The f and g Series

Devendra Ghate

November 21, 2026

Table of contents

Where we are going **1**

1 A Taylor series, and a bookkeeping device **2**
1.1 The series . . . 2
1.2 Every derivative is a combination of **r** and **v** . . . 3
1.3 The recurrence . . . 3
1.4 From the recurrence to F and G . . . 4

2 Three scalars that close the system **4**
2.1 The definitions . . . 4
2.2 Their time derivatives . . . 5
2.3 Why this closes the problem . . . 5

3 Turning the crank **6**
3.1 The first coefficients . . . 6
3.2 The series . . . 7
3.3 The velocity series . . . 7

4 Two checks **7**
4.1 The circular orbit . . . 7
4.2 Conservation of angular momentum . . . 8

5 How good is it? **8**
5.1 A worked example . . . 8
5.2 Over longer steps . . . 9
5.3 Where the series is used . . . 10

6 Summary **10**

Exercises **10**

Further reading **11**

Where we are going

The question of the universal-variables note was:

> A spacecraft is at position $\mathbf{r}_0$ with velocity $\mathbf{v}_0$ at time t_0. Where is it a time τ later?

Its answer is exact and works for any τ, but it needs an iteration (the universal Kepler equation) before anything else can be computed. When τ is **short**, a small fraction of an orbit, there is a cheaper answer: the Lagrange coefficients F and G can be written as **explicit polynomials in** τ, with coefficients computed directly

from $\mathbf{r}_0$ and $\mathbf{v}_0$. No Kepler equation, no orbital elements, and no test of which conic we are on. These polynomials are the f **and** g **series**, and they are the first step of Gauss's method of orbit determination (the orbit-determination note), where the state is not yet known and an iteration is impossible.

The plan:

1. Section 1 expands $\mathbf{r}(t_0+\tau)$ in a Taylor series and shows that every derivative of $\mathbf{r}$ is a combination of $\mathbf{r}$ and $\mathbf{v}$. This gives a **recurrence** for the coefficients.
2. Section 2 introduces three scalars u, p, q whose time derivatives are again functions of u, p, q, so that the recurrence becomes pure algebra.
3. Section 3 turns the crank up to $n = 5$ and assembles the series, with the velocity series.
4. Section 4 checks the result on a circular orbit and against conservation of angular momentum.
5. Section 5 measures how good the truncated series is, with a worked example and an interactive figure.

The same material is also available as a slide deck, with animated derivations.

Names. Textbooks write the Lagrange coefficients f and g, hence "the f and g series". In these notes f is the true anomaly, so we write F and G, as the universal-variables and Lambert notes do. The letters u, p, q below are those of the orbit-determination texts (Bate, Mueller, and White 1971; Escobal 1965).

Table 1: Notation

Symbol	Meaning
$\mathbf{r}_0$, $\mathbf{v}_0$	position and velocity at t_0
$\tau = t - t_0$	the time step
F, G, $\dot{F}$, $\dot{G}$	Lagrange coefficients, $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$, $\mathbf{v} = \dot{F}\mathbf{r}_0 + \dot{G}\mathbf{v}_0$
$\mathbf{r}^{(n)}$	the n-th time derivative of $\mathbf{r}$ ($\mathbf{r}^{(0)} = \mathbf{r}$, $\mathbf{r}^{(1)} = \mathbf{v}$)
F_n, G_n	scalar coefficients of $\mathbf{r}^{(n)} = F_n\mathbf{r} + G_n\mathbf{v}$ (Equation 3)
$u = \mu/r^3$	"local mean motion squared"; n^2 on a circle
$p = \mathbf{r}\cdot\mathbf{v}/r^2 = \dot{r}/r$	relative rate of change of r; **not** the semi-latus rectum here
$q = v^2/r^2 - u$	departure of the speed from circular speed

Throughout, $\mu = 398\,600.4418\ \mathrm{km}^3/\mathrm{s}^2$ (the Earth). Symbols shared by all the notes are on the notation page.

1 A Taylor series, and a bookkeeping device

1.1 The series

Expand the position about t_0:

$$\mathbf{r}(t_0+\tau) = \mathbf{r}_0 + \tau\,\dot{\mathbf{r}}_0 + \frac{\tau^2}{2!}\ddot{\mathbf{r}}_0 + \frac{\tau^3}{3!}\dddot{\mathbf{r}}_0 + \cdots = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\,\mathbf{r}_0^{(n)}. \tag{1}$$

Here the subscript 0 means "evaluated at t_0". The first two derivatives are given. The second comes from the equation of motion,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\,\mathbf{r} = -u\,\mathbf{r}, \qquad u = \frac{\mu}{r^3}. \tag{2}$$

The third and higher come from differentiating Equation 2 again and again. Done naively, this quickly becomes unmanageable: $\dddot{\mathbf{r}} = -\dot{u}\,\mathbf{r} - u\,\mathbf{v}$ needs $\dot{u}$, the fourth derivative needs $\ddot{u}$, and so on. We need a way to organise the work.

