac{r\sin u\cos i}{h\sin i} \\[2ex] \dfrac{b\,(p\cos f - 2re)}{ahe} & -\dfrac{b\,(p+r)\sin f}{ahe} & 0 \end{pmatrix}}_{\mathbf{B}(a,\, e,\, i,\, \omega,\, M)} \begin{pmatrix} a_R \\ a_S \\ a_W \end{pmatrix}
\tag{16}
$$

The matrix depends on a, e, i, ω and M through r, f and u, but never on Ω. That is the Kepler problem's symmetry about the polar axis; the force itself may still depend on Ω.

By the fundamental rule, $\mathbf{B}$ is the velocity gradient of the elements expressed in RSW axes. A small impulse $\Delta\mathbf{v}$, resolved in RSW, therefore changes the elements by $\Delta\alpha \approx \mathbf{B}\,\Delta\mathbf{v}$.

11 Tangential–normal (TNW) form

Resolving the force along and across the velocity puts the entire change of the semi-major axis into the tangential component. This is the natural frame for thrust and drag.

11.1 The frame

$\hat{\mathbf{T}}$ points along the velocity and $\hat{\mathbf{W}}$ along the orbit normal. $\hat{\mathbf{N}}_{\text{in}} = \hat{\mathbf{W}} \times \hat{\mathbf{T}}$ lies in the plane and points to the inside of the path, towards the centre of curvature (Figure 1, left; it is written $\hat{\mathbf{N}}$ in the figure, not to be confused with the node direction of Section 3.2). With the flight-path angle γ defined by $v_R = v\sin\gamma$ and $v_S = v\cos\gamma$,

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{v} = \sin\gamma\,\hat{\mathbf{R}} + \cos\gamma\,\hat{\mathbf{S}}, \qquad \hat{\mathbf{N}}_{\text{in}} = \hat{\mathbf{W}} \times \hat{\mathbf{T}} = -\cos\gamma\,\hat{\mathbf{R}} + \sin\gamma\,\hat{\mathbf{S}},$$

$$\sin\gamma = \frac{\mu e \sin f}{hv}, \qquad \cos\gamma = \frac{h}{rv}, \qquad v^2 = \mu\Big(\frac{2}{r} - \frac{1}{a}\Big).$$

The force components transform by the same rotation:

$$a_R = \sin\gamma\, a_T - \cos\gamma\, a_N, \qquad a_S = \cos\gamma\, a_T + \sin\gamma\, a_N.$$

11.2 Converting the equations

The semi-major axis needs no algebra. The energy result $\dot{a} = (2a^2/\mu)\,\mathbf{v}\cdot\mathbf{a}_d$ already depends only on $\mathbf{v}\cdot\mathbf{a}_d = v\,a_T$.

For e, substitute and use $\mu p/h^2 = 1$ and $p/r = 1 + e\cos f$. The a_N coefficient also needs (I1) in the form $er(p+r)\cos f = p^2 - r^2$:

$$a_T: \quad \frac{p\sin f\sin\gamma + \big[(p+r)\cos f + re\big]\cos\gamma}{h} = \frac{e\sin^2 f + (2+e\cos f)\cos f + e}{v} = \frac{2\,(e+\cos f)}{v}$$

$$a_N: \quad \frac{-p\sin f\cos\gamma + \big[(p+r)\cos f + re\big]\sin\gamma}{h} = \frac{\sin f}{prv}\Big[-p^2 + er\,(p+r)\cos f + e^2r^2\Big] = -\frac{r}{av}\,\sin f$$

