The Circular Restricted Three-Body Problem

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Where we are going

The N-body note ended on a hard fact: for three or more bodies there is no general solution, and the only simplification that makes the problem solvable is the two-body one. A great deal of spaceflight lives between the two. A spacecraft near the Moon feels both the Earth and the Moon, and neither can be ignored. A spacecraft at the Sun–Earth L_1 point feels the Sun and the Earth at comparable strength, which is exactly why it can *stay* there. The patched-conics note handled this by cutting the trajectory into two-body pieces. This note does it differently, by keeping three bodies and making the one assumption that restores some order: the third body is a **spacecraft**, too light to disturb the other two, and the other two move on **circles**. The result is the **circular restricted three-body problem** (CR3BP). It has no closed-form solution either, but it has an integral of motion, five equilibrium points, and families of periodic orbits around them, and it is the model in which the orbits of SOHO, the James Webb Space Telescope, Aditya-L1 and many lunar missions are designed.

> What you should be able to answer at the end
>
> (1) What are the assumptions of the problem, and what are the units in which its equations look the same for every pair of bodies? (2) How do the equations of motion in the rotating frame follow from Newton's law, and which terms are the Coriolis and centrifugal ones? (3) What is the Jacobi constant, why is it conserved, and what does it forbid a spacecraft to do? (4) Where are the five Lagrange points, how do you find them, and why is L_1 about 0.01 AU from the Earth? (5) Which of them are stable, and why is the answer "L_4 and L_5 only if $\mu < 0.0385$"? (6) What are halo orbits, and why do missions fly them? (7) How is the Hill sphere related to the sphere of influence of the patched-conics note?

The required background is the N-body and two-body notes, the rotating-frame kinematics of the CW note (the same Coriolis and centrifugal terms appear here), and the sphere of influence of the patched-conics note. A mission that uses this model is worked in Problem 6 of the Indian missions problems; this note supplies the background that problem assumes. As in the other notes, a blue **Try it** box asks you to do a small calculation before reading on, and a green **Check yourself** box asks a short question; each has a hidden answer. Symbols and constants shared by all the notes are on the notation page; the study plan gives the order in which to read the notes. Here m_1 is the large primary, m_2 the small one, and $\mu = m_2/(m_1 + m_2)$; the letters μ (mass parameter), C (Jacobi constant) and Ω (pseudo-potential) are local to this note.

1 The restricted problem

1.1 The assumptions and the units

Two **primaries**, of masses $m_1 \geq m_2$, move on circular orbits about their common centre of mass, as the two-body problem says they may. A third body, of negligible mass, moves in their gravitational field without affecting them. It may move in three dimensions; the primaries' orbit defines the xy plane. Take the frame that **rotates** with the primaries, with its origin at their centre of mass, the x axis through both of them, and the z axis along their angular momentum. In that frame the primaries are at rest.

The equations look the same for every pair of primaries if the units are chosen well. Take the distance between the primaries as the unit of length, the sum $m_1 + m_2$ of their masses as the unit of mass, and $G = 1$. Then $G(m_1 + m_2) = 1$, Kepler's third law gives the primaries' angular rate as $n = \sqrt{G(m_1 + m_2)/l^3} = 1$, so the unit of time is $1/n$ (the orbital period is 2π), and the unit of speed is $l\,n$, the relative orbital speed of the primaries. The only parameter left is the **mass parameter**

$$\mu = \frac{m_2}{m_1 + m_2} \qquad (0 < \mu \leq \tfrac{1}{2}),$$

and the primaries sit at $(-\mu, 0, 0)$ for m_1 and $(1 - \mu, 0, 0)$ for m_2 (so that their centre of mass is the origin).

Table 1: Two systems in the standard units.

system	μ	length unit l	time unit $1/n$	speed unit $l\,n$
Earth–Moon	0.0121506	384 400 km	4.3425 d	1.0245 km/s
Sun–(Earth+Moon)	3.0404×10^{-6}	1 AU	58.132 d	29.785 km/s

In the Sun–Earth problem the Earth and the Moon together play the role of the small primary (their centre of mass moves on the circle); this is the model of Problem 6 in the Indian missions problems.