1.2 Every derivative is a combination of r and v

Look at the first few derivatives:

$$\mathbf{r}^{(0)} = 1\cdot\mathbf{r} + 0\cdot\mathbf{v}, \qquad \mathbf{r}^{(1)} = 0\cdot\mathbf{r} + 1\cdot\mathbf{v}, \qquad \mathbf{r}^{(2)} = -u\,\mathbf{r} + 0\cdot\mathbf{v}, \qquad \mathbf{r}^{(3)} = -\dot{u}\,\mathbf{r} - u\,\mathbf{v}.$$

Each is a combination of the **current** $\mathbf{r}$ and $\mathbf{v}$ with scalar coefficients $F_n(t)$, $G_n(t)$. This is true for every n:

$$\boxed{\mathbf{r}^{(n)} = F_n\,\mathbf{r} + G_n\,\mathbf{v}} \tag{3}$$

The geometric reason is that two-body motion stays in the plane spanned by $\mathbf{r}$ and $\mathbf{v}$, so every derivative of $\mathbf{r}$ lies in that plane too, and $\mathbf{r}$, $\mathbf{v}$ are a basis of it. Figure 1 shows this: whatever the point of the orbit and whatever n, the vector $\mathbf{r}^{(n)}$ is the diagonal of a parallelogram built on $F_n\mathbf{r}$ and $G_n\mathbf{v}$.

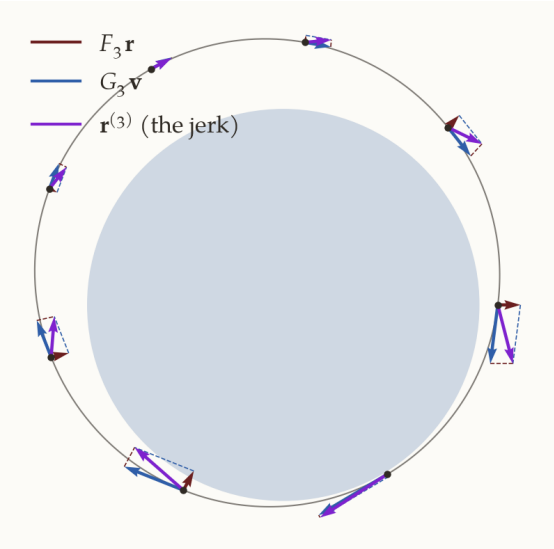


Figure 1: **Every derivative lives in the plane of r and v.** The jerk $\mathbf{r}^{(3)}$ (purple) at eight points of the orbit of Section 5.1, drawn as the diagonal of the parallelogram on $F_3\mathbf{r}$ (maroon) and $G_3\mathbf{v}$ (blue). The web version of this note animates it for $n = 2, \ldots, 5$. *Script: fig-fg-series.py.*

1.3 The recurrence

To prove Equation 3, and to compute F_n, G_n, assume it holds for some n and differentiate once more. Each term is a product of a scalar and a vector, so the product rule gives two terms:

$$\frac{d}{dt}(F_n\,\mathbf{r}) = \dot{F}_n\,\mathbf{r} + F_n\,\dot{\mathbf{r}} = \dot{F}_n\,\mathbf{r} + F_n\,\mathbf{v}, \qquad \frac{d}{dt}(G_n\,\mathbf{v}) = \dot{G}_n\,\mathbf{v} + G_n\,\dot{\mathbf{v}}.$$

Adding them,

$$\mathbf{r}^{(n+1)} = \dot{F}_n\,\mathbf{r} + F_n\,\mathbf{v} + \dot{G}_n\,\mathbf{v} + G_n\,\dot{\mathbf{v}}.$$

The last term contains $\dot{\mathbf{v}}$, which is not $\mathbf{r}$ or $\mathbf{v}$. But the equation of motion Equation 2 says $\dot{\mathbf{v}} = -u\,\mathbf{r}$, so that term points along $\mathbf{r}$:

$$\mathbf{r}^{(n+1)} = \dot{F}_n\,\mathbf{r} + F_n\,\mathbf{v} + \dot{G}_n\,\mathbf{v} - u\,G_n\,\mathbf{r}.$$

Now collect the terms along $\mathbf{r}$ and the terms along $\mathbf{v}$:

$$\mathbf{r}^{(n+1)} = (\dot{F}_n - u\,G_n)\,\mathbf{r} + (F_n + \dot{G}_n)\,\mathbf{v}.$$

This is of the form Equation 3 with $n+1$ in place of n, which completes the induction (it starts at $n = 0$ with $F_0 = 1$, $G_0 = 0$). Reading off the coefficients:

$$\boxed{F_{n+1} = \dot{F}_n - u\,G_n, \qquad G_{n+1} = F_n + \dot{G}_n, \qquad F_0 = 1, \quad G_0 = 0.} \tag{4}$$

