# Relative Motion and the Clohessy–Wiltshire Equations

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## Table of contents

**1 Where this note sits** **2**

**2 The target-centred rotating frame** **3**
2.1 The three axes . . . 3
2.2 How fast does the frame turn? . . . 4
2.3 Why a rotating frame at all? . . . 5

**3 The exact equation of relative motion** **5**
3.1 Subtracting two two-body equations . . . 5
3.2 Differentiating in a rotating frame, component by component . . . 6
3.3 The same result from the transport theorem . . . 6
3.4 The exact equations in the rotating frame . . . 7

**4 Linearisation** **7**
4.1 Small parameters and the binomial series . . . 7
4.2 Second route: the gravity-gradient tensor . . . 8
4.3 The Clohessy–Wiltshire equations . . . 9
4.4 Reading the equations . . . 10
4.5 Out of the plane . . . 10
4.6 Two first integrals . . . 10

**5 The solution** **11**
5.1 Reduction to one equation . . . 11
5.2 The radial motion . . . 12
5.3 The velocities . . . 12
5.4 The along-track motion . . . 12
5.5 Checking the solution . . . 13
5.6 The state transition matrix . . . 13
5.7 Where the drift comes from: the eigenstructure . . . 14

**6 The anatomy of a natural motion** **15**
6.1 Drift or no drift . . . 15
6.2 Guiding centre and epicycle . . . 16
6.3 Why the guiding centre is the difference in semi-major axis . . . 18

**7 Worked examples, checked against Kepler** **18**
7.1 Example 1: a 1 m/s prograde burn . . . 18
7.2 Check against Kepler . . . 19
7.3 The co-circular drift . . . 19
7.4 Example 2: a 1 m/s radial burn . . . 20

**8 How far can we trust it?** **20**

8.1 cw against the exact two-body motion . . . . . 20
8.2 Rules of thumb . . . . . 21

**9 Summary** **22**

**Further reading** **22**

# 1 Where this note sits

Lambert's problem (the Lambert note) joins two points **anywhere** on two orbits, and it is the right tool when the chaser is thousands of kilometres from its target. A few kilometres from the target it is the wrong one, for two reasons.

- **Arithmetic.** At 420 km altitude gravity is $g = 8.625\ \mathrm{m/s^2}$. A chaser 10 km above the target feels a gravitational acceleration that differs from the target's by only $0.025\ \mathrm{m/s^2}$: the difference of two vectors of $8.6\ \mathrm{m/s^2}$ each, which agree to within 0.29%. (In the frame that turns with the target, the centrifugal term adds half as much again, and the net relative acceleration is $0.038\ \mathrm{m/s^2}$, as we will find.) Likewise a rendezvous burn of a few m/s is a tiny fraction (one part in 7700 per m/s) of the 7.66 km/s orbital speed. Everything can be computed, but the answer lives in the last digits of two large numbers.
- **Insight.** A Lambert solver returns numbers. It does not say *why* burning forward makes you fall behind, *why* a radial burn closes after exactly one orbit, or what a given miss distance will cost.

The cure is to describe the chaser **relative to the target**, in a frame that **rotates with the target**, and to **linearise**. The reward is a pair of equations that are linear with constant coefficients and can be solved with sines and cosines. The equations are named after Clohessy and Wiltshire (Clohessy and Wiltshire 1960), and are Hill's lunar equations (Hill 1878) with the attraction of the body at the origin switched off.

**History: two names, 82 years apart.** George William Hill (1838–1914) wrote down equations of this form in 1878 for the Moon's motion relative to the Earth, in a frame turning with the Sun. W. H. Clohessy and R. S. Wiltshire derived them again in 1960 for one spacecraft moving relative to another, with rendezvous guidance in mind. The literature calls them *Hill's*, *Clohessy–Wiltshire* (CW) or *Hill–Clohessy–Wiltshire* (HCW) equations; here: CW.
Hill worked for the US Nautical Almanac Office from 1861 and spent most of his career at his family farm in West Nyack, New York. Poincaré said that his 1878 memoir contains "the germ of most of the progress" that celestial mechanics has made since.

The route is short, and every step will be done in full:

1. **The frame** (Section 2). Define the target-centred rotating frame and find how fast it turns.
2. **The exact equation** (Section 3). Subtract two two-body equations, and re-express the result in the rotating frame. No approximation has been made yet.
3. **Linearisation** (Section 4). Expand the gravity difference for a nearby chaser. This is the only approximation in the note, and we will do it twice, in two independent ways.
4. **The solution** (Section 5). Integrate in closed form, check the answer by substitution, and package it as a state transition matrix.
5. **The anatomy of a motion** (Section 6). Every solution is a *drifting centre* carrying a *2:1 epicycle*. We find what sets each, and link them to the semi-major axis.
6. **Worked examples** (Section 7) checked against Kepler, and **limits of validity** (Section 8).

**The setting, and what is assumed**

The target is on a **circular** orbit at 420 km (ISS-like). Motion is treated **in-plane** except in Section 4.5. We assume:

1. **Two-body gravity** from a point-mass Earth: no $J_2$, no drag, no third bodies.
2. The target's circular orbit has **constant** rate $n$ (this is where "circular"

is used).
3. The chaser is **close**: $\rho \ll r_t$ (this is where the linearisation is used).
4. The spacecraft's **mutual attraction is neglected**. A $4 \times 10^5$ kg target pulls a chaser 10 km away with $2.7 \times 10^{-13}$ m/s$^2$, eleven orders of magnitude below the 0.038 m/s$^2$ we care about.
5. Burns are **impulsive**: they change velocity instantly and leave position alone.

Table 1: Notation

| Symbol | Meaning | Value |
|---|---|---|
| $\mu$ | Earth's gravitational parameter | 398 600.4418 km$^3$/s$^2$ |
| $r_t$ | radius of the target's circular orbit (6378.137 + 420 km) | 6798.137 km |
| $n = \sqrt{\mu / r_t^3}$ | orbital rate of the target, and of the rotating frame | $1.126378 \times 10^{-3}$ rad/s |
| $T = 2\pi / n$ | period | 5578.2 s = 92.97 min |
| $v_t = n r_t$ | target's speed | 7.65727 km/s |
| $\rho = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ | chaser's position **relative to the target** | |
| $\mathbf{X} = (x, y, \dot{x}, \dot{y})^{\mathsf{T}}$ | in-plane relative state | |
| $\tau = nt$, $s = \sin\tau$, $c = \cos\tau$ | shorthand used in the matrices | |

A dot over a symbol means $d/dt$ **as seen in the rotating frame**. A derivative taken in the inertial frame carries the subscript $\mathcal{I}$.

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

# 2 The target-centred rotating frame

## 2.1 The three axes

At time $t$ the target is at $\mathbf{r}_t$ with velocity $\mathbf{v}_t$. Its orbit normal is $\mathbf{h}_t = \mathbf{r}_t \times \mathbf{v}_t$. Define

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_t}{r_t}, \qquad \hat{\mathbf{k}} = \frac{\mathbf{h}_t}{h_t}, \qquad \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}. \tag{1}$$

These three unit vectors are orthonormal and right-handed ($\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$), and they ride with the target (Figure 1). On a circular orbit the velocity is perpendicular to the radius at every instant, so $\hat{\mathbf{j}}$ points **along the velocity**. The coordinates of the chaser are

- $x$ — **radial**: positive means *above* the target (farther from the Earth);
- $y$ — **along-track**: positive means *ahead* of the target;
- $z$ — **cross-track**: out of the orbit plane.

The same three directions appear elsewhere in the course under other names: they are the polar basis $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{h}}$ of the two-body note, and the $\hat{\mathbf{R}}, \hat{\mathbf{S}}, \hat{\mathbf{W}}$ axes of the Gauss variational equations (see the notation page). Here they are attached to the *target*, and positions are measured from it.

**Common mistake: someone else's axes**

Other books order these axes differently. NASA's LVLH, for example, takes $x$ roughly along the velocity and $z$ towards the Earth. Every matrix in this note, including the state-transition matrix of Section 5.6, then has its rows and columns permuted and some signs flipped. Before using a matrix from another source, check which way its axes point.

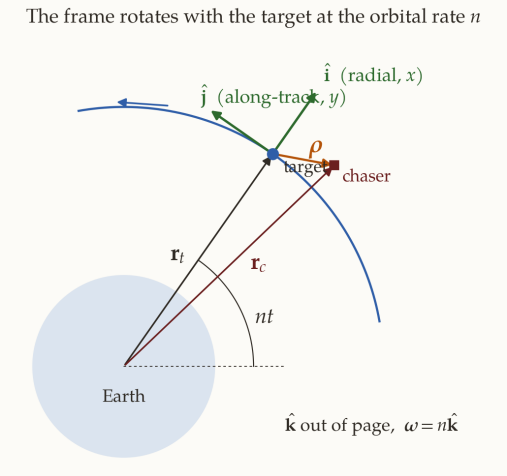



Figure 1: The local-vertical local-horizontal (LVLH) frame: origin on the target, $\hat{\mathbf{i}}$ radially outward, $\hat{\mathbf{j}}$ along the direction of motion, $\hat{\mathbf{k}}$ along the orbit normal. It rotates at the target's orbital rate $n$. *Script: fig-cw-frame.py.*

To be concrete, choose inertial axes $(X, Y, Z)$ with the orbit in the $XY$ plane and the target on the $X$ axis at $t = 0$. The target's polar angle is then $nt$, and the unit vectors, written in the inertial basis, are

$$\hat{\mathbf{i}} = \begin{pmatrix} \cos nt \\ \sin nt \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{j}} = \begin{pmatrix} -\sin nt \\ \cos nt \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

(Check: $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = (0 \cdot 0 - 1 \cdot \sin nt,\ 1 \cdot \cos nt - 0 \cdot 0,\ 0) = \hat{\mathbf{j}}$, as Equation 1 requires.)

**Naming.** This frame is called **LVLH** (local vertical, local horizontal) in NASA documents, **RSW** or **RTN** (radial, along-track or transverse, normal) in much of the astrodynamics literature, and the **Hill frame** in Hill's honour. Axis conventions differ between sources — NASA's own LVLH, for example, takes $x$ roughly along the velocity and $z$ towards the Earth — so check the axes before using someone else's matrix. In ISS operations the line along the velocity is the *V-bar* and the line along the radius vector is the *R-bar*; an approach along either is a V-bar or an R-bar approach.