The ω and M coefficients reduce the same way:

$$\omega,\ a_T: \quad \frac{-p\cos f\sin\gamma + (p+r)\sin f\cos\gamma}{he} = \frac{2\sin f}{ev}$$

$$\omega,\ a_N: \quad \frac{p\cos f\cos\gamma + (p+r)\sin f\sin\gamma}{he} = \frac{1}{ev}\Big(2e + \frac{r}{a}\cos f\Big)$$

$$M,\ a_T: \quad \frac{b}{ahe}\Big[(p\cos f - 2re)\sin\gamma - (p+r)\sin f\cos\gamma\Big] = -\frac{2b}{eav}\Big(1 + \frac{e^2r}{p}\Big)\sin f$$

$$M,\ a_N: \quad \frac{b}{ahe}\Big[-(p\cos f - 2re)\cos\gamma - (p+r)\sin f\sin\gamma\Big] = -\frac{b}{eav}\,\frac{r}{a}\cos f$$

11.3 The TNW equations

The cross-track terms are untouched, since $\hat{\mathbf{W}}$ is common to both frames:

$$
\begin{aligned}
\dot{a} &= \frac{2a^2 v}{\mu}\, a_T \\
\dot{e} &= \frac{1}{v}\Big[2\,(e+\cos f)\, a_T - \frac{r}{a}\,\sin f\, a_N\Big] \\
\dot{i} &= \frac{r\cos u}{h}\, a_W, \qquad \dot{\Omega} = \frac{r\sin u}{h\sin i}\, a_W \\
\dot{\omega} &= \frac{1}{ev}\Big[2\sin f\, a_T + \Big(2e + \frac{r}{a}\cos f\Big) a_N\Big] - \frac{r\sin u\cos i}{h\sin i}\, a_W \\
\dot{M} &= n - \frac{b}{eav}\Big[2\Big(1+\frac{e^2 r}{p}\Big)\sin f\, a_T + \frac{r}{a}\cos f\, a_N\Big]
\end{aligned}
\tag{17}
$$

Some texts take the in-plane normal outwards, $\hat{\mathbf{T}} \times \hat{\mathbf{W}}$. Every a_N coefficient then changes sign.

11.4 What the TNW form shows

Only a_T changes a, and at a rate proportional to the speed. A given tangential acceleration therefore raises a fastest at periapsis: the Oberth effect again.

At an apsis, $\sin f = 0$. A tangential force there changes a and e but does not turn the apse line. A normal force there leaves a and e alone and only turns the apse line (Figure 1).

	radial and transverse (RSW)	tangential and normal (TNW)
in-plane axes	$\hat{\mathbf{R}}$ along $\mathbf{r}$, $\hat{\mathbf{S}} \perp \mathbf{r}$	$\hat{\mathbf{T}}$ along $\mathbf{v}$, $\hat{\mathbf{N}}_{\text{in}} \perp \mathbf{v}$
natural for	gravity, which depends on position	thrust and drag, which follow the velocity
$\dot{a}$	needs both a_R and a_S	needs only a_T
the two agree	on a circle, or at an apsis, where $\gamma = 0$	

12 Structure, physical interpretation and singularities

The equations split cleanly by force direction, and each singularity sits exactly where an element's geometric definition fails.

12.1 Which force moves which element

Element	Radial a_R	Transverse a_S	Cross-track a_W
a	yes, $\propto e\sin f$	yes, $\propto p/r$	no
e	yes, $\propto \sin f$	yes, $\propto (p+r)\cos f + re$	no
h, p	no	yes, $\propto r$	no
i	no	no	yes, $\propto r\cos u$
Ω	no	no	yes, $\propto r\sin u/\sin i$

Element	Radial a_R	Transverse a_S	Cross-track a_W
ω	yes, $\propto -\cos f/e$	yes, $\propto (p+r)\sin f/e$	yes, $-\cos i$ times the Ω coefficient
M	yes, $\propto p\cos f - 2re$	yes, $\propto -(p+r)\sin f$	no

Three facts organise the table. The semi-major axis responds only to the power $\mathbf{v}\cdot\mathbf{a}_d$, and the size of the angular momentum only to the torque ra_S. A radial force therefore trades a against e at fixed $p = a(1-e^2)$. Only a_W moves the plane, and it reaches ω solely because the node, the origin of ω, moves.