1.4 From the recurrence to F and G

Evaluate Equation 3 at t_0, where $\mathbf{r} = \mathbf{r}_0$ and $\mathbf{v} = \mathbf{v}_0$, and put it into the Taylor series Equation 1:

$$\mathbf{r}(t_0 + \tau) = \sum_n \frac{\tau^n}{n!}(F_{n,0}\,\mathbf{r}_0 + G_{n,0}\,\mathbf{v}_0).$$

The vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are the same in every term, so they can be taken out of the sums:

$$\mathbf{r}(t_0 + \tau) = \Big(\sum_n F_{n,0}\frac{\tau^n}{n!}\Big)\mathbf{r}_0 + \Big(\sum_n G_{n,0}\frac{\tau^n}{n!}\Big)\mathbf{v}_0.$$

Comparing with $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$ identifies the Lagrange coefficients:

$$F(\tau) = \sum_{n=0}^{\infty} F_{n,0}\,\frac{\tau^n}{n!}, \qquad G(\tau) = \sum_{n=0}^{\infty} G_{n,0}\,\frac{\tau^n}{n!}. \tag{5}$$

Everything vector-valued has been dealt with. What remains is to compute the **numbers** $F_{n,0}$ and $G_{n,0}$. The recurrence needs the time derivatives $\dot{F}_n$, $\dot{G}_n$, and for that we must know what F_n and G_n are functions of.

2 Three scalars that close the system

2.1 The definitions

$$u = \frac{\mu}{r^3}, \qquad p = \frac{\mathbf{r}\cdot\mathbf{v}}{r^2} = \frac{\dot{r}}{r}, \qquad q = \frac{v^2}{r^2} - u = \frac{v^2 - \mu/r}{r^2}. \tag{6}$$

The second form of p uses $r\dot{r} = \mathbf{r}\cdot\mathbf{v}$, which follows from differentiating $r^2 = \mathbf{r}\cdot\mathbf{r}$: $2r\dot{r} = 2\,\mathbf{r}\cdot\mathbf{v}$.

What they mean:

- u has units s^{-2}. On a circular orbit of radius r the mean motion is $n = \sqrt{\mu/r^3}$, so $u = n^2$; in general $1/\sqrt{u}$ is the time scale of the motion at the current radius.
- p has units s^{-1}. It is the **relative** rate at which the distance changes, and it is zero at periapsis and apoapsis, and everywhere on a circle.

- q has units s^{-2}. Circular speed at radius r satisfies $v^2 = \mu/r$, so $q = 0$ when the speed is circular, whatever the direction of the velocity.

Because $u\tau^2$, $p\tau$ and $q\tau^2$ are dimensionless, they are the natural small parameters of a short step.

2.2 Their time derivatives

Three short calculations, each using only $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = -\mu\mathbf{r}/r^3$ and $r\dot{r} = \mathbf{r}\cdot\mathbf{v}$.

The derivative of u. By the chain rule,

$$\dot{u} = \frac{d}{dt}(\mu\, r^{-3}) = -3\mu\, r^{-4}\,\dot{r} = -3\,\frac{\mu}{r^3}\cdot\frac{\dot{r}}{r} = -3\,u\,p.$$

The derivative of p. Write $p = N/D$ with $N = \mathbf{r}\cdot\mathbf{v}$ and $D = r^2$. Then

$$\dot{N} = \mathbf{v}\cdot\mathbf{v} + \mathbf{r}\cdot\dot{\mathbf{v}} = v^2 - \frac{\mu}{r^3}\,\mathbf{r}\cdot\mathbf{r} = v^2 - \frac{\mu}{r}, \qquad \dot{D} = 2r\dot{r} = 2\,\mathbf{r}\cdot\mathbf{v}.$$

By the quotient rule, $\dot{p} = \dot{N}/D - N\dot{D}/D^2$:

$$\dot{p} = \frac{v^2 - \mu/r}{r^2} - \frac{(\mathbf{r}\cdot\mathbf{v})\cdot 2\,(\mathbf{r}\cdot\mathbf{v})}{r^4} = \Big(\frac{v^2}{r^2} - \frac{\mu}{r^3}\Big) - 2\Big(\frac{\mathbf{r}\cdot\mathbf{v}}{r^2}\Big)^2 = q - 2p^2.$$