## 2.2 How fast does the frame turn?

A circular orbit of radius $r_t$ and angular rate $n$ needs a centripetal acceleration $r_t n^2$, and the only force is gravity, $\mu / r_t^2$. Equating,

$$r_t\, n^2 = \frac{\mu}{r_t^2} \qquad \Longrightarrow \qquad n = \sqrt{\frac{\mu}{r_t^3}}, \tag{3}$$

which is Kepler's third law for a circle. Now differentiate Equation 2, remembering that $n$ is a **constant**:

$$\begin{aligned} \frac{d\hat{\mathbf{i}}}{dt} &= n\,(-\sin nt,\ \cos nt,\ 0)^{\mathsf{T}} = n\hat{\mathbf{j}}, \\ \frac{d\hat{\mathbf{j}}}{dt} &= -n\,(\cos nt,\ \sin nt,\ 0)^{\mathsf{T}} = -n\hat{\mathbf{i}}, \\ \frac{d\hat{\mathbf{k}}}{dt} &= \mathbf{0}. \end{aligned} \tag{4}$$

All three are of the form $d\hat{\mathbf{e}}/dt = \omega \times \hat{\mathbf{e}}$ with the **angular velocity** $\omega = n\hat{\mathbf{k}}$, because $n\hat{\mathbf{k}} \times \hat{\mathbf{i}} = n\hat{\mathbf{j}}$, $n\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -n\hat{\mathbf{i}}$ and $n\hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$. So **the frame turns about the orbit normal at the constant rate** $n$: once per orbit, $T = 92.97$ min.

### 2.3 Why a rotating frame at all?

Suppose the chaser sits motionless next to the target: 3 km above and 4 km behind, say. In the rotating frame its coordinates are $(x, y) = (3, -4)$ km, **forever**. Seen from an inertial frame, the same offset $\rho = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ has components, from Equation 2,

$$\rho_X = x \cos nt - y \sin nt, \qquad \rho_Y = x \sin nt + y \cos nt,$$

which swing between $\pm\sqrt{x^2 + y^2} = \pm 5$ km once per orbit (Figure 2). The interesting information — how far above, how far ahead — is in the *slowly varying* rotating-frame numbers; in the inertial frame it is buried in a rotation. This is the first reason to use the rotating frame. The second, which will turn out to matter even more, is that in this frame **the target is at rest and the problem has a symmetry** (Section 4.4).

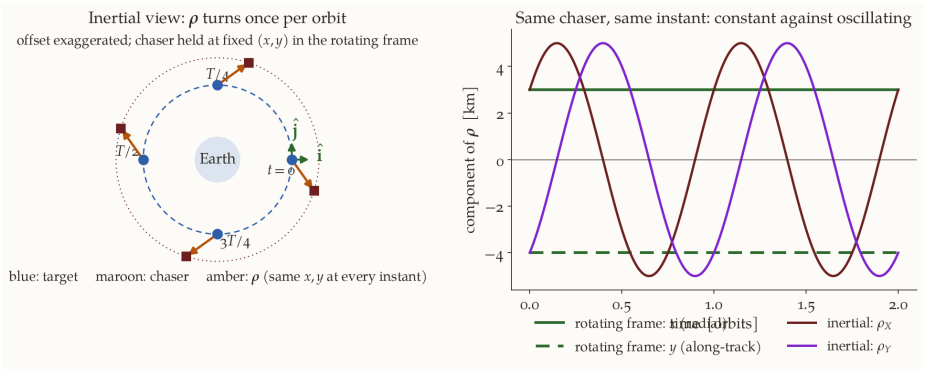



Figure 2: A chaser held at a fixed offset from the target. **Left:** inertial view, with the offset exaggerated: $\rho$ turns once per orbit. **Right:** its components for a real offset of 3 km above and 4 km behind. In the rotating frame they are constant (green); in the inertial frame they oscillate with amplitude $|\rho| = 5$ km. *Script: fig-cw-inertial.py.*

## 3 The exact equation of relative motion

### 3.1 Subtracting two two-body equations

Let the chaser be at $\mathbf{r}_c = \mathbf{r}_t + \rho$. Each spacecraft obeys the two-body equation in the **inertial** frame:

$$\ddot{\mathbf{r}}_t\big|_{\mathcal{I}} = -\frac{\mu\,\mathbf{r}_t}{r_t^3}, \qquad \ddot{\mathbf{r}}_c\big|_{\mathcal{I}} = -\frac{\mu\,\mathbf{r}_c}{r_c^3},$$

where $|_{\mathcal{I}}$ marks a derivative taken in the inertial frame (a plain dot will soon mean the rotating-frame derivative). Subtracting the first from the second, $\ddot{\rho}|_{\mathcal{I}} = \ddot{\mathbf{r}}_c|_{\mathcal{I}} - \ddot{\mathbf{r}}_t|_{\mathcal{I}}$, gives

$$\ddot{\rho}\big|_{\mathcal{I}} = -\frac{\mu\,(\mathbf{r}_t + \rho)}{|\mathbf{r}_t + \rho|^3} + \frac{\mu\,\mathbf{r}_t}{r_t^3}. \tag{5}$$

Equation 5 is **exact**. It used neither the circularity of the target's orbit nor the smallness of $\rho$. It is also useless as it stands, for two reasons: the right-hand side is a difference of two nearly equal vectors (Section 1), and the left-hand side is an acceleration seen from the *inertial* frame, in which (Section 2.3) the coordinates of $\rho$ oscillate even when the chaser is at rest. We fix the second by changing frame, and the first by expanding.

## 3.2 Differentiating in a rotating frame, component by component

Write $\rho = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ (the $z$ axis is treated in Section 4.5). Both the coordinates $x(t)$, $y(t)$ and the unit vectors change with time, so the product rule gives two terms for every derivative. Using Equation 4, $\dot{\hat{\mathbf{i}}} = n\hat{\mathbf{j}}$ and $\dot{\hat{\mathbf{j}}} = -n\hat{\mathbf{i}}$:

**First derivative.**

$$\begin{aligned}\dot{\rho}\big|_{\mathcal{I}} &= \dot{x}\,\hat{\mathbf{i}} + \dot{y}\,\hat{\mathbf{j}} + x\,\frac{d\hat{\mathbf{i}}}{dt} + y\,\frac{d\hat{\mathbf{j}}}{dt} \\ &= \dot{x}\,\hat{\mathbf{i}} + \dot{y}\,\hat{\mathbf{j}} + nx\,\hat{\mathbf{j}} - ny\,\hat{\mathbf{i}} \\ &= (\dot{x} - ny)\,\hat{\mathbf{i}} + (\dot{y} + nx)\,\hat{\mathbf{j}}.\end{aligned} \tag{6}$$

**Second derivative.** Differentiate Equation 6 in the same way. The coefficient of $\hat{\mathbf{i}}$ is $(\dot{x} - ny)$, with derivative $(\ddot{x} - n\dot{y})$ and, from the moving $\hat{\mathbf{i}}$, an extra $(\dot{x} - ny)\,n\hat{\mathbf{j}}$; the coefficient of $\hat{\mathbf{j}}$ is $(\dot{y} + nx)$, with derivative $(\ddot{y} + n\dot{x})$ and, from the moving $\hat{\mathbf{j}}$, an extra $-(\dot{y} + nx)\,n\hat{\mathbf{i}}$. Collecting the $\hat{\mathbf{i}}$ terms and the $\hat{\mathbf{j}}$ terms:

$$\begin{aligned}\ddot{\rho}\big|_{\mathcal{I}} &= \Big[(\ddot{x} - n\dot{y}) - n(\dot{y} + nx)\Big]\hat{\mathbf{i}} + \Big[(\ddot{y} + n\dot{x}) + n(\dot{x} - ny)\Big]\hat{\mathbf{j}} \\ &= (\ddot{x} - 2n\dot{y} - n^2x)\hat{\mathbf{i}} + (\ddot{y} + 2n\dot{x} - n^2y)\hat{\mathbf{j}}.\end{aligned} \tag{7}$$

Nothing here is approximate. The terms that would be there if the frame did not turn are $\ddot{x}\,\hat{\mathbf{i}} + \ddot{y}\,\hat{\mathbf{j}}$; everything else comes from the rotation.

## 3.3 The same result from the transport theorem

The computation above is a special case of a general rule. For *any* vector $\mathbf{q} = q_i\hat{\mathbf{i}} + q_j\hat{\mathbf{j}} + q_k\hat{\mathbf{k}}$ expressed in a frame that rotates with angular velocity $\omega$,

$$\frac{d\mathbf{q}}{dt}\Big|_{\mathcal{I}} = \underbrace{\dot{q}_i\hat{\mathbf{i}} + \dot{q}_j\hat{\mathbf{j}} + \dot{q}_k\hat{\mathbf{k}}}_{\frac{d\mathbf{q}}{dt}\big|_{\mathcal{R}}} + q_i\,\omega\times\hat{\mathbf{i}} + q_j\,\omega\times\hat{\mathbf{j}} + q_k\,\omega\times\hat{\mathbf{k}} = \frac{d\mathbf{q}}{dt}\Big|_{\mathcal{R}} + \omega\times\mathbf{q}. \tag{8}$$

The first step is the product rule with $\dot{\hat{\mathbf{e}}} = \omega\times\hat{\mathbf{e}}$ (Section 2.2); the second is linearity of the cross product. Apply Equation 8 to $\mathbf{q} = \rho$, and then again to $\mathbf{q} = \dot{\rho}|_{\mathcal{I}}$, with $\dot{\omega} = \mathbf{0}$ because the target's orbit is circular:

$$\begin{aligned}\dot{\rho}\big|_{\mathcal{I}} &= \dot{\rho} + \omega\times\rho, \\ \ddot{\rho}\big|_{\mathcal{I}} &= (\ddot{\rho} + \omega\times\dot{\rho}) + \omega\times(\dot{\rho} + \omega\times\rho) = \ddot{\rho} + 2\,\omega\times\dot{\rho} + \omega\times(\omega\times\rho).\end{aligned}$$

For $\omega = n\hat{\mathbf{k}}$ and $\rho = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, with $\hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}}$:

**History: where the rotating frame comes from.** Gaspard-Gustave Coriolis worked out the complete equations of motion in a rotating frame in 1835 (Coriolis 1835), while studying machines with rotating parts, water wheels in particular. He showed that besides the centrifugal term a body *moving* in the frame feels an acceleration at right angles to its velocity. The name "Coriolis force" came into use only in the early twentieth century. The effect had been used earlier: George Hadley (1735) explained the trade winds by the Earth's rotation, and Foucault's pendulum (1851) made it visible.

$$\omega \times \rho = n(x\hat{\mathbf{j}} - y\hat{\mathbf{i}}),$$
$$\omega \times \dot{\rho} = n(\dot{x}\hat{\mathbf{j}} - \dot{y}\hat{\mathbf{i}}),$$
$$\omega \times (\omega \times \rho) = n^2\,\hat{\mathbf{k}} \times (x\hat{\mathbf{j}} - y\hat{\mathbf{i}}) = -n^2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$

Substituting reproduces Equation 7 line for line. In this language the two extra terms have names: $2\omega\times\dot{\rho}$ is the **Coriolis** acceleration, and $\omega\times(\omega\times\rho) = -n^2\rho$ is the **centripetal** one. Moved to the other side of the equation of motion they appear as the familiar apparent forces: the Coriolis term $-2\omega \times \dot{\rho}$ (at right angles to the relative velocity, so it does no work) and the centrifugal term $+n^2\rho$ (directed away from the target, the origin of the rotating frame).