1.2 The equations of motion

Let $\mathbf{r} = (x, y, z)$ be the position of the spacecraft in the rotating frame, and let

$$r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$$

be its distances from the primaries. In an inertial frame with the same origin, Newton's law with the two gravitational pulls says (in our units, where $Gm_1 = 1 - \mu$ and $Gm_2 = \mu$)

$$\ddot{\mathbf{r}}\big|_{\mathcal{I}} = -\frac{(1-\mu)(\mathbf{r}-\mathbf{r}_1)}{r_1^3} - \frac{\mu(\mathbf{r}-\mathbf{r}_2)}{r_2^3}.$$

The rotating frame turns at the rate $n = 1$ about $\hat{\mathbf{z}}$. Exactly as in the CW note, the inertial acceleration in terms of the rotating-frame ones is $\ddot{\mathbf{r}}|_{\mathcal{I}} = \ddot{\mathbf{r}} + 2\omega \times \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r})$ with $\omega = \hat{\mathbf{z}}$. In components, $2\omega \times \dot{\mathbf{r}} = (-2\dot{y},\ 2\dot{x},\ 0)$ and $\omega \times (\omega \times \mathbf{r}) = (-x,\ -y,\ 0)$. Moving the centrifugal term to the right (the Coriolis term stays on the left),

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} &= x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = \Omega_x, \\ \ddot{y} + 2\dot{x} &= y - \frac{(1-\mu)\,y}{r_1^3} - \frac{\mu\,y}{r_2^3} = \Omega_y, \\ \ddot{z} &= -\frac{(1-\mu)\,z}{r_1^3} - \frac{\mu\,z}{r_2^3} = \Omega_z, \end{aligned}} \tag{1}$$

where the subscripts are partial derivatives of the **pseudo-potential**

$$\Omega(x,y,z) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}. \tag{2}$$

The first part of Ω is the centrifugal potential and the rest is the gravitational one; the Coriolis force, $(2\dot{y}, -2\dot{x}, 0)$ on the left, does no work and is not part of Ω. These are the equations that every figure of this note integrates (_cr3bp.py).

Where in Equation 1 would the CW equations come from?

Answer

Put the spacecraft close to one primary, so that the other primary's field is the Sun-like tide of the CW note: the Coriolis terms $-2\dot{y}$, $+2\dot{x}$ and the centrifugal term are exactly the CW ones. The CW frame rotates with a *spacecraft's* orbit about its primary; the CR3BP frame rotates with the two primaries about each other. The structure of the equations is the same.

History: the restricted problem. Euler (1767) found the three collinear equilibrium points; Lagrange (1772) found the two triangular

2 The Jacobi constant

The equations Equation 1 depend explicitly on neither the time nor the angle, so there should be an integral of motion. There is no angular momentum conservation, because the primaries break the symmetry, and no separate energy conservation, because the frame rotates. But multiply the three equations by $\dot{x}, \dot{y}, \dot{z}$ and add. The Coriolis terms cancel, since $-2\dot{y}\dot{x} + 2\dot{x}\dot{y} = 0$, and

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \Omega_x\dot{x} + \Omega_y\dot{y} + \Omega_z\dot{z} = \frac{d\Omega}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\tfrac{1}{2}v^2 - \Omega\right) = 0.$$

So $v^2 - 2\Omega$ is constant. Its negative is the **Jacobi constant**,

$$\boxed{C = 2\Omega - v^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - v^2} \tag{3}$$

with v the speed in the *rotating* frame. (Some books add the constant $\mu(1-\mu)$, which makes $C = 3$ at L_4; we follow the usual convention of spaceflight, without it.) It is an energy-like quantity, but of the rotating frame. In terms of the inertial energy per unit mass $E = \frac{1}{2}v_I^2 - (1-\mu)/r_1 - \mu/r_2$ and the inertial angular momentum per unit mass h_z about the z axis, it is exactly $C = 2(h_z - E)$: neither is conserved alone (the primaries' field is not static in an inertial frame), but their difference is. A large C means a spacecraft that is deep in a potential well or moving slowly; a small C means a spacecraft with energy to spare.