The derivative of q. First the speed: $\dfrac{d}{dt}v^2 = 2\,\mathbf{v}\cdot\dot{\mathbf{v}} = -\dfrac{2\mu}{r^3}\,\mathbf{r}\cdot\mathbf{v}$. Then, by the quotient rule,

$$\frac{d}{dt}\Big(\frac{v^2}{r^2}\Big) = \frac{1}{r^2}\frac{dv^2}{dt} - \frac{2v^2\dot{r}}{r^3} = -\frac{2\mu}{r^3}\cdot\frac{\mathbf{r}\cdot\mathbf{v}}{r^2} - 2\,\frac{v^2}{r^2}\cdot\frac{\dot{r}}{r} = -2up - 2\,(q+u)\,p,$$

where the last step uses $v^2/r^2 = q + u$ from Equation 6. Subtracting $\dot{u} = -3up$,

$$\dot{q} = -2up - 2pq - 2pu + 3up = -up - 2pq = -p\,(u + 2q).$$

Collecting the three results:

$$\boxed{\dot{u} = -3\,u\,p, \qquad \dot{p} = q - 2p^2, \qquad \dot{q} = -p\,(u + 2q).} \tag{7}$$

2.3 Why this closes the problem

The right-hand sides of Equation 7 are polynomials in u, p, q. By the product rule, the time derivative of **any** polynomial in u, p, q is again a polynomial in u, p, q:

$$\frac{d}{dt}P(u,p,q) = \frac{\partial P}{\partial u}\,\dot{u} + \frac{\partial P}{\partial p}\,\dot{p} + \frac{\partial P}{\partial q}\,\dot{q} = -3up\,\frac{\partial P}{\partial u} + (q - 2p^2)\,\frac{\partial P}{\partial p} - p\,(u + 2q)\,\frac{\partial P}{\partial q}. \tag{8}$$

The recurrence Equation 4 starts from the constants $F_0 = 1$, $G_0 = 0$ and uses only addition, multiplication by u, and time differentiation. So, by induction, **every F_n and G_n is a polynomial in u, p, q**, and Equation 8 differentiates it with no reference to the vectors. The whole computation has become algebra.

3 Turning the crank

3.1 The first coefficients

Apply Equation 4 step by step, differentiating with Equation 8.

$n = 1$. $F_1 = \dot{F}_0 - uG_0 = 0 - 0 = 0$ and $G_1 = F_0 + \dot{G}_0 = 1 + 0 = 1$. So $\mathbf{r}^{(1)} = \mathbf{v}$, as it should be.

$n = 2$. $F_2 = \dot{F}_1 - uG_1 = 0 - u = -u$ and $G_2 = F_1 + \dot{G}_1 = 0 + 0 = 0$. So $\mathbf{r}^{(2)} = -u\,\mathbf{r}$, the equation of motion.

$n = 3$. $F_3 = \dot{F}_2 - uG_2 = -\dot{u} - 0 = 3up$ and $G_3 = F_2 + \dot{G}_2 = -u + 0 = -u$. So

$$\dddot{\mathbf{r}} = 3up\,\mathbf{r} - u\,\mathbf{v},$$

the **jerk**. Check it directly: $\frac{d}{dt}(-u\mathbf{r}) = -\dot{u}\,\mathbf{r} - u\,\mathbf{v} = 3up\,\mathbf{r} - u\,\mathbf{v}$.

$n = 4$. For F_4 differentiate $F_3 = 3up$ by the product rule, then use Equation 7:

$$\dot{F}_3 = 3(\dot{u}\,p + u\,\dot{p}) = 3(-3up \cdot p + u\,(q - 2p^2)) = -9up^2 + 3uq - 6up^2 = 3uq - 15up^2.$$

Then

$$F_4 = \dot{F}_3 - u\,G_3 = 3uq - 15up^2 - u\,(-u) = u^2 + 3uq - 15up^2.$$

For G_4, $\dot{G}_3 = -\dot{u} = 3up$, so $G_4 = F_3 + \dot{G}_3 = 3up + 3up = 6up$.

$n = 5$. Differentiate $F_4 = u^2 + 3uq - 15up^2$ term by term:

$$\begin{aligned}
\frac{d}{dt}(u^2) &= 2u\,\dot{u} = -6u^2p, \\
\frac{d}{dt}(3uq) &= 3(\dot{u}\,q + u\,\dot{q}) = 3(-3upq - up\,(u + 2q)) = -3u^2p - 15upq, \\
\frac{d}{dt}(-15up^2) &= -15(\dot{u}\,p^2 + 2up\,\dot{p}) = -15(-3up^3 + 2up\,(q - 2p^2)) = 105up^3 - 30upq.
\end{aligned}$$

Adding, $\dot{F}_4 = -9u^2p - 45upq + 105up^3$, and

$$F_5 = \dot{F}_4 - u\,G_4 = -9u^2p - 45upq + 105up^3 - 6u^2p = 15up\,(7p^2 - 3q - u).$$

For G_5: $\dot{G}_4 = 6(\dot{u}\,p + u\,\dot{p}) = 6(-3up^2 + uq - 2up^2) = 6uq - 30up^2$, so

$$G_5 = F_4 + \dot{G}_4 = u^2 + 3uq - 15up^2 + 6uq - 30up^2 = u^2 + 9uq - 45up^2.$$

Table 2: The coefficients of $\mathbf{r}^{(n)} = F_n\mathbf{r} + G_n\mathbf{v}$. The verify script generates them to $n = 8$.