> Where "circular" has been used
>
> $\dot{\omega} = \mathbf{0}$ is the only place the circularity of the target's orbit entered the kinematics. For an eccentric target the frame's rate $\dot{f}$ (the rate of change of the true anomaly) is not constant, there is an extra Euler term $\dot{\omega} \times \rho$, and the coefficients below depend on time. The linearised equations still exist (Section 8), but they are no longer this simple.

### 3.4 The exact equations in the rotating frame

In LVLH components the target is at $\mathbf{r}_t = r_t\hat{\mathbf{i}}$ and the chaser at $\mathbf{r}_t + \rho = (r_t + x)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, so $|\mathbf{r}_t + \rho|^2 = (r_t + x)^2 + y^2 = r_c^2$. Equate Equation 7 with the right-hand side of Equation 5, component by component, and use $\mu/r_t^2 = n^2 r_t$ from Equation 3:

$$\begin{aligned} \ddot{x} - 2n\dot{y} - n^2 x &= -\frac{\mu\,(r_t + x)}{r_c^3} + n^2 r_t, \\ \ddot{y} + 2n\dot{x} - n^2 y &= -\frac{\mu\, y}{r_c^3}. \end{aligned} \tag{9}$$

These are the **exact** (nonlinear) equations of planar relative motion about a circular reference orbit. We will use them as the "truth" against which CW is judged in Section 8. All that remains is to simplify the right-hand side.

## 4 Linearisation

### 4.1 Small parameters and the binomial series

Measure the offset in units of the orbit radius: $\varepsilon = x/r_t$ and $\eta = y/r_t$, both small. Then $r_c^2 = r_t^2 D$, with

$$D = (1 + \varepsilon)^2 + \eta^2 = 1 + 2\varepsilon + \varepsilon^2 + \eta^2.$$

Using $\mu = n^2 r_t^3$ again, the right-hand sides of Equation 9, divided by $n^2 r_t$, are

$$\frac{\mathrm{RHS}_x}{n^2 r_t} = 1 - \frac{1 + \varepsilon}{D^{3/2}}, \qquad \frac{\mathrm{RHS}_y}{n^2 r_t} = -\frac{\eta}{D^{3/2}}.$$

Look at the first of these. The "1" is the target's own gravity and the second term is the chaser's: at $\varepsilon = \eta = 0$ they cancel exactly. **What survives is the first correction in $\varepsilon$ and $\eta$**, and this is why a numerical difference of two large accelerations loses digits, while an algebraic expansion does not.

To expand $D^{-3/2}$ we need the binomial series, valid for $|u| < 1$:

$$(1+u)^{-3/2} = 1 - \tfrac{3}{2}\,u + \tfrac{15}{8}\,u^2 - \tfrac{35}{16}\,u^3 + \cdots, \tag{10}$$

(the coefficient of $u^k$ is $\binom{-3/2}{k}$; for example $\binom{-3/2}{2} = \frac{(-3/2)(-5/2)}{2} = \frac{15}{8}$). Here $u = 2\varepsilon + \varepsilon^2 + \eta^2$, which is of **first** order in the small quantities because of the $2\varepsilon$. Keeping only first-order terms, $u \approx 2\varepsilon$, so

$$D^{-3/2} \approx 1 - 3\varepsilon.$$

Then

$$\frac{\mathrm{RHS}_x}{n^2 r_t} \approx 1 - (1+\varepsilon)(1-3\varepsilon) \;\to\; 2\varepsilon,$$
$$\frac{\mathrm{RHS}_y}{n^2 r_t} \approx -\eta\,(1-3\varepsilon) \;\to\; -\eta,$$

where the arrows mean "discard terms of second order": multiplying out $(1+\varepsilon)(1-3\varepsilon)$ produces an $\varepsilon^2$ that is *not* meaningful, since we dropped second-order terms from $D^{-3/2}$ itself. Multiplying by $n^2 r_t$ and returning to $x = r_t\varepsilon$, $y = r_t\eta$:

$$\mathrm{RHS} \;\approx\; (2n^2 x,\; -n^2 y). \tag{11}$$

What we threw away: the second-order terms

Keep one more order. From Equation 10 with $u = 2\varepsilon + \varepsilon^2 + \eta^2$:

$$-\tfrac{3}{2}u = -3\varepsilon - \tfrac{3}{2}\varepsilon^2 - \tfrac{3}{2}\eta^2, \qquad \tfrac{15}{8}u^2 = \tfrac{15}{8}\,(4\varepsilon^2 + \cdots) = \tfrac{15}{2}\varepsilon^2 + \cdots,$$

so $D^{-3/2} = 1 - 3\varepsilon + 6\varepsilon^2 - \frac{3}{2}\eta^2 + O(3)$. Then $(1+\varepsilon)D^{-3/2} = 1 - 2\varepsilon + 3\varepsilon^2 - \frac{3}{2}\eta^2 + O(3)$, and

$$\frac{\mathrm{RHS}_x}{n^2 r_t} = 2\varepsilon - 3\varepsilon^2 + \tfrac{3}{2}\eta^2 + O(3), \qquad \frac{\mathrm{RHS}_y}{n^2 r_t} = -\eta + 3\varepsilon\eta + O(3).$$

In dimensional form the *exact* equations are, to second order,

$$\ddot{x} - 2n\dot{y} - 3n^2 x = \frac{3n^2}{2r_t}(y^2 - 2x^2), \qquad \ddot{y} + 2n\dot{x} = \frac{3n^2}{r_t}\,xy.$$

The correction is smaller than what we keep by a factor of order $\rho/r_t$: $1.5 \times 10^{-3}$ at 10 km. Figure 3 (panels b and c) shows how the exact curves peel away from the straight line as $\varepsilon$ and $\eta$ grow. (cw-verify.py checks this series symbolically.)

## 4.2 Second route: the gravity-gradient tensor

Equation 11 can be reached without expanding anything, and the second derivation shows what it *means*. Gravity is a vector field $\mathbf{a}(\mathbf{r}) = -\mu\mathbf{r}/r^3$. The difference between the field at the chaser and at the target is, to first order, the **Jacobian** of the field times the displacement:

$$\mathbf{a}(\mathbf{r}_t + \rho) - \mathbf{a}(\mathbf{r}_t) \approx \mathsf{G}\,\rho, \qquad G_{ij} = \left.\frac{\partial a_i}{\partial r_j}\right|_{\mathbf{r}_t}.$$

Differentiating $a_i = -\mu\, r_i\, r^{-3}$ with $\partial r/\partial r_j = r_j/r$ (the product rule, one term for each factor):

$$\frac{\partial a_i}{\partial r_j} = -\mu\Big[\frac{\delta_{ij}}{r^3} - \frac{3\,r_i r_j}{r^5}\Big] = \frac{\mu}{r^3}\big(3\,\hat r_i \hat r_j - \delta_{ij}\big).$$

At the target $\hat{\mathbf{r}} = \hat{\mathbf{i}}$, so $\hat r_i \hat r_j$ is 1 for $i = j =$ radial and 0 otherwise, and $\mu/r_t^3 = n^2$:

$$\mathsf{G} = n^2\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \delta\mathbf{g} = \mathsf{G}\rho = n^2\,(2x,\ -y,\ -z). \tag{12}$$

The in-plane part is Equation 11. Three things are worth reading off.

- **Radial stretch, $+2n^2x$.** Gravity falls as $1/r^2$, so a chaser *above* the target is pulled *less* than the target. Relative to the target it falls away outward. Quantitatively, $\partial(-\mu/r^2)/\partial r = +2\mu/r^3 = 2n^2$.
- **Transverse squeeze, $-n^2y$.** Gravity points at the Earth's centre. A chaser displaced a distance $y$ along-track sees the centre in a direction rotated by the small angle $y/r_t$, so it feels a sideways component $-g\,(y/r_t) = -n^2y$ back towards the target's radial line.
- **Zero trace.** $2 - 1 - 1 = 0$: the field is divergence-free ($\nabla \cdot \mathbf{a} = 0$ in empty space, Laplace's equation), so a small ring of test masses is stretched in one direction exactly as fast as it is squeezed in the other two (Figure 3, panel a).