A radial impulse at periapsis shows the in-plane coupling most cleanly (Figure 1, right):

$$f = 0: \qquad \Delta a = 0, \qquad \Delta e = 0, \qquad \Delta\omega = -\frac{p}{he}\,\Delta v_R, \qquad \Delta f = +\frac{p}{he}\,\Delta v_R.$$

The conic keeps its size and shape and turns backwards. The body, with its position unchanged, now sits just past its new periapsis.

A satellite on an eccentric orbit fires its engine briefly along $\hat{\mathbf{S}}$ at apoapsis. Which elements change, and does the apse line turn?

Answer

At apoapsis $f = 180°$, so $\sin f = 0$: the a_S entries for ω and M vanish, and so does the a_R entry for a. The burn changes a (through p/r), e (through $(p+r)\cos f + re = -(p+r) + re = -p - r(1-e) < 0$: e *decreases* for a forward burn) and h and p, but not ω, i or Ω. At apoapsis $\hat{\mathbf{S}}$ is also along the velocity, so this is the circularisation burn of a Hohmann transfer: a up, e down, apse line fixed.

12.2 Link to Lagrange's planetary equations

For a conservative perturbation $\mathbf{a}_d = \nabla\mathcal{R}$, with $\mathcal{R}$ the disturbing function (the $\mathcal{R}$ of the gravity note), the Gauss form turns into Lagrange's. On the osculating conic the position depends on time only through M, which advances at rate n:

$$\mathbf{v}\cdot\nabla\mathcal{R} = \frac{\partial\phi}{\partial t}\cdot\nabla\mathcal{R} = n\,\frac{\partial\phi}{\partial M}\cdot\nabla\mathcal{R} = n\,\frac{\partial\mathcal{R}}{\partial M} \quad\Longrightarrow\quad \dot a = \frac{2a^2}{\mu}\,n\,\frac{\partial\mathcal{R}}{\partial M} = \frac{2}{na}\,\frac{\partial\mathcal{R}}{\partial M}.$$

That is Lagrange's equation for a. The other five follow the same way, but each needs the partial derivatives $\partial\phi/\partial\alpha$ of the position with respect to the elements (or, equivalently, the Lagrange brackets), which is longer than this note needs. We quote the results used in the gravity note, with the derivation in Vallado, chapter 9:

$$\dot\Omega = \frac{1}{na^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\mathcal{R}}{\partial i}, \qquad \dot\omega = \frac{\sqrt{1-e^2}}{na^2 e}\,\frac{\partial\mathcal{R}}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\mathcal{R}}{\partial i}.$$

Section 13 checks the averaged $\dot\omega$ that these give against a direct average of Equation 12, with no Lagrange equations involved.

12.3 Singularities

Condition	Equations that blow up	Cause	Regular replacement
$e \to 0$	ω, f, M, through $1/e$	periapsis undefined	u or mean longitude as the fast angle; $e\cos\varpi$, $e\sin\varpi$
$i \to 0$	Ω, ω, through $1/\sin i$	node undefined	$\varpi = \Omega + \omega$; $\tan\frac{i}{2}\cos\Omega$, $\tan\frac{i}{2}\sin\Omega$
$e \to 1$	a and M become undefined: $a \to \infty$, $n \to 0$	the conic becomes a parabola	p in place of a; true longitude

(At $e \to 1$ the coefficient of $\dot{M}$ does not blow up: $b/(ahe) = \sqrt{1-e^2}/(he) \to 0$. What fails is that a and n stop being usable variables.) The fixes for $i \to 0$ fail at $i = 180°$, where a retrograde variant is used. In their p, r, f forms, the equations for a, e, i, Ω, ω and f also hold for hyperbolic orbits, with $a < 0$.