2.1 Zero-velocity curves and forbidden regions

Since $v^2 \geq 0$, a spacecraft with Jacobi constant C can only be where $2\Omega(x, y, z) \geq C$. The boundary, $2\Omega = C$, is the **zero-velocity surface**: it can be approached only with $v \to 0$. For planar motion it is a curve, and Figure 1 draws it for the Earth–Moon system at five values of C. The forbidden region is grey.

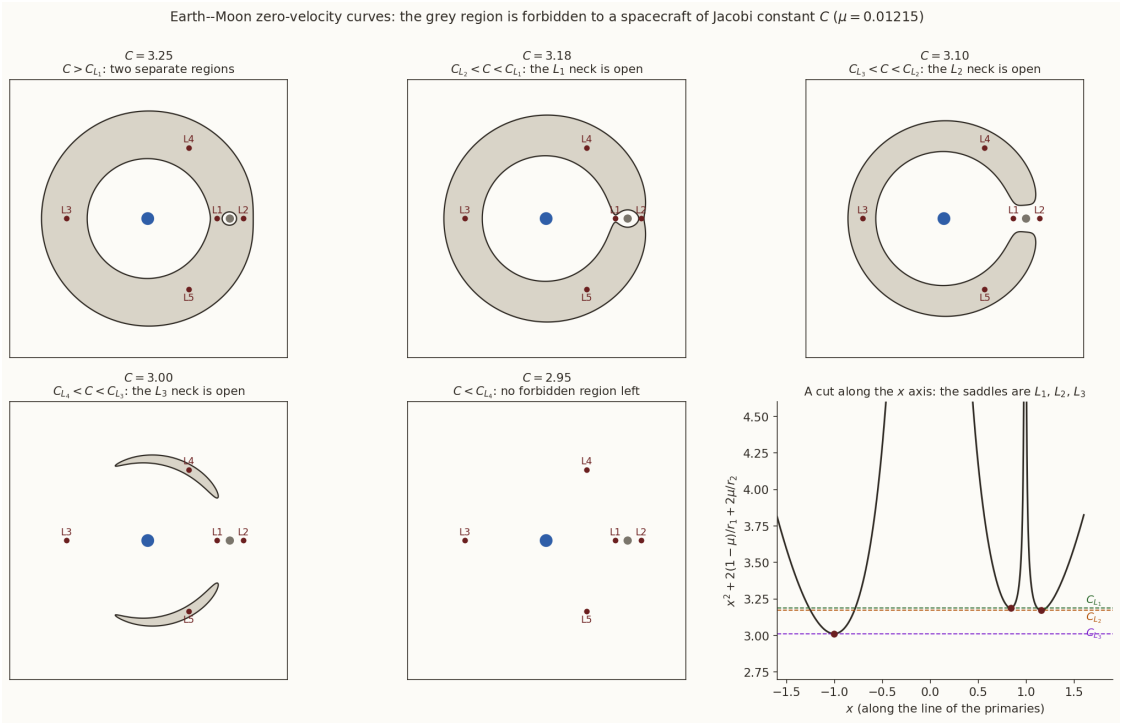


Figure 1: The Earth–Moon zero-velocity curves at five values of the Jacobi constant. For a large C the spacecraft is confined to a region about the Earth, a region about the Moon, or the outside; as C falls the forbidden region opens first at the L_1 neck (so that the spacecraft can travel between Earth and Moon), then at L_2, then at L_3, and finally vanishes. The last panel is a cut along the x axis; L_1, L_2, L_3 are the saddles, and each has its own C_{L_i}. *Script: fig-cr3bp.py*.