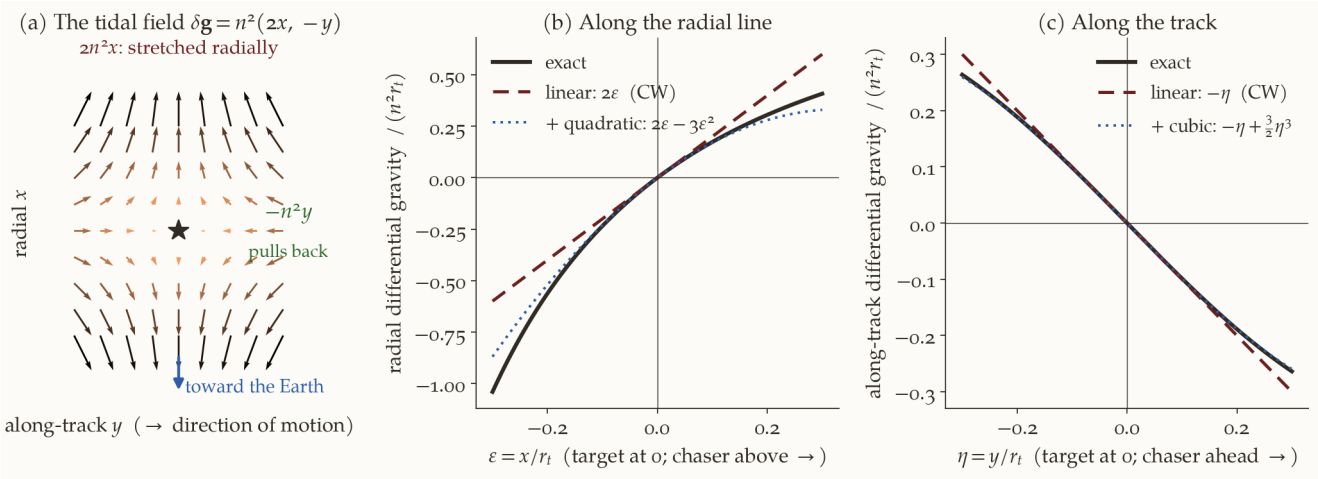



Figure 3: **(a)** The differential gravity around the target in the LVLH plane (vertical: radial; horizontal: along-track). **(b)** Along the radial line: exact differential acceleration, the linear term used by CW, and the first correction (second order in $\varepsilon$). **(c)** The same along the track (the first correction is cubic in $\eta$). Plotted to $\pm 0.3\,r_t$, far beyond the range where CW is used. *Script: fig-cw-tidal.py.*

## 4.3 The Clohessy–Wiltshire equations

Equate Equation 7 with Equation 11:

$$\ddot x - 2n\dot y - n^2x = 2n^2x, \qquad \ddot y + 2n\dot x - n^2y = -n^2y.$$

In the second equation $-n^2y$ appears on both sides and cancels. In the first, move $n^2x$ to the right: $2n^2x + n^2x = 3n^2x$. The result is

$$\boxed{\ddot x - 2n\dot y - 3n^2x = 0, \qquad \ddot y + 2n\dot x = 0.} \tag{13}$$

**History: tides and the 3.** Newton's *Principia* (1687) explained the ocean tides as the *difference* between the Moon's pull on the near side, the centre and the far side of the Earth (Newton 1687). That difference is exactly the gravity gradient of this section. Édouard Roche (1848) used it to show that a body held together only by its own gravity cannot survive inside a critical distance of a planet: the Roche limit.
Hill's lunar problem turns the same tidal term into the **Hill sphere**: the distance $r_H = (\mu_0/3n^2)^{1/3}$ at which the pull of a body of parameter $\mu_0$ equals the $3n^2x$ of Equation 13 below. The 3 is ours.

## 4.4 Reading the equations

- **Linear, with constant coefficients.** Both facts come from the same two assumptions: a circular target (constant $n$) and a nearby chaser (linearised gravity). This is what makes closed-form solution possible.
- **The $-3n^2x$ is a net term.** It is $2n^2x$ of gravity gradient plus $n^2x$ of centrifugal acceleration. The two are of the same order, which is why a rotating frame and the tide must be treated together.
- **Coupled through Coriolis.** The $x$ equation contains $\dot{y}$ and the $y$ equation contains $\dot{x}$: radial motion drives along-track acceleration and vice versa. This coupling is the origin of every counter-intuitive result in rendezvous.
- **$y$ never appears undifferentiated.** Neither equation contains $y$ itself, only $\dot{y}$ and $\ddot{y}$. Shifting the chaser along the orbit by any amount $y_0$ changes nothing: **every point of the circular orbit is equivalent**, which is the symmetry of the problem. It is also the reason a first integral exists (Section 4.6).
- **No damping.** The equations conserve energy-like quantities (Section 4.6); there is no friction, so no solution decays.

**History: Hill's equations with the Earth removed.** Hill's lunar equations, in a frame turning at the Sun's mean motion $n$, with the origin at the Earth and $x$ along the Sun–Earth line, read

$$\ddot{x}-2n\dot{y}-3n^2x=-\frac{\mu_0\,x}{r^3},\qquad \ddot{y}+2n\dot{x}=-\frac{\mu_0\,y}{r^3},$$

where $\mu_0$ is the gravitational parameter of the body at the origin and $r^2=x^2+y^2$. Remove the right-hand sides, the attraction of the body at the origin, and you have Equation 13. **The CW equations are Hill's equations for a target of negligible mass**, with the Earth in the role of Hill's Sun.
Hill's form is the more general one: put $\mu_0$ back and the radial line has equilibria at $x=\pm(\mu_0/3n^2)^{1/3}$, the Hill sphere.

## 4.5 Out of the plane

For the $\hat{\mathbf{k}}$ component, $\omega\parallel\hat{\mathbf{k}}$ gives $\omega\times(z\hat{\mathbf{k}})=\mathbf{0}$ and $\omega\times(\dot{z}\hat{\mathbf{k}})=\mathbf{0}$, so the kinematics contributes only $\ddot{z}$. The gravity difference contributes $-n^2z$ from Equation 12 (the chaser's gravity points at the Earth's centre, which is tilted by $z/r_t$ from the target's). Hence

$$\ddot{z}+n^2z=0\qquad\Longrightarrow\qquad z(t)=z_0\cos nt+\frac{\dot{z}_0}{n}\sin nt. \tag{14}$$

A simple harmonic oscillator at the orbital frequency, **decoupled** from $x$ and $y$. Physically it is just a slightly inclined orbit: two orbits of different inclination cross twice per revolution. We stay in-plane from here on.

## 4.6 Two first integrals

A *first integral* is a combination of the unknowns and their first derivatives that stays constant. Two exist, and each tells us something without solving anything.

**The momentum integral.** The second CW equation is an exact derivative:

$$\ddot{y}+2n\dot{x}=\frac{d}{dt}(\dot{y}+2nx)=0,$$

so that

$$\dot{y}+2nx=\dot{y}_0+2nx_0\equiv K. \tag{15}$$

What is $K$? Compute the chaser's angular momentum about the Earth's centre in the inertial frame. Its position is $(r_t+x)\hat{\mathbf{i}}+y\hat{\mathbf{j}}$ and, by Equation 6, its velocity is $\mathbf{v}_t+\dot{\rho}|_{\mathcal{I}}=(\dot{x}-ny)\hat{\mathbf{i}}+(nr_t+\dot{y}+nx)\hat{\mathbf{j}}$. The $\hat{\mathbf{k}}$ component of the cross product is

$$\begin{aligned}h_c&=(r_t+x)(nr_t+\dot{y}+nx)-y(\dot{x}-ny)\\&=nr_t^2+r_t(\dot{y}+2nx)+[x\dot{y}-y\dot{x}+n(x^2+y^2)].\end{aligned}$$

The target has $h_t=nr_t^2$. So $h_c-h_t=r_t(\dot{y}+2nx)+(\text{second order})$. Each spacecraft conserves its own angular momentum exactly, so $h_c-h_t$ is exactly constant, and to first order $K=(h_c-h_t)/r_t$: the momentum integral is the linearised statement that *the chaser's angular momentum, relative to the target's, is conserved*. It is what

makes along-track and radial motion inseparable: you cannot change $\dot{y}$ without paying for it in $x$, and the reverse.

**The Jacobi (energy) integral.** Multiply the $x$ equation by $\dot{x}$ and the $y$ equation by $\dot{y}$, and add:

$$\dot{x}\ddot{x} - 2n\dot{x}\dot{y} - 3n^2x\dot{x} \;+\; \dot{y}\ddot{y} + 2n\dot{x}\dot{y} = 0.$$

The two Coriolis terms $\mp 2n\dot{x}\dot{y}$ cancel (they are the "does no work" statement), leaving a perfect derivative:

$$\frac{d}{dt}\Big[\tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \tfrac{3}{2}n^2x^2\Big] = 0,$$

so that

$$J = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \tfrac{3}{2}n^2x^2 = \text{const.} \tag{16}$$

**History.** Jacobi found the analogous constant of motion for the restricted three-body problem in a rotating frame in 1836. Hill used it in 1877–78 to draw **curves of zero velocity**: closed curves that the Moon cannot cross, which prove, within his model, that the Moon stays bound to the Earth for ever. Equation 16 is the same integral in the limit that Hill's problem becomes CW.

The two integrals together give the whole radial motion at once. Substitute $\dot{y} = K - 2nx$ into $J$:

$$J = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(K-2nx)^2 - \tfrac{3}{2}n^2x^2 = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}n^2x^2 - 2nKx + \tfrac{1}{2}K^2.$$

Complete the square, $\tfrac{1}{2}n^2x^2 - 2nKx = \tfrac{1}{2}n^2(x-2K/n)^2 - 2K^2$, to get

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}n^2\Big(x - \frac{2K}{n}\Big)^2 = J + \tfrac{3}{2}K^2 = \text{const.} \tag{17}$$

This is the energy of a **harmonic oscillator of frequency $n$ centred at $x_c = 2K/n$.** We have just proved, without solving anything, that the radial motion oscillates harmonically at the orbital frequency about a centre fixed by $K$. The tide pulls outward (Section 4.2), and yet $x$ cannot run away: the Coriolis coupling converts the outward push into along-track velocity, which by $\dot{y} + 2nx = K$ pushes back.

# 5 The solution

## 5.1 Reduction to one equation

The integral Equation 15 gives $\dot{y} = K - 2nx$ in terms of $x$ alone. Substitute it into the $x$ equation of Equation 13:

$$\ddot{x} - 2n(K-2nx) - 3n^2x = 0 \quad\Longrightarrow\quad \ddot{x} + n^2x = 2nK.$$

Define the constant

$$x_c \equiv \frac{2K}{n} = 4x_0 + \frac{2\dot{y}_0}{n}, \tag{18}$$

so that the equation reads $\ddot{x} + n^2(x - x_c) = 0$: a harmonic oscillator, centred at $x_c$, as Equation 17 said. (The equality on the right of Equation 18 is $K = \dot{y}_0 + 2nx_0$ inserted.)