n	F_n	G_n
0	1	0
1	0	1
2	$-u$	0
3	$3up$	$-u$

n	F_n	G_n
4	$u^2 + 3uq - 15up^2$	$6up$
5	$15up\,(7p^2 - 3q - u)$	$u^2 + 9uq - 45up^2$
6	$-u\,(945p^4 - 630p^2q - 210p^2u + 45q^2 + 24qu + u^2)$	$30up\,(14p^2 - 6q - u)$

3.2 The series

Divide each coefficient by $n!$ and put it into Equation 5. Term by term for F: $F_2/2! = -u/2$, $F_3/3! = 3up/6 = up/2$, $F_4/4! = \frac{u}{24}(u + 3q - 15p^2)$, and $F_5/5! = 15up(7p^2 - 3q - u)/120 = \frac{up}{8}(7p^2 - 3q - u)$. For G: $G_3/3! = -u/6$, $G_4/4! = 6up/24 = up/4$, and $G_5/5! = \frac{u}{120}(u + 9q - 45p^2)$. So

$$\boxed{\begin{aligned} F &= 1 - \frac{u}{2}\tau^2 + \frac{up}{2}\tau^3 + \frac{u}{24}(u + 3q - 15p^2)\tau^4 + \frac{up}{8}(7p^2 - 3q - u)\tau^5 + \cdots \\ G &= \tau - \frac{u}{6}\tau^3 + \frac{up}{4}\tau^4 + \frac{u}{120}(u + 9q - 45p^2)\tau^5 + \cdots \end{aligned}} \tag{9}$$

with u, p, q evaluated at t_0, computed from $\mathbf{r}_0$ and $\mathbf{v}_0$.

The input is the state and the step, the output is $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$. There is no iteration, no element, and no distinction between ellipse, parabola and hyperbola.

3.3 The velocity series

$\mathbf{r}_0, \mathbf{v}_0$ do not depend on τ, so differentiate $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$ term by term, using $\frac{d}{d\tau}\tau^n = n\tau^{n-1}$ (for example $\frac{d}{d\tau}\frac{up}{2}\tau^3 = \frac{3up}{2}\tau^2$):

$$\mathbf{v}(t_0 + \tau) = \frac{dF}{d\tau}\,\mathbf{r}_0 + \frac{dG}{d\tau}\,\mathbf{v}_0,$$

$$\begin{aligned} \dot{F} &= -u\,\tau + \frac{3up}{2}\tau^2 + \frac{u}{6}(u + 3q - 15p^2)\tau^3 + \frac{5up}{8}(7p^2 - 3q - u)\tau^4 + \cdots \\ \dot{G} &= 1 - \frac{u}{2}\tau^2 + up\,\tau^3 + \frac{u}{24}(u + 9q - 45p^2)\tau^4 + \cdots \end{aligned} \tag{10}$$

Note that $\dot{F}$ here means $dF/d\tau$, with u, p, q **held at their** t_0 **values**. It is **not** the $\dot{F}_n$ of the recurrence, which is a time derivative along the orbit, with u, p, q changing.

4 Two checks

4.1 The circular orbit

On a circle, $\dot{r} = 0$ and $v^2 = \mu/r$, so $p = 0$ and $q = 0$, and $u = n^2$ with n the mean motion. Set $p = 0$ in Equation 9: the terms $\frac{up}{2}\tau^3$, $\frac{up}{8}(\cdots)\tau^5$ of F and $\frac{up}{4}\tau^4$ of G vanish, and the τ^4 and τ^5 brackets lose their p^2 parts:

$$F = 1 - \frac{u}{2}\tau^2 + \frac{u}{24}\,(u + 3q)\,\tau^4 - \cdots, \qquad G = \tau - \frac{u}{6}\tau^3 + \frac{u}{120}\,(u + 9q)\,\tau^5 - \cdots$$

Now set $q = 0$:

$$F = 1 - \frac{u}{2}\tau^2 + \frac{u^2}{24}\tau^4 - \cdots, \qquad G = \tau - \frac{u}{6}\tau^3 + \frac{u^2}{120}\tau^5 - \cdots$$

and finally $u = n^2$, writing $2 = 2!$, $24 = 4!$, $6 = 3!$, $120 = 5!$:

$$F = 1 - \frac{(n\tau)^2}{2!} + \frac{(n\tau)^4}{4!} - \cdots = \cos n\tau, \qquad G = \tau - \frac{n^2\tau^3}{3!} + \frac{n^4\tau^5}{5!} - \cdots = \frac{\sin n\tau}{n}.$$

So $\mathbf{r} = \mathbf{r}_0 \cos n\tau + (\mathbf{v}_0/n)\sin n\tau$: uniform circular motion. (The verify script checks the agreement through τ^8.) The odd powers of F and the even powers of G all carry a factor p, so p **measures the departure from circular motion**: it is what makes the series of an eccentric orbit differ from cos and sin.