12.4 Regular combinations

Sums of angles that do not need the missing reference direction stay finite. The longitude of periapsis $\varpi = \Omega + \omega$ is regular at $i = 0$, because $1 - \cos i = \sin i \tan(i/2)$:

$$\dot{\varpi} = \dot{\Omega} + \dot{\omega} = \frac{1}{he}\Big[-p\cos f\, a_R + (p+r)\sin f\, a_S \Big] + \frac{r\sin u}{h}\,\tan\frac{i}{2}\,a_W.$$

The mean longitude $\lambda = M + \omega + \Omega$ is regular at $e = 0$ as well. In the compact $\dot{M}$, the apse-line rate is multiplied by $1 - \sqrt{1-e^2} \approx e^2/2$, which cancels its $1/e$:

$$\dot{\lambda} = n - \frac{2r}{\sqrt{\mu a}}\,a_R + \big(1 - \sqrt{1-e^2}\big)(\dot{\omega} + \cos i\,\dot{\Omega}) + \frac{r\sin u}{h}\,\tan\frac{i}{2}\,a_W.$$

A fully regular set follows by applying $\mathcal{D}$ to state functions with no singular geometry. The *modified equinoctial elements* of Walker, Ireland and Owens (1985) are the common choice. Here f, g, h and k are the standard names of these elements; they have nothing to do with the true anomaly and the angular momentum:

$$p, \qquad f = e\cos(\omega+\Omega), \qquad g = e\sin(\omega+\Omega), \qquad h = \tan\tfrac{i}{2}\cos\Omega, \qquad k = \tan\tfrac{i}{2}\sin\Omega, \qquad L = \Omega+\omega+f_{\text{true}}.$$

The results above convert directly. With $w = 1 + f\cos L + g\sin L = p/r$, the p equation is already in equinoctial form, and the true longitude follows from $\dot{L} = \dot{\Omega} + \dot{u}$:

$$\dot{p} = \frac{2p}{w}\sqrt{\frac{p}{\mu}}\,a_S, \qquad \dot{L} = \frac{\sqrt{\mu p}}{r^2} + (1-\cos i)\,\dot{\Omega} = \sqrt{\mu p}\,\Big(\frac{w}{p}\Big)^2 + \frac{1}{w}\sqrt{\frac{p}{\mu}}\,(h\sin L - k\cos L)\,a_W.$$

The full equinoctial set is singular only at $i = 180°$.

13 Worked check: the J_2 secular rates by orbit averaging

Averaging the node and periapsis equations over one orbit recovers the classical J_2 rates of the gravity note, directly from the Gauss equations.

13.1 The J_2 force in RSW components

The J_2 disturbing function depends on position only through r and the height above the equator, $z = r \sin i \sin u$, with $R_\oplus$ the Earth's equatorial radius:

$$\mathcal{R} = -\frac{\mu J_2 R_\oplus^2}{2}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right), \qquad z = \hat{\mathbf{K}} \cdot \mathbf{r} = r \sin i \sin u.$$

Its gradient therefore has only a radial part and a polar-axis part. The $\hat{\mathbf{K}}$ projections of Section 3.2 resolve it into RSW:

$$\mathbf{a}_d = \nabla \mathcal{R} = \left.\frac{\partial \mathcal{R}}{\partial r}\right|_z \hat{\mathbf{R}} + \left.\frac{\partial \mathcal{R}}{\partial z}\right|_r \hat{\mathbf{K}}, \qquad \left.\frac{\partial \mathcal{R}}{\partial z}\right|_r = -\frac{3\mu J_2 R_\oplus^2 z}{r^5},$$

$$\begin{aligned} a_R &= \frac{3\mu J_2 R_\oplus^2}{2r^4}\left(3\sin^2 i \sin^2 u - 1\right) \\ a_S &= -\frac{3\mu J_2 R_\oplus^2}{r^4} \sin^2 i \, \sin u \cos u \\ a_W &= -\frac{3\mu J_2 R_\oplus^2}{r^4} \sin i \cos i \, \sin u \end{aligned} \tag{18}$$