## 5.2 The radial motion

The general solution of $\ddot{x} + n^2(x - x_c) = 0$ is a particular solution plus the homogeneous one. A particular solution is the constant $x = x_c$ (substitute: $\ddot{x} = 0$ and $n^2(x_c - x_c) = 0$). The homogeneous solutions are $\cos nt$ and $\sin nt$. So

$$x(t) = x_c + C_1 \cos nt + C_2 \sin nt.$$

The two constants come from the initial conditions:

$$\begin{aligned} x(0) = x_c + C_1 = x_0 \;&\Rightarrow\; C_1 = x_0 - x_c = -\Big(3x_0 + \frac{2\dot{y}_0}{n}\Big) \equiv -\alpha, \\ \dot{x}(0) = nC_2 = \dot{x}_0 \;&\Rightarrow\; C_2 = \frac{\dot{x}_0}{n} \equiv \beta, \end{aligned}$$

where $\alpha = 3x_0 + 2\dot{y}_0/n$ and $\beta = \dot{x}_0/n$ will keep reappearing. Written out in the form needed for the matrix below (collect the $x_0$ terms: $4x_0 - 3x_0 \cos nt$; the $\dot{y}_0$ terms: $\frac{2\dot{y}_0}{n}(1 - \cos nt)$):

$$\begin{aligned} x(t) &= x_c - \alpha \cos nt + \beta \sin nt \\ &= (4 - 3\cos nt)\, x_0 + \frac{\sin nt}{n}\, \dot{x}_0 + \frac{2(1 - \cos nt)}{n}\, \dot{y}_0. \end{aligned} \tag{19}$$

## 5.3 The velocities

Differentiate Equation 19:

$$\begin{aligned} \dot{x}(t) &= n\alpha \sin nt + n\beta \cos nt \\ &= 3n \sin nt\; x_0 + \cos nt\; \dot{x}_0 + 2 \sin nt\; \dot{y}_0. \end{aligned} \tag{20}$$

For $\dot{y}$ we do not integrate anything: Equation 15 gives $\dot{y} = K - 2nx$, and $K = nx_c/2$. So $\dot{y} = K - 2n(x_c - \alpha \cos nt + \beta \sin nt)$. Since $2nx_c = 4K$, this is $\dot{y} = -3K + 2n\alpha \cos nt - 2n\beta \sin nt$. Expand $K = \dot{y}_0 + 2nx_0$ and $\alpha = 3x_0 + 2\dot{y}_0/n$:

$$\begin{aligned} \dot{y}(t) &= -3(\dot{y}_0 + 2nx_0) + (6nx_0 + 4\dot{y}_0) \cos nt - 2\dot{x}_0 \sin nt \\ &= -6n(1 - \cos nt)\, x_0 - 2 \sin nt\; \dot{x}_0 + (4 \cos nt - 3)\, \dot{y}_0. \end{aligned} \tag{21}$$

## 5.4 The along-track motion

Integrate Equation 21 from 0 to $t$ with $y(0) = y_0$. The three integrals needed are

$$\begin{aligned} \int_0^t (1 - \cos nt')\, dt' &= t - \frac{\sin nt}{n}, \\ \int_0^t \sin nt'\, dt' &= \frac{1 - \cos nt}{n}, \\ \int_0^t (4 \cos nt' - 3)\, dt' &= \frac{4 \sin nt}{n} - 3t. \end{aligned}$$

Hence, term by term,

$$y(t) = y_0 - 6n\Big(t - \frac{\sin nt}{n}\Big)x_0 - \frac{2(1 - \cos nt)}{n}\, \dot{x}_0 + \Big(\frac{4 \sin nt}{n} - 3t\Big)\dot{y}_0,$$

which is, with $-6nt\, x_0 + 6 \sin nt\, x_0 = 6(\sin nt - nt)\, x_0$,

$$y(t) = 6(\sin nt - nt)\,x_0 + y_0 \\ -\frac{2(1-\cos nt)}{n}\,\dot{x}_0 + \frac{4\sin nt - 3nt}{n}\,\dot{y}_0. \tag{22}$$

The terms in $nt$ are the only ones that grow without bound; we will come back to them in Section 6.

## 5.5 Checking the solution

A derived formula should be checked by an independent route. Take each initial condition on its own (the equations are linear, so contributions add) and substitute into Equation 13. Write $c = \cos nt$, $s = \sin nt$.

**Unit $x_0$:** $x = 4 - 3c$, $y = 6(s - nt)$, so $\ddot{x} = 3n^2 c$, $\dot{x} = 3ns$, $\dot{y} = 6n(c-1)$, $\ddot{y} = -6n^2 s$:

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 3n^2 c + 12n^2(1-c) - 3n^2(4-3c) \\ = (3 - 12 + 9)\,n^2 c + (12 - 12)\,n^2 = 0 \;\checkmark$$
$$\ddot{y} + 2n\dot{x} = -6n^2 s + 2n\,(3ns) = 0 \;\checkmark$$

**Unit $\dot{x}_0$:** $x = s/n$, $y = -2(1-c)/n$, so $\ddot{x} = -ns$, $\dot{x} = c$, $\dot{y} = -2s$, $\ddot{y} = -2nc$:

$$\ddot{x} - 2n\dot{y} - 3n^2 x = -ns + 4ns - 3ns = 0 \;\checkmark$$
$$\ddot{y} + 2n\dot{x} = -2nc + 2nc = 0 \;\checkmark$$

**Unit $\dot{y}_0$:** $x = 2(1-c)/n$, $y = (4s - 3nt)/n$, so $\ddot{x} = 2nc$, $\dot{x} = 2s$, $\dot{y} = 4c - 3$, $\ddot{y} = -4ns$:

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 2nc - 2n(4c-3) - 6n(1-c) \\ = (2 - 8 + 6)\,nc + (6-6)\,n = 0 \;\checkmark$$
$$\ddot{y} + 2n\dot{x} = -4ns + 4ns = 0 \;\checkmark$$

**Unit $y_0$:** $x = 0$, $y = 1$ trivially. And the **initial conditions**: at $t = 0$, $c = 1$, $s = 0$, so Equation 19, Equation 20, Equation 22 and Equation 21 return $x_0$, $\dot{x}_0$, $y_0$, $\dot{y}_0$ ($\checkmark$). A last check at **small times**: expanding $4 - 3\cos nt \approx 1 + \frac{3}{2}n^2 t^2$ and $2(1-\cos nt)/n \approx nt^2$ gives $\ddot{x}(0) = 3n^2 x_0 + 2n\dot{y}_0$, which is exactly what the first equation of Equation 13 says at $t = 0$. For $t \ll 1/n$ the motion is ballistic, and the first correction is the equation itself. (cw-verify.py repeats all of this symbolically.)

## 5.6 The state transition matrix

Collect the in-plane state as $\mathbf{X} = (x,\ y,\ \dot{x},\ \dot{y})^{\mathsf{T}}$. The cw equations are a first-order linear system

$$\dot{\mathbf{X}} = \mathsf{A}\,\mathbf{X}, \qquad \mathsf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & -2n & 0 \end{bmatrix}, \tag{23}$$

where rows 3 and 4 are the two cw equations solved for $\ddot{x}$ and $\ddot{y}$. Because the solution is **linear in the initial state**, it can be written $\mathbf{X}(t) = (t)\mathbf{X}_0$, where the $j$-th column of  is the solution that starts from the $j$-th unit vector. Reading the columns off Equation 19, Equation 20, Equation 22 and Equation 21 ($\tau = nt$, $s = \sin\tau$, $c = \cos\tau$):

$$(t) = \begin{bmatrix} rr & rv \\ vr & vv \end{bmatrix},$$

$$rr = \begin{bmatrix} 4-3c & 0 \\ 6(s-\tau) & 1 \end{bmatrix}, \qquad rv = \frac{1}{n}\begin{bmatrix} s & 2(1-c) \\ -2(1-c) & 4s-3\tau \end{bmatrix},$$

$$vr = \begin{bmatrix} 3ns & 0 \\ -6n(1-c) & 0 \end{bmatrix}, \qquad vv = \begin{bmatrix} c & 2s \\ -2s & 4c-3 \end{bmatrix}. \tag{24}$$

(The subscripts $r$ and $v$ stand for position $\mathbf{r} = (x, y)$ and velocity $\mathbf{v} = (\dot{x}, \dot{y})$. For example $_{rv}$ says how the final *position* depends on the initial *velocity*.)

**What to know about .**

- $(0) = \mathsf{I}$**.** Obvious from the formulas, and what a "transition matrix" must do at zero elapsed time.
- $^{\cdot} = \mathsf{A}$**.** Each column is a solution of $\dot{\mathbf{X}} = \mathsf{A}\mathbf{X}$. Formally $(t) = e^{\mathsf{A}t}$, the matrix exponential.
- **Group property,** $(t_1 + t_2) = (t_2)(t_1)$**.** The system is autonomous (the coefficients do not depend on time), and solutions of a linear ODE are unique, so propagating for $t_1$ and then $t_2$ is the same as propagating for $t_1 + t_2$. The trigonometry confirms it. For example, the $(\dot{x}, \dot{x})$ entry of $(t_2)(t_1)$ is row 3 of $(t_2)$ times column 3 of $(t_1)$:

$$\begin{aligned}(3ns_2)\frac{s_1}{n} + 0 + c_2c_1 + (2s_2)(-2s_1) &= 3s_1s_2 + c_1c_2 - 4s_1s_2 \\ &= c_1c_2 - s_1s_2 = \cos(\tau_1 + \tau_2),\end{aligned}$$

  which is the $(\dot{x}, \dot{x})$ entry of $(t_1 + t_2)$.
- **Inverse.** $(t)^{-1} = (-t)$ (the group property with $t_2 = -t_1$): one matrix multiply also propagates backwards.
- $\det = 1$**.** For $= e^{\mathsf{A}t}$, $\det = e^{t\,\mathrm{tr}\,\mathsf{A}}$ (Liouville's formula), and $\mathrm{tr}\,\mathsf{A} = 0$. Areas in the $(\mathbf{r}, \mathbf{v})$ phase space are preserved: relative motion has no sinks.
- **A numerical example.** At half an orbit, $\tau = \pi$, $c = -1$, $s = 0$, and in the scaled variables $(x,\ y,\ \dot{x}/n,\ \dot{y}/n)$ the matrix is

$$\begin{bmatrix} 7 & 0 & 0 & 4 \\ -6\pi & 1 & -4 & -3\pi \\ 0 & 0 & -1 & 0 \\ -12 & 0 & 0 & -7 \end{bmatrix}.$$

> **Important**
>
> Propagating a relative state over any time is **one matrix multiply** — no Kepler equation, no iteration, no loss of digits. This is what the chase note uses in its terminal phase.