This is the main use of the Jacobi constant. It is a *rigorous* statement about where the spacecraft cannot go. The Jacobi constants of the equilibrium points (Table 2 below) are the thresholds. For $C > C_{L_1}$ the Earth region and the Moon region are separated by a wall: a spacecraft in lunar orbit cannot reach the Earth, however it coasts, and a transfer needs a burn that lowers C. For $C_{L_2} < C < C_{L_1}$ the L_1 neck is open, and a transfer is *allowed*: the spacecraft is not forbidden to cross, though the dynamics need not take it there.

A spacecraft is at perigee 300 km above the Earth, on the Earth–Moon line on the side towards the Moon, moving prograde (so in the $+y$ direction). What perigee speed (inertial) makes $C = C_{L_1}$, so that the L_1 neck just opens? Compare with the escape speed and with the perigee speed of a Hohmann transfer to the Moon's distance.

What you should find

The rotating-frame speed is the inertial speed minus $n\,r_p$, with $n = 2.6653 \times 10^{-6}$ rad/s, so $v_{\text{rot}} = v_{\text{in}} - 0.0178$ km/s. Solving $C = C_{L_1} = 3.1883$ gives

$v_{\text{in}} = 10.79$ km/s. That is below the escape speed (10.926 km/s) and just below the Hohmann perigee speed to the Moon's distance (10.83 km/s): the neck at L_1 is closer than the Moon, and the Moon's pull helps.

3 The five equilibrium points

An **equilibrium point** is a point where a spacecraft at rest in the rotating frame stays at rest: $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = 0$, so $\Omega_x = \Omega_y = \Omega_z = 0$, which says that the gravitational pull of the primaries exactly provides the centripetal force that the rotation needs. There are five.

3.1 The triangular points

Look first for points off the axis, $y \neq 0$. The condition $\Omega_z = 0$ gives $z = 0$. The condition $\Omega_y = 0$, divided by y, gives

$$1 = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}.$$

Substituting this "1" into $\Omega_x = 0$,

$$\Omega_x = x\left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right] - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = \mu(1-\mu)\left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right) = 0,$$

so $r_1 = r_2$, and then the first condition gives $r_1 = r_2 = 1$. The third body is at the same distance from both primaries as they are from each other: the three bodies form an **equilateral triangle**. There are two such points, one on each side of the axis:

$$\boxed{L_{4,5} = \left(\tfrac{1}{2} - \mu,\ \pm\tfrac{\sqrt{3}}{2},\ 0\right)} \tag{4}$$

(Figure 2, left). They exist for every μ, and their position does not depend on the masses at all, in these units.

3.2 The collinear points

On the axis ($y = z = 0$) only the x equation is left:

$$f(x) \equiv x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} = 0. \tag{5}$$

This is a quintic in disguise. f goes to $\mp\infty$ at each primary and $\pm\infty$ at infinity, so there is exactly one root in each of the three intervals outside the singularities: L_1 between the primaries, L_2 beyond the small one, and L_3 beyond the large one. They are found numerically (a bracketed root-finder is enough).

An estimate near the small primary. Let the distance of L_1 (or L_2) from m_2 be γ, small when μ is small. Take L_1, with $x = 1 - \mu - \gamma$. In Equation 5, the first two terms are $(1-\mu-\gamma) - (1-\mu)(1-\gamma)^{-2} \approx -3\gamma$ (the centrifugal term contributes $-\gamma$ and the large primary's field gradient -2γ: together exactly the "tidal plus centrifugal" combination of a rotating frame). The third term is $-\mu(-\gamma)/\gamma^3 = +\mu/\gamma^2$, the pull of the small primary. Balancing them,

$$-3\gamma + \frac{\mu}{\gamma^2} = 0 \quad \Longrightarrow \quad \boxed{\gamma \approx \left(\frac{\mu}{3}\right)^{1/3}} \tag{6}$$

the **Hill radius** in these units (the same result holds for L_2 at leading order). Sun–Earth: $(3.0404 \times 10^{-6}/3)^{1/3} = 0.0100447$ AU $= 1.5027 \times 10^6$ km, against the exact L_1 at 1.4976×10^6 km and L_2 at 1.5077×10^6 km, so the Hill value is the average of the two, to the accuracy of the approximation. This is the "1.5 million km" of every description of SOHO, JWST and Aditya-L1.