13.2 The node and the inclination

Inserting a_W into the node and inclination equations gives the instantaneous rates:

$$\dot{\Omega} = -\frac{3\mu J_2 R_\oplus^2 \cos i}{h\, r^3} \sin^2 u, \qquad \dot{i} = -\frac{3\mu J_2 R_\oplus^2 \sin i \cos i}{h\, r^3} \sin u \cos u.$$

To first order, average over one osculating orbit with the elements frozen. Time converts to true anomaly through $dt = r^2\, df/h$, and $h^2 = \mu p$:

$$\langle \dot{\Omega} \rangle = \frac{n}{2\pi}\int_0^{2\pi} \dot{\Omega}\, \frac{r^2}{h}\, df = -\frac{3nJ_2 R_\oplus^2 \cos i}{2\pi\, p^2}\int_0^{2\pi} (1 + e\cos f)\sin^2(\omega + f)\, df.$$

The integral is π, because $e\cos f \sin^2(\omega + f)$ has no constant part. The inclination integrand, $(1 + e\cos f)\sin u \cos u$, has none at all:

$$\boxed{\langle \dot{\Omega} \rangle = -\frac{3}{2} n J_2 \left(\frac{R_\oplus}{p}\right)^2 \cos i} \qquad \langle \dot{i} \rangle = 0. \tag{19}$$

13.3 The apse line

The same averaging applied to Equation 12 gives the rotation of the apse line. Write $k_2 = \frac{3}{2}\mu J_2 R_\oplus^2$ and $s = \sin i$, so that $a_R = (k_2/r^4)(3s^2\sin^2 u - 1)$ and $a_S = -(2k_2/r^4)\,s^2 \sin u \cos u$. The cross-track part of $\dot\omega$ is $-\cos i\,\dot\Omega$, whose average we already have. The in-plane part is $(1/he)[-p\cos f\,a_R + (p+r)\sin f\,a_S]$, and its average is

$$\left\langle \dot\omega + \cos i\,\dot\Omega \right\rangle = \frac{n}{2\pi h}\int_0^{2\pi} \frac{1}{he}\Big[-p\cos f\,a_R + (p+r)\sin f\,a_S\Big] r^2\,df.$$

Use $1/r^2 = (1+e\cos f)^2/p^2$ and $p+r = p\,(2+e\cos f)/(1+e\cos f)$. The integrand becomes $-(k_2/p)[\,\cos f\,(1+e\cos f)^2\,(3s^2\sin^2 u - 1) + 2s^2\sin f\sin u\cos u\,(2+e\cos f)(1+e\cos f)\,]$, a trigonometric polynomial in f. Write $\sin^2 u = \frac{1}{2}(1-\cos 2u)$ and $\sin u\cos u = \frac{1}{2}\sin 2u$, with $u = \omega + f$, and keep only the terms with a constant part:

$$I_1 \equiv \int_0^{2\pi} \cos f\,(1+e\cos f)^2\,(3s^2\sin^2 u - 1)\,df = \left(\tfrac{3}{2}s^2 - 1\right)2\pi e - \tfrac{3}{2}s^2\,\pi e\cos 2\omega,$$
$$I_2 \equiv \int_0^{2\pi} \sin f\sin u\cos u\,(2+e\cos f)(1+e\cos f)\,df = \tfrac{3}{4}\,\pi e\cos 2\omega.$$

(In I_1, only $2e\cos^2 f$ survives against the constant, and $2e\cos^2 f\cos 2f$ against $\cos 2u$; in I_2, only $\sin f\sin 2f\cdot 3e\cos f$, whose average is that of $6e\sin^2 f\cos^2 f$.) The $\cos 2\omega$ terms cancel in the combination $I_1 + 2s^2 I_2$, leaving

$$\left\langle \dot\omega + \cos i\,\dot\Omega \right\rangle = -\frac{n}{2\pi h^2 e}\,\frac{k_2}{p}\,\pi e\,(3s^2-2) = \frac{3}{4}\,nJ_2\left(\frac{R_\oplus}{p}\right)^2 (3\cos^2 i - 1).$$

Adding $\cos i\,\langle\dot\Omega\rangle = -\frac{3}{2}nJ_2(R_\oplus/p)^2\cos^2 i$ back:

$$\boxed{\langle\dot\omega\rangle = \frac{3}{4}\,nJ_2\left(\frac{R_\oplus}{p}\right)^2 (5\cos^2 i - 1)} \tag{20}$$

the same result the gravity note obtains from the Lagrange equations. It vanishes at the critical inclinations 63.4° and 116.6°.