## 5.7 Where the drift comes from: the eigenstructure

Why does a *linear oscillating* system contain a term growing in $t$? Look at the eigenvalues of $\mathsf{A}$. Expand $\det(\lambda\mathsf{I} - \mathsf{A})$ along the second column, which has a single non-zero entry $\lambda$:

$$\begin{aligned}\det(\lambda\mathsf{I} - \mathsf{A}) &= \lambda \det\begin{bmatrix} \lambda & -1 & 0 \\ -3n^2 & \lambda & -2n \\ 0 & 2n & \lambda \end{bmatrix} \\ &= \lambda\big[\lambda(\lambda^2 + 4n^2) - 3n^2\lambda\big] = \lambda^2(\lambda^2 + n^2).\end{aligned} \tag{25}$$

**History.** Peano (1888), and Baker (1905) after him, wrote the solution of a linear differential equation with time-varying coefficients as a series of repeated integrals. With constant coefficients that series sums to the matrix exponential used here. The phrase "state transition" entered engineering with the state-space methods of 1960 — the same year as Clohessy and Wiltshire — when Kalman published his celebrated paper on filtering (Kalman 1960).

The eigenvalues are $\lambda = \pm in$ (simple, giving the oscillation) and $\lambda = 0$ (**double**). No eigenvalue has a positive real part, so nothing grows exponentially. The double zero is the interesting one. Solving $\mathsf{A}\mathbf{v} = \mathbf{0}$ gives only **one** eigenvector, $\mathbf{v} = (0, 1, 0, 0)^{\mathsf{T}}$ (a pure along-track shift), so $\mathsf{A}$ has a **Jordan block** and a second, *generalised* eigenvector $\mathbf{w}$ with $\mathsf{A}\mathbf{w} = -\frac{3}{2}n\,\mathbf{v}$:

$$\mathbf{w} = (1,\ 0,\ 0,\ -\tfrac{3}{2}n)^{\mathsf{T}}: \qquad \mathsf{A}\mathbf{w} = (0,\ -\tfrac{3}{2}n,\ 0,\ 0)^{\mathsf{T}} = -\tfrac{3}{2}n\,\mathbf{v}.$$

The solution that starts at $\mathbf{w}$ is $e^{\mathsf{A}t}\mathbf{w} = \mathbf{w} + t\,\mathsf{A}\mathbf{w} = (1,\ -\frac{3}{2}nt,\ 0,\ -\frac{3}{2}n)^{\mathsf{T}}$: the chaser sits 1 km higher than the target and drifts steadily backwards at $\frac{3}{2}n$ per km of altitude. That is **a circular orbit at a different radius**, whose period differs from the target's (Section 6.3). So the four fundamental motions are:

Table 2: The four independent solutions of the in-plane cw equations.

| Mode | State | Meaning |
|---|---|---|
| shift | $(0, 1, 0, 0)$ | the same circle, 1 km ahead: an **equilibrium** |
| shear | $(1, -\frac{3}{2}nt,\ 0, -\frac{3}{2}n)$ | a neighbouring circle: **steady drift** |
| epicycle, 2 modes | $\cos nt,\ \sin nt$ | the **oscillation** at frequency $n$ |

Any chaser motion is a combination of these, and the next section shows that picture explicitly.

# 6 The anatomy of a natural motion

## 6.1 Drift or no drift

The only terms in  that grow without bound are those proportional to $\tau = nt$, and they all sit in the $y$ row: $-6nt\,x_0$ from Equation 22 and $-3t\,\dot{y}_0$. Collecting,

$$y(t) = \underbrace{-\,3\,(2nx_0 + \dot{y}_0)\,t}_{\text{secular drift}} + \text{bounded terms} = -3K\,t + \text{bounded terms}.$$

Hence relative motion is bounded if and only if $K = 0$:

$$\boxed{\text{Relative motion is bounded} \iff \dot{y}_0 = -2n\,x_0.} \tag{26}$$

The condition says: **match the target's period**. We will see in Section 6.3 that $\dot{y}_0 + 2nx_0$ measures the difference in semi-major axis between the chaser's orbit and the target's. Any chaser whose along-track velocity differs from $-2nx_0$ has a different semi-major axis, and drifts.

Bounded is not the same as *around the target*. With $K = 0$ the motion is a closed 2:1 ellipse, but Section 6.2 shows that its centre sits at the along-track position $y_c = y_0 - 2\dot{x}_0/n$. It encircles the target only if, in addition, $y_c = 0$; with $\dot{x}_0 = 0$ that means $y_0 = 0$.

## 6.2 Guiding centre and epicycle

Equation 19 and Equation 20 suggest rewriting everything in terms of the constants we met: $x_c = 4x_0 + 2\dot{y}_0/n$, $\alpha = 3x_0 + 2\dot{y}_0/n$ and $\beta = \dot{x}_0/n$. The solution for $y$, from Equation 22, is

$$y(t) = y_c - \tfrac{3}{2}\, n\, x_c\, t + 2\alpha \sin nt + 2\beta \cos nt, \qquad y_c \equiv y_0 - 2\beta = y_0 - \frac{2\dot{x}_0}{n},$$

(to see this, integrate $\dot{y} = -3K + 2n\alpha\cos nt - 2n\beta \sin nt$ from Section 5.3, which gives $y = y_0 - 3Kt + 2\alpha \sin nt - 2\beta(1 - \cos nt)$, and use $K = \frac{1}{2}nx_c$ and $y_c = y_0 - 2\beta$). Now combine $\alpha = A\cos\phi$, $\beta = A\sin\phi$, with $A = \sqrt{\alpha^2 + \beta^2}$. Then $-\alpha\cos nt + \beta \sin nt = -A\cos(nt + \phi)$ and $2\alpha \sin nt + 2\beta\cos nt = 2A\sin(nt + \phi)$, so

$$\boxed{\begin{aligned} x(t) &= x_c - A\cos(nt + \phi), \\ y(t) &= y_c - \tfrac{3}{2}\, n\, x_c\, t + 2A\sin(nt + \phi). \end{aligned}} \tag{27}$$

Eliminating the phase $nt + \phi$:

$$\left(\frac{x - x_c}{A}\right)^2 + \left(\frac{y - y_c + \frac{3}{2}nx_c\, t}{2A}\right)^2 = 1.$$

This says, in words, that **every natural motion is a 2:1 ellipse** — radial semi-axis $A$, along-track semi-axis $2A$, traversed once per orbit — **whose centre drifts along-track at the constant speed** $-\frac{3}{2}nx_c$. The four constants $(x_c, y_c, A, \phi)$ are the four initial conditions in a better basis:

| Constant | What it sets | Mode of Table 2 |
|---|---|---|
| $y_c$ | where the guiding centre starts along-track | shift |
| $x_c = 4x_0 + 2\dot{y}_0/n$ | the altitude of the guiding centre, and its drift rate $-\frac{3}{2}nx_c$ | shear |
| $A = \sqrt{(3x_0 + 2\dot{y}_0/n)^2 + (\dot{x}_0/n)^2}$ | the size of the epicycle | epicycle |
| $\phi = \arg(\alpha + i\beta)$ | where on the epicycle the chaser starts | epicycle |

**Which burn changes which constant.** An impulse changes $\dot{x}_0$ and $\dot{y}_0$ but not the position. From the formulas above, an along-track burn $\Delta\dot{y}$ moves $x_c$ by $2\Delta\dot{y}/n$ (and so changes the drift) and leaves $y_c$ alone; a radial burn $\Delta\dot{x}$ moves $y_c$ by $-2\Delta\dot{x}/n$ and leaves $x_c$, hence the drift, alone. Both change the epicycle $(A, \phi)$. So: **to stop a drift, burn along-track; to shift the centre of a closed ellipse, burn radially.**

The shape of the track depends on one ratio, $x_c/A$ (Figure 4). The along-track velocity is $\dot{y} = -\frac{3}{2}nx_c + 2nA\cos(nt + \phi)$. It changes sign if and only if $2A > \frac{3}{2}|x_c|$:

- $|x_c| > \frac{4}{3}A$: $\dot{y}$ never changes sign. The path is **wavy**, always moving the same way along-track.

- $|x_c| = \frac{4}{3}A$: $\dot{y}$ touches zero once per orbit. The path has **cusps**. (A chaser released from rest above the target, $\mathbf{X}_0 = (x_0, 0, 0, 0)$, has $x_c = 4x_0$ and $A = 3x_0$, which is exactly this case.)
- $|x_c| < \frac{4}{3}A$: the path **loops**.
- $x_c = 0$: no drift, a closed 2:1 ellipse centred on the guiding centre at $y_c$.

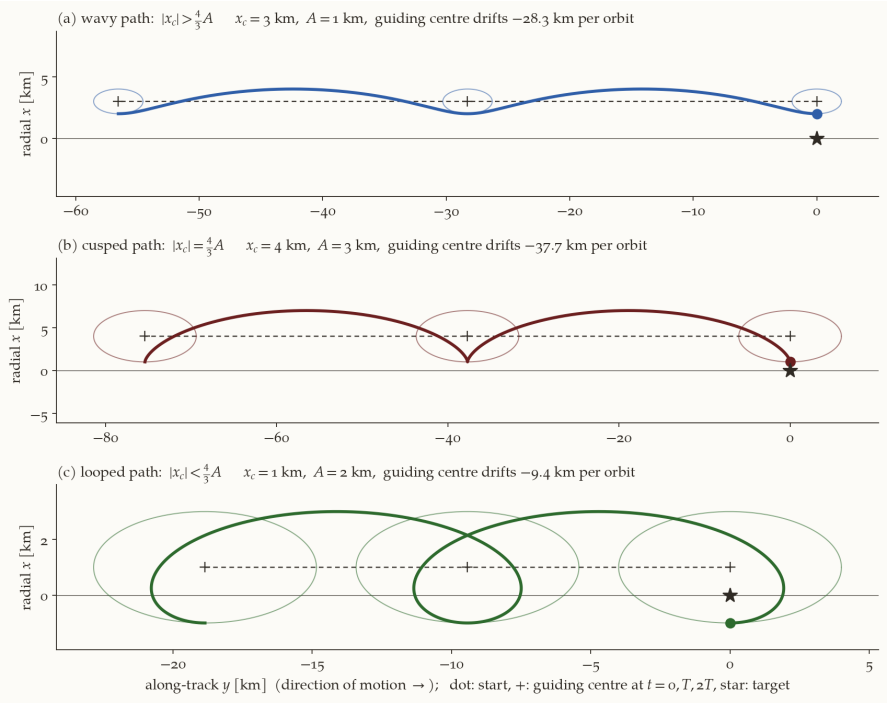



Figure 4: Three natural motions over two orbits, in the (along-track, radial) plane. Thick line: the chaser. Thin ellipses: the epicycle drawn about the guiding centre (+, dashed line) at $t = 0$, $T$, $2T$. In every case the guiding centre drifts at $-\frac{3}{2}nx_c$, i.e. $-3\pi x_c$ per orbit, while the chaser circles it. *Script: fig-cw-epicycle.py*.