For the Earth–Moon system ($\mu = 0.0121506$) find L_1 and L_2 numerically from Equation 5, express their distances from the Moon in km, and compare with Equation 6.

What you should find

L_1 is at $x = 0.836915$ and L_2 at $x = 1.155682$ (and L_3 at -1.005063). The distances from the Moon are $\gamma_1 = 0.150934$ and $\gamma_2 = 0.167833$ in units of 384 400 km, i.e. 58 019 km and 64 515 km. The Hill estimate is $(\mu/3)^{1/3} = 0.159401$, i.e. 61 274 km, which is the average of γ_1 and γ_2 (0.159384) to 0.01%, though each is 5% away because μ is not very small. L_1 is 326 381 km from the Earth.

Table 2: The equilibrium points and their Jacobi constants.

	Earth–Moon	Sun–(Earth+Moon)
L_1 (x)	0.836915	0.989986
L_2 (x)	1.155682	1.010075
L_3 (x)	−1.005063	−1.000001
distance of L_1 from the small primary	58 019 km	1 497 621 km

	Earth–Moon	Sun–(Earth+Moon)
distance of L_2 from the small primary	64 515 km	1 507 683 km
C_{L_1}	3.188341	3.000898
C_{L_2}	3.172160	3.000894
C_{L_3}	3.012147	3.000003
$C_{L_4} = C_{L_5}$ $(= 3 - \mu(1-\mu))$	2.987997	2.999997

4 Stability of the equilibrium points

A spacecraft placed near an equilibrium point either stays near it (stable) or drifts away (unstable). To tell, write $\mathbf{r} = \mathbf{r}_{\text{eq}} + \xi$ in Equation 1 and keep the terms linear in $\xi = (\xi, \eta, \zeta)$:

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta, \qquad \ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta, \qquad \ddot{\zeta} = \Omega_{zz}\zeta,$$

with the second derivatives of Ω evaluated at the point ($\Omega_{xz} = \Omega_{yz} = 0$ there, since $z = 0$). For solutions $e^{\lambda t}$ the planar pair gives the characteristic equation

$$\lambda^4 + (4 - \Omega_{xx} - \Omega_{yy})\lambda^2 + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = 0. \tag{7}$$

4.1 The collinear points are unstable

On the axis $\Omega_{xy} = 0$, and with

$$c_2 = \frac{\mu}{|x - 1 + \mu|^3} + \frac{1-\mu}{|x + \mu|^3} \tag{8}$$

the second derivatives are $\Omega_{xx} = 1 + 2c_2$, $\Omega_{yy} = 1 - c_2$ and $\Omega_{zz} = -c_2$. Then Equation 7 becomes

$$\lambda^4 + (2 - c_2)\lambda^2 + (1 + c_2 - 2c_2^2) = 0.$$

Because $c_2 > 1$ at the collinear points (it is 5.15 at the Earth–Moon L_1 and 4.06 at the Sun–Earth L_1), the constant term is negative, so the quadratic in λ^2 has one positive and one negative root: two real eigenvalues $\pm\lambda$, and a pair of imaginary ones $\pm i\nu$. The out-of-plane motion is $\ddot{\zeta} = -c_2\zeta$, a harmonic

oscillation of frequency $\nu_z = \sqrt{c_2}$. So each collinear point is a **saddle** × **centre** × **centre**: a spacecraft near it oscillates in