13.4 Numbers

Prograde orbits regress and retrograde orbits advance. Take a circular orbit at $a = 7000$ km and $i = 98°$, with the constants of the notation page ($J_2 = 1.08263\times10^{-3}$):

$$\langle\dot\Omega\rangle = +1.001328°/\text{day}.$$

That is close to the sun-synchronous rate, 360° per 365.2422 days or 0.985647°/day. This orbit reaches it at $i = 97.874°$.

14 Numerical verification

Every equation in this note passes three independent numerical tests, all in gauss-variational-verify.py.

Velocity gradients. The fundamental rule turns verification into a finite-difference exercise. The perturbative part of each rate is the element change per unit velocity kick at fixed position:

$$\frac{\alpha(\mathbf{r},\ \mathbf{v}+\epsilon\hat{\mathbf{d}}) - \alpha(\mathbf{r},\ \mathbf{v}-\epsilon\hat{\mathbf{d}})}{2\epsilon} = \frac{\partial\alpha}{\partial\mathbf{v}}\cdot\hat{\mathbf{d}} + O(\epsilon^2).$$

On 500 random orbits (a from 7000 to 42 000 km, e from 0.02 to 0.9, i from 0.1 to $\pi - 0.1$ rad, uniform angles), with $\epsilon = 10^{-6}v$ along each RSW and TNW axis, every row of $\mathbf{B}$ in both frames agrees with the central differences to better than 2×10^{-7}, relative to the row's largest entry. The rows for $\dot{f}$ agree to 2×10^{-9}, and for $\dot{u}$ to 1.4×10^{-6} (its $1/\sin i$ amplifies the rounding at small i). The rate of the eccentricity vector, the $\cos E$ form of $\dot{e}$, the compact $\dot{M}$ and the J_2 components Equation 18 (against the Cartesian J_2 acceleration) agree as well.

End to end. The gradient test cannot see the Keplerian parts, so a second test integrates the complete equations. Start from $a = 9000$ km, $e = 0.3$, $i = 50°$, $\Omega = 30°$, $\omega = 40°$, $f = 10°$; apply J_2 plus a time-varying thrust with RSW components $2\times10^{-6}\cos(t/900\text{ s})$, 3×10^{-6} and $2.5\times10^{-6}\sin(t/1300\text{ s})$ km/s^2; and integrate the Cartesian equations, and separately the Gauss equations, for five periods (DOP853, tolerances 10^{-12}). Carrying M (with Kepler's equation solved at each step) or f as the sixth element, the final positions differ from the Cartesian one by 0.18 mm and 0.56 mm at a radius of 7875 km: a relative agreement of 2×10^{-11} and 7×10^{-11}, at the integrator's tolerance.

J_2 averages. Averaging the instantaneous rates numerically over one orbit reproduces Equation 19 and Equation 20 to every digit shown, and the averaged inclination rate is zero to rounding:

a (km)	e	i	ω	$\langle\dot{\Omega}\rangle$ (°/day)	$\langle\dot{\omega}\rangle$ (°/day)
7000	0	98°	—	+1.001328	undefined at $e = 0$
9000	0.3	50°	40°	−2.317396	+1.921369
12 000	0.4	63°	25°	−0.701808	+0.023603
26 000	0.7	30°	250°	−0.242576	+0.385141

The near-zero apsidal rate at 63° is the critical inclination at work. (At $e = 0$ the *rate formula* Equation 20 is perfectly regular; it is ω itself that has no meaning on a circle.)