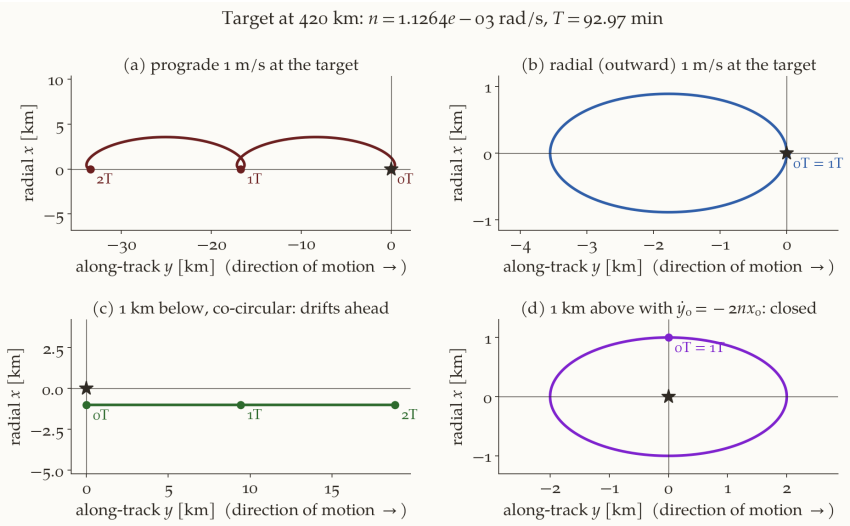



Figure 5: Natural motion about the target (star), plotted with the direction of motion to the right and "up" radially outward. (a) A prograde burn sends the chaser **up and behind**. (b) A radial burn gives a closed 2:1 ellipse that returns after one orbit. (c) A lower co-circular orbit drifts **ahead**. (d) With $\dot{y}_0 = -2nx_0$ the ellipse closes around the target. *Script: fig-cw-motions.py*.

### 6.3 Why the guiding centre is the difference in semi-major axis

The guiding-centre altitude $x_c$ has an exact orbital meaning. Compare the specific energies of the chaser and the target, to first order in the offset. The chaser's inertial speed squared is, using Equation 6,

$$v_c^2 = (\dot{x} - ny)^2 + (nr_t + \dot{y} + nx)^2 \approx n^2 r_t^2 + 2nr_t(\dot{y} + nx),$$

and its distance from the Earth is $|\mathbf{r}_t + \rho| \approx r_t + x$, so $-\mu/|\mathbf{r}_c| \approx -\mu/r_t + \mu x/r_t^2 = -\mu/r_t + n^2 r_t x$. Hence

$$E_c - E_t \approx nr_t(\dot{y} + nx) + n^2 r_t x = nr_t\,(\dot{y} + 2nx) = nr_t\,K.$$

With $E = -\mu/2a$ we have $\delta E = \mu\,\delta a/2a^2$, so $\delta a = 2a^2\,\delta E/\mu = 2r_t^2\,nr_t K/(n^2 r_t^3)$:

$$\boxed{\delta a = \frac{2K}{n} = x_c = 4x_0 + \frac{2\dot{y}_0}{n}.} \tag{28}$$

**The guiding centre sits at the difference of semi-major axes**, and the momentum integral $K$ is the energy difference per unit $nr_t$ (for neighbouring circular orbits $\delta E = n\,\delta h$, so the angular-momentum and the energy descriptions of $K$ agree). Now Kepler's third law, $n \propto a^{-3/2}$, gives the chaser's angular rate $n + \delta n$ with $\delta n = -\frac{3}{2} n\,\delta a/r_t$. The chaser's angular position drifts relative to the target at $\delta n$, so along-track it drifts at

$$r_t\,\delta n = -\tfrac{3}{2}\,n\,\delta a = -\tfrac{3}{2}\,n\,x_c,$$

exactly the guiding-centre drift in Equation 27. **The cw drift is Kepler's third law**: a *higher* orbit ($x_c > 0$) is *slower* and falls behind; a lower one gets ahead. This identity was derived here independently of cw, from energy and Kepler alone, and (cw-verify.py) it holds numerically: for a small test offset the exact $\Delta a$ from the energy of the chaser agrees with $x_c$ to six figures.

**History: epicycles again.** The word is Greek astronomy's: in the Ptolemaic system a planet moves on a small circle (the epicycle) whose centre moves on a larger one (the deferent). The decomposition into a drifting guiding centre and a small oscillation about it is the same idea, and it came back in the twentieth century. The *epicyclic approximation* of galactic dynamics describes stars on near-circular orbits this way, and the axis ratio $2:1$ above is $2\Omega/\kappa$ with $\kappa = \Omega$, which holds for a Kepler (inverse-square) force.
Hill's local approximation, extended to disks, became the *shearing sheet* used for galactic disks (Goldreich and Lynden-Bell, 1965; Julian and Toomre, 1966) and for planetary rings.

**History.** Kepler published the third law in 1619 (*Harmonices Mundi*) as an empirical regularity of the planets; Newton showed in the *Principia* (1687) that it follows from an inverse-square force (Newton 1687). Using it to predict how fast two spacecraft drift apart is the same arithmetic Kepler did for the planets, now run for the ISS.

## 7 Worked examples, checked against Kepler

### 7.1 Example 1: a 1 m/s prograde burn

**Given:** the chaser starts at the target, $\mathbf{X}_0 = (0,\,0,\,0,\,\Delta v)^{\mathsf{T}}$, with $\Delta v = 1$ m/s $= 10^{-3}$ km/s. Note $\Delta v/n = 0.88780$ km. Only the last column of Equation 24 acts:

$$x(t) = \frac{2(1 - \cos nt)}{n}\,\Delta v, \qquad y(t) = \frac{4\sin nt - 3nt}{n}\,\Delta v.$$

At half an orbit $\tau = \pi$ ($c = -1$, $s = 0$) and at a full orbit $\tau = 2\pi$ ($c = 1$, $s = 0$):

| | $t = T/2$ | $t = T$ |
|---|---|---|
| $x$ [km] | $4\Delta v/n = \mathbf{3.551}$ | $0$ |
| $y$ [km] | $-3\pi\Delta v/n = -8.367$ | $-6\pi\Delta v/n = \mathbf{-16.735}$ |

In the language of Section 6.2: $x_c = 2\Delta v/n = 1.776$ km, $A = \alpha = 2\Delta v/n = 1.776$ km, so the chaser rises from 0 to $2x_c = 3.55$ km and the guiding centre drifts $-3\pi x_c = -3T\Delta v$ per orbit. A convenient memory aid:

Speeding up makes you fall behind: $3T\,\Delta v = 16.7$ km per orbit per m/s.

## 7.2 Check against Kepler

Nothing in cw is new physics: it must agree with vis-viva and Kepler's third law to first order. Here are the numbers by the other route.

**Step 1: the new orbit.** Vis-viva, $v^2 = \mu(2/r - 1/a)$, differentiated at fixed $r$ with $v = r\,n$ on the circle, gives $2v\,\delta v = \mu\,\delta a/a^2$, so

$$\Delta a = \frac{2a^2 v\,\Delta v}{\mu} = \frac{2a\,\Delta v}{v} = \frac{2\Delta v}{n} = 1.7756 \text{ km}.$$

(using $\mu = n^2a^3$ and $v = na$.) This equals the cw $x_c$, as Equation 28 said it must.

**Step 2: the far side of the new orbit.** The burn point is the new orbit's pericentre (a burn along the velocity at a circular orbit raises the opposite side), so its apocentre is $a + 2\Delta a$, i.e. $x = 2\Delta a = 3.551$ km above the target (✓ the table, at $t = T/2$).

**Step 3: the period and the lag.** $T \propto a^{3/2}$ gives $\Delta T = \frac{3}{2}T\,\Delta a/a = \frac{3}{2}(5578.2\text{ s})(1.7756/6798.137) = 2.1855$ s. After one target period the chaser has not completed its longer orbit, so it is behind by $v\,\Delta T = 7.65727 \times 2.1855 = 16.735$ km (✓). Algebraically the two routes are the same number: $v\,\Delta T = \frac{3}{2}\,v\,T\,\Delta a/a = \frac{3}{2}(na)(2\pi/n)(2\Delta v/n)/a = 6\pi\Delta v/n$, which is the table's $-6\pi\Delta v/n$.

**How good is the first-order theory?** Integrating both spacecraft exactly (cw-verify.py):

| | cw | exact two-body |
|---|---|---|
| $x(T/2)$ [km] | 3.5512 | 3.5472 |
| $y(T/2)$ [km] | −8.3673 | −8.3673 |
| $x(T)$ [km] | 0 | −0.0206 |
| $y(T)$ [km] | −16.7347 | −16.7434 |

The along-track lag after one orbit differs by 0.05%; the neglected terms are of relative order $\Delta a/a \sim 3 \times 10^{-4}$.

## 7.3 The co-circular drift

Suppose instead the chaser is on a circular orbit 1 km *below* the target, $\mathbf{X}_0 = (-1,\ 0,\ 0,\ \dot y_0)$. What $\dot y_0$ makes the orbit circular? The chaser's inertial speed is $n'(a + x)$ with $n' = n\,(1 + x/a)^{-3/2} \approx n(1 - \frac{3}{2}x/a)$, while the rotating frame, at the same radius, moves at $n(a + x)$. The difference is the velocity *relative to the frame*:

$$\dot y_0 = (n' - n)(a + x_0) \approx -\tfrac{3}{2}\,n\,x_0.$$

For $x_0 = -1$ km this is $\dot y_0 = +1.690$ m/s (ahead). With this value $K = \dot y_0 + 2nx_0 = \frac{1}{2}nx_0$, i.e. $x_c = x_0$: the chaser *is* its own guiding centre ($\alpha = 3x_0 + 2\dot y_0/n = 0$ and $\beta = 0$, so $A = 0$), and it drifts at $-\frac{3}{2}nx_0$, which for $x_0 = -1$ km is $3\pi = 9.425$ km per orbit **ahead**, exactly the Kepler value $a\,(n' - n)\,T$. This is panel (c) of Figure 5.