4.2 Conservation of angular momentum

Cross $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$ with $\mathbf{v} = \dot{F}\mathbf{r}_0 + \dot{G}\mathbf{v}_0$. Since $\mathbf{r}_0 \times \mathbf{r}_0 = \mathbf{v}_0 \times \mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_0 \times \mathbf{r}_0 = -\mathbf{r}_0 \times \mathbf{v}_0$,

$$\mathbf{r} \times \mathbf{v} = F\dot{G}\,\mathbf{r}_0 \times \mathbf{v}_0 + G\dot{F}\,\mathbf{v}_0 \times \mathbf{r}_0 = (F\dot{G} - \dot{F}G)\,\mathbf{r}_0 \times \mathbf{v}_0.$$

The angular momentum is constant, so $F\dot{G} - \dot{F}G = 1$ exactly. Check it on the series through τ^3. Multiply out, keeping powers up to τ^3:

$$\begin{aligned} F\dot{G} &= \Big(1 - \frac{u}{2}\tau^2 + \frac{up}{2}\tau^3\Big)\Big(1 - \frac{u}{2}\tau^2 + up\,\tau^3\Big) = 1 - \frac{u}{2}\tau^2 - \frac{u}{2}\tau^2 + \frac{up}{2}\tau^3 + up\,\tau^3 + O(\tau^4) \\ &= 1 - u\tau^2 + \frac{3}{2}up\,\tau^3 + O(\tau^4), \\ \dot{F}G &= \Big(-u\tau + \frac{3}{2}up\,\tau^2\Big)\Big(\tau - \frac{u}{6}\tau^3\Big) = -u\tau^2 + \frac{3}{2}up\,\tau^3 + O(\tau^4). \end{aligned}$$

The difference is $1 + O(\tau^4)$. A truncated series conserves angular momentum only to its own order; Exercise 3 carries the check one order further.

5 How good is it?

5.1 A worked example

Given. A satellite in low Earth orbit has $\mathbf{r}_0 = (7000,\ 0,\ 0)$ km and $\mathbf{v}_0 = (1,\ 7.8,\ 0)$ km/s. **Find** its position $\tau = 60$ s later from the series through τ^4, and compare with the exact answer.

Step 1: the three scalars. $r_0 = 7000$ km, $\mathbf{r}_0 \cdot \mathbf{v}_0 = 7000$ km^2/s, $v_0^2 = 61.84$ km^2/s^2, and $r_0^2 = 4.9 \times 10^7$ km^2:

$$u = \frac{398\,600.4418}{7000^3} = 1.162100{\times}10^{-6}\ \mathrm{s}^{-2}, \qquad p = \frac{7000}{4.9 \times 10^7} = 1.428571{\times}10^{-4}\ \mathrm{s}^{-1},$$

$$q = \frac{61.84}{4.9 \times 10^7} - u = 1.262041 \times 10^{-6} - 1.162100 \times 10^{-6} = 9.99404 \times 10^{-8}\ \mathrm{s}^{-2}.$$

The time scale $1/\sqrt{u} = 928$ s, so $\tau = 60$ s is short. (For the record, $a = 7659$ km and the period is 111 min.)

Step 2: the coefficients. The bracket of the τ^4 term is $u + 3q - 15p^2 = 1.155799 \times 10^{-6}$ s^{-2}. With $\tau = 60$ s, term by term,

	τ^0 or τ^1	τ^2	τ^3	τ^4	sum
F	1	-2.091781×10^{-3}	1.792955×10^{-5}	7.253035×10^{-7}	0.997926874
G [s]	60	—	-4.183561×10^{-2}	5.378865×10^{-4}	59.958702272

The terms shrink fast: the τ^2 term is 2×10^{-3}, and each further power of τ brings a factor of order $\sqrt{u}\,\tau \approx 0.065$ or $p\tau \approx 0.009$.

Step 3: the position. $\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0 = (7045.44682,\ 467.67788,\ 0)$ km.

Check. The exact coefficients (universal variables) are $F = 0.997926853$, $G = 59.958710410$ s, giving $(7045.44668,\ 467.67794,\ 0)$ km. The series is off by 0.15 **m** after 60 s. The velocity from Equation 10, through τ^3, is off by 1.3 cm/s.

5.2 Over longer steps

Figure 2 repeats the computation continuously in τ and for several truncation orders, and Table 4 lists a few values.