**History: Gemini 4, June 1965.** James McDivitt tried to fly close to the spent Titan II second stage that had put him in orbit. Simply thrusting towards the stage changed his orbit rather than his position relative to it, the stage drew away, and after using almost half his thruster propellant he gave up to concentrate on the spacewalk. The lesson changed how NASA planned rendezvous: later missions approached "upward", from a lower, faster orbit. The first successful rendezvous was Gemini 6A and 7 in December 1965. Buzz Aldrin, who flew on Gemini 12, wrote his 1963 MIT doctoral thesis on manned rendezvous guidance (Aldrin 1963) and was nicknamed "Dr. Rendezvous".

### 7.4 Example 2: a 1 m/s radial burn

**Given** $\mathbf{X}_0 = (0,\ 0,\ \Delta v,\ 0)^{\mathsf{T}}$. The third column of Equation 24 gives

$$x(t) = \frac{\Delta v}{n}\sin nt, \qquad y(t) = -\frac{2\Delta v}{n}(1-\cos nt).$$

This satisfies Equation 26 ($x_0 = \dot{y}_0 = 0$, so $K = 0$ and $x_c = 0$): no drift. In the form of Equation 27, $\alpha = 0$, $\beta = \Delta v/n$, so $A = \Delta v/n = 0.888$ km and $y_c = -2\Delta v/n = -1.776$ km: **a closed ellipse of semi-axes** 0.888 **km (radial) and** 1.776 **km (along-track), centred** 1.776 **km behind the target**, returning to the target after exactly one orbit.

> **Note**
>
> A radial burn does no work on a circular orbit ($\mathbf{v} \perp \Delta\mathbf{v}$), so to first order the energy, the semi-major axis and hence the period are unchanged. That is why the motion closes, and why $x_c = \delta a = 0$: it is the same statement as Equation 26.

## 8 How far can we trust it?

### 8.1 cw against the exact two-body motion

The only approximation was the linearisation of gravity in Section 4.1, and the cost of that approximation is not subtle. We can measure it, by integrating the *exact* equations of motion of both spacecraft inertially and comparing with the cw prediction for the same initial state. Figure 6 uses a radial kick that produces a cw ellipse of along-track extent $d$ (so $A = d/4$).

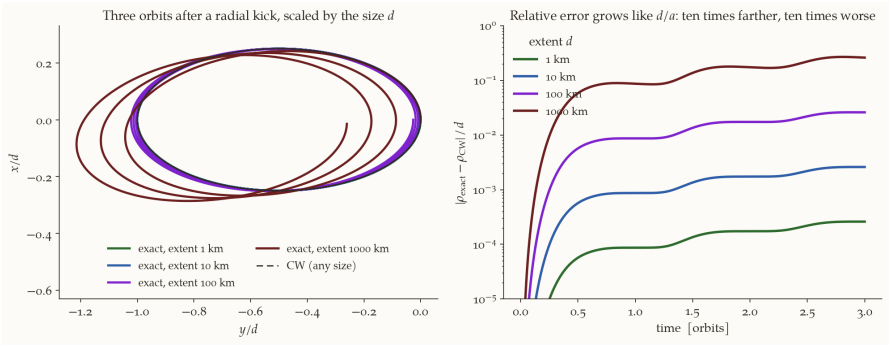



Figure 6: A radial kick producing a cw ellipse of along-track extent $d$, integrated exactly (two-body, both spacecraft) and compared with cw. Left: the exact trajectories, scaled by $d$; at 1000 km the ellipse has visibly opened into a drift. Right: the error, relative to $d$, grows in proportion to $d$. *Script: fig-cw-validity.py.*

For the numbers (cw-verify.py):

| extent $d$ | $d/r_t$ | error after $T$ | after $3T$ | error$(T)$ / $d$ |
|---|---|---|---|---|
| 1 km | $1.5\times10^{-4}$ | 0.087 m | 0.26 m | $8.7\times10^{-5}$ |
| 10 km | $1.5\times10^{-3}$ | 8.7 m | 26 m | $8.7\times10^{-4}$ |
| 100 km | $1.5\times10^{-2}$ | 0.87 km | 2.6 km | $8.7\times10^{-3}$ |
| 1000 km | 0.15 | 87 km | 261 km | $8.7\times10^{-2}$ |

The error after one orbit is $0.59\,d^2/r_t$ to within 0.5% over three decades in $d$: quadratic in the size of the motion, as the second-order terms of Section 4.1 suggest, and *linear* in time (compare the $T$ and $3T$ columns). The coefficient can be derived exactly, and doing so shows where the error comes from.

**Where the error comes from.** For this radial kick cw has $K = \dot{y}_0 + 2nx_0 = 0$ and hence $\delta a = x_c = 0$ (Equation 28): the motion closes forever. The exact chaser is not so lucky. A radial kick $\Delta v$ is at right angles to the velocity $v_t$, so the speed becomes $\sqrt{v_t^2 + \Delta v^2}$ and the specific energy rises by $\Delta E = \frac{1}{2}\Delta v^2$ — a **second-order** change that the linearisation cannot see. With $\delta E = \mu\,\delta a/2a^2$ and $\mu = r_t v_t^2$,

$$\delta a = \frac{2r_t^2\,\Delta E}{\mu} = r_t\Big(\frac{\Delta v}{v_t}\Big)^2, \qquad \Delta v = \frac{nd}{4} \;\Rightarrow\; \frac{\Delta v}{v_t} = \frac{nd}{4\,nr_t} = \frac{d}{4r_t}$$

(the kick $\Delta v = nd/4$ is what gives a relative ellipse of along-track extent $d$: its radial semi-axis is $A = \Delta v/n = d/4$ and its along-track semi-axis $2A$, Section 6.2).

The chaser is on a slightly higher orbit, so by Kepler's third law (Section 6.3) its period is longer by $\Delta T = \frac{3}{2}T\,\delta a/r_t$, and after one target period it lags the target by

$$v_t\,\Delta T = \tfrac{3}{2}\,(v_t T)\,\frac{\delta a}{r_t} = \tfrac{3}{2}\,(2\pi r_t)\,\frac{d^2}{16\,r_t^2} = \frac{3\pi}{16}\,\frac{d^2}{r_t} = 0.589\,\frac{d^2}{r_t}.$$

This matches the numerical coefficient, 0.589, to three figures, and the sign: the exact chaser falls *behind* cw's. **cw's error here is a missing second-order change of semi-major axis, and any $\delta a$ becomes an along-track drift that grows linearly in time.**

## 8.2 Rules of thumb

- **The neglected terms are of order $\rho/r_t$ relative to those kept** (Section 4.1). For an orbit-sized relative ellipse of extent $d$, the position error after one orbit is about $0.6\,d^2/r_t$: about 9 m at $d = 10$ km. (The coefficient $3\pi/16$ is exact for the radial-kick family above; other shapes of relative motion have other coefficients of the same size, but always proportional to $d^2$.)
- **Errors accumulate.** The neglected terms change the semi-major axis at second order, $\delta a \sim \rho^2/r_t$, and by Kepler's third law (Section 6.3) any $\delta a$ is a *steady* drift. So long propagations degrade, even at short range: the error is linear in the number of orbits.
- **Along-track distances are measured along a straight line.** The linearisation uses a straight $y$ axis, whereas the target's circle curves. A chaser *on the same circular orbit*, 100 km ahead, is at $x = -s^2/2r_t = -0.735$ km, not at $x = 0$; at 10 km the offset is 7 m. This is a second-order term ($\eta^2$) and cw does not contain it. Formulations that use curvilinear coordinates remove this particular error (Alfriend et al. 2009).
- **The target orbit must be circular.** For an eccentric target the frame rate is not constant and the coefficients depend on time. The linear relative-motion equations about an elliptic orbit of any eccentricity were obtained by Tschauner and Hempel (Tschauner and Hempel 1965); they reduce to cw as $e \to 0$.
- **Only the point-mass force is modelled.** Differential drag and the Earth's oblateness ($J_2$) act on the two spacecraft differently and are not in Equation 13; for long propagations they matter more than the second-order gravity terms.

# 9 Summary

1. In the target's rotating frame ($\omega = n\hat{\mathbf{k}}$), relative motion obeys the **cw equations** Equation 13: $\ddot{x} - 2n\dot{y} - 3n^2x = 0$, $\ddot{y} + 2n\dot{x} = 0$ — linear, constant coefficients, coupled through Coriolis. The $3n^2$ is $2n^2$ of tide plus $n^2$ of centrifugal term; the equations are Hill's, with a target of negligible mass.
2. The second equation integrates once: $\dot{y} + 2nx = K$, the **angular-momentum** (equivalently energy) difference. With the Jacobi integral it shows that $x$ oscillates at the orbital frequency about $x_c = 2K/n = 4x_0 + 2\dot{y}_0/n$.
3. The solution is the **state transition matrix** Equation 24: one matrix multiply propagates a relative state, and  has the group property, the inverse $(-t)$, and det $= 1$.
4. Every natural motion is a **2:1 epicycle on a drifting guiding centre**. The guiding centre sits at $x_c = \delta a$, the difference of semi-major axes, and drifts at $-\frac{3}{2}nx_c$: **the drift is Kepler's third law.**
5. Relative motion is **bounded iff** $\dot{y}_0 = -2nx_0$, the condition $\delta a = 0$; otherwise the chaser drifts at $-3(2nx_0 + \dot{y}_0)$.
6. Counter-intuitive but Keplerian: **a prograde burn makes you fall behind** ($3T\,\Delta v$ per orbit), a lower orbit drifts ahead, a radial burn closes after one orbit.
7. CW is accurate to relative order $\rho/r_t$: for a radial kick giving a relative ellipse of extent $d$ the error is exactly the missing second-order $\delta a$, $\frac{3\pi}{16}d^2/r_t \approx 0.6\,d^2/r_t$ per orbit. Excellent for the last tens of kilometres, which is exactly where the chase note uses it.

**Everything in this note can be reproduced** with cw-verify.py: the symbolic checks of the expansions, the solution, the state transition matrix and its eigenstructure, and the numerical examples, including the exact two-body comparisons.

# Further reading

For the standard textbook treatments see Curtis (2014) and Vallado (2013); for relative motion in all its forms, with curvilinear coordinates, eccentric targets and $J_2$, Alfriend et al. (2009).


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