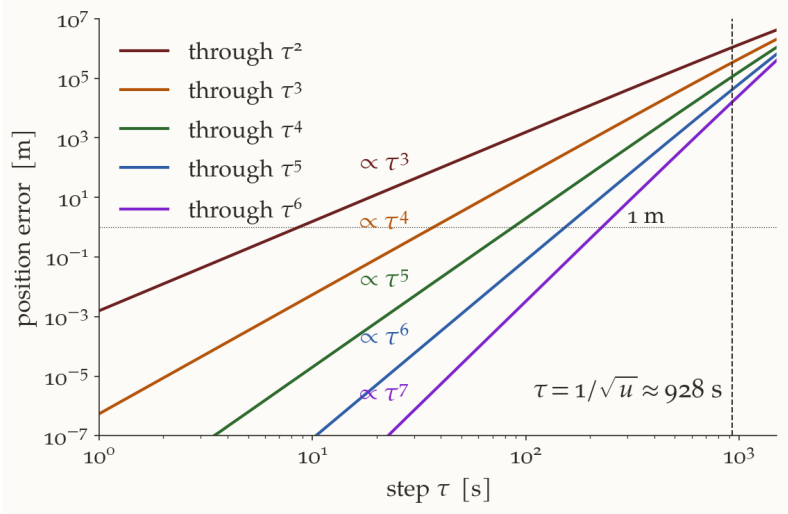


Figure 2: **The truncated series against the exact motion,** for the orbit of Section 5.1: position error of the series truncated after $\tau^2, \ldots, \tau^6$, against the exact Lagrange coefficients. The web version of this note has an interactive version that also draws the predicted paths. *Script: fig-fg-series.py.*

Table 4: Position error of the truncated series for the example orbit, against the exact Lagrange coefficients.

τ	through τ^2	through τ^3	through τ^4	through τ^6
10 s	1.6 m	5.4 mm	0.02 mm	< 1 μm
60 s	334 m	6.9 m	0.15 m	0.09 mm
300 s	40 km	4.1 km	458 m	6.8 m

Two rules can be read off Figure 2:

1. **Each extra term buys one power of τ.** Truncating after τ^k leaves an error proportional to τ^{k+1}, the straight lines of slope $k+1$ on the log-log plot.
2. **The scale is** $1/\sqrt{u}$ (928 s here, comparable to $T/2\pi = 1061$ s). For $\tau \ll 1/\sqrt{u}$, a few degrees of arc, the series is excellent. As τ approaches $1/\sqrt{u}$ all the truncations fail together, and the predicted paths peel away from the true orbit.

The series is therefore a **short-step** tool. For an eccentric orbit it also has a finite radius of convergence, so adding terms cannot rescue a long step. For long steps use the universal-variable coefficients.

5.3 Where the series is used

In Gauss's method of angles-only orbit determination the state at the middle observation is unknown, so the exact F and G cannot be computed. Keeping only the first two terms of Equation 9,

$$F \approx 1 - \frac{u}{2}\tau^2, \qquad G \approx \tau - \frac{u}{6}\tau^3,$$

leaves $u = \mu/r^3$ as the only unknown, a function of the **distance** alone. That reduces the method to one polynomial equation in r; the orbit-determination note does this and then replaces the series by the exact coefficients.

History. Gauss's first approximation in *Theoria Motus* (1809) is equivalent to keeping these leading terms; the method grew out of his 1801 recovery of Ceres from a short arc (Gauss 1809).

6 Summary

1. Two-body motion is planar, so every derivative of $\mathbf{r}$ is a combination $\mathbf{r}^{(n)} = F_n\mathbf{r} + G_n\mathbf{v}$ of the current position and velocity (Equation 3, Figure 1).
2. Differentiating once more and using $\dot{\mathbf{v}} = -u\mathbf{r}$ gives the recurrence $F_{n+1} = \dot{F}_n - uG_n$, $G_{n+1} = F_n + \dot{G}_n$ (Equation 4), and the Taylor series of $\mathbf{r}$ becomes $F\mathbf{r}_0 + G\mathbf{v}_0$ with $F = \sum F_{n,0}\tau^n/n!$, $G = \sum G_{n,0}\tau^n/n!$.
3. With $u = \mu/r^3$, $p = \mathbf{r}\cdot\mathbf{v}/r^2$, $q = v^2/r^2 - u$, the rules $\dot{u} = -3up$, $\dot{p} = q - 2p^2$, $\dot{q} = -p(u + 2q)$ (Equation 7) make every F_n, G_n a polynomial in u, p, q.
4. The result is the f and g series Equation 9 and its derivative Equation 10.
5. It reduces to $\cos n\tau$, $\sin n\tau/n$ on a circle, and conserves angular momentum to its own order.
6. Truncated after τ^k its error grows as τ^{k+1}; it is accurate for $\tau \ll 1/\sqrt{u}$, and it is the first step of Gauss's method.

Everything in this note can be reproduced with fg-series-verify.py: the derivative rules from $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3$ (sympy), the coefficients to $n = 8$, the circular limit and the angular-momentum identity, and the numbers of Section 5.1 and Table 4 against the exact coefficients of _uv.py.

Exercises

1. **Directly.** Differentiate $\ddot{\mathbf{r}} = -u\mathbf{r}$ twice, using Equation 7, and confirm $\mathbf{r}^{(4)} = (u^2 + 3uq - 15up^2)\,\mathbf{r} + 6up\,\mathbf{v}$.

Solution. Once: $\dddot{\mathbf{r}} = -\dot{u}\,\mathbf{r} - u\,\mathbf{v} = 3up\,\mathbf{r} - u\,\mathbf{v}$. Twice:

$$\mathbf{r}^{(4)} = 3(\dot{u}\,p + u\,\dot{p})\,\mathbf{r} + 3up\,\mathbf{v} - \dot{u}\,\mathbf{v} - u\,\dot{\mathbf{v}} = 3(-3up^2 + uq - 2up^2)\mathbf{r} + 3up\,\mathbf{v} + 3up\,\mathbf{v} + u^2\,\mathbf{r},$$

which is $(u^2 + 3uq - 15up^2)\,\mathbf{r} + 6up\,\mathbf{v}$. This is the recurrence done by hand: the term $-u\dot{\mathbf{v}} = u^2\mathbf{r}$ is the $-uG_3$ of Equation 4.

2. **The next coefficients.** Derive F_6 and G_6 from the recurrence and compare with Table 2.

Hint. $G_6 = F_5 + \dot{G}_5$ is the easier one: $\dot{G}_5 = 2u\dot{u} + 9(\dot{u}q + u\dot{q}) - 45(\dot{u}p^2 + 2up\dot{p})$. Expand each product with Equation 7, collect, and add F_5; you should get $30up(14p^2 - 6q - u)$. For $F_6 = \dot{F}_5 - uG_5$ write $F_5 = 105up^3 - 45upq - 15u^2p$ first.

3. **Angular momentum, one order further.** Keep all four series, Equation 9 and Equation 10, through τ^4, and show that $F\dot{G} - \dot{F}G = 1 + O(\tau^5)$.

Solution. Through τ^3 the check was done in the text. For τ^4, collect every product of a τ^i term and a τ^{4-i} term (F and $\dot{G}$ have no τ^1 term, G has no τ^2 term):

$$F\dot{G}\big|_{\tau^4} = \underbrace{\frac{u}{24}(u + 9q - 45p^2)}_{1\cdot\dot{G}_{\tau^4}} + \underbrace{\frac{u^2}{4}}_{(-\frac{u}{2})(-\frac{u}{2})} + \underbrace{\frac{u}{24}(u + 3q - 15p^2)}_{F_{\tau^4}\cdot 1} = \frac{u^2}{3} + \frac{uq}{2} - \frac{5up^2}{2},$$

$$\dot{F}G\big|_{\tau^4} = \underbrace{\frac{u^2}{6}}_{(-u)(-\frac{u}{6})} + \underbrace{\frac{u}{6}(u + 3q - 15p^2)}_{\dot{F}_{\tau^3}\cdot\tau} = \frac{u^2}{3} + \frac{uq}{2} - \frac{5up^2}{2}.$$

They are equal, so the difference has no τ^4 term. (The verify script checks the identity through τ^7 with the coefficients to $n = 8$.)

4. **At periapsis.** At periapsis $p = 0$. Write F and G through τ^4 in terms of the periapsis radius r_p, speed v_p and μ. Which terms of the circular-orbit series are changed by the eccentricity?

Solution. With $p = 0$: $F = 1 - \frac{u}{2}\tau^2 + \frac{u}{24}(u + 3q)\tau^4$ and $G = \tau - \frac{u}{6}\tau^3$, where $u = \mu/r_p^3$ and $q = v_p^2/r_p^2 - \mu/r_p^3$. Only the τ^4 term of F (and higher terms) feels the eccentricity, through q: at periapsis $v_p^2 > \mu/r_p$, so $q > 0$ and F decreases more slowly than $\cos n\tau$ would. The first deviation of the motion from a circle at periapsis is a fourth-order effect.

5. **How long a step?** For the orbit of Section 5.1, how large can τ be if a position error of 1 m is acceptable and the series is cut after τ^4? Check your answer against the figure.

Answer. About 85 s. The error through τ^4 is 0.15 m at 60 s and grows as τ^5, so it reaches 1 m at $60 \times (1/0.154)^{1/5} \approx 87$ s; Figure 2 shows the green curve crossing the 1 m line there.

Further reading

The f and g series, and its use in orbit determination, are treated in the orbit-determination literature (Escobal 1965; Bate, Mueller, and White 1971; Vallado 2013); the orbit-determination note uses it in Gauss's method, and the universal-variables note gives the exact coefficients that replace it for long steps.

Bate, Roger R., Donald D. Mueller, and Jerry E. White. 1971. *Fundamentals of Astrodynamics*. New York: Dover.

Escobal, Pedro Ramon. 1965. *Methods of Orbit Determination*. New York: Wiley.

Gauss, Carl Friedrich. 1809. *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium*. Hamburg: Perthes et Besser.

Vallado, David A. 2013. *Fundamentals of Astrodynamics and Applications*. 4th ed. Hawthorne, CA: Microcosm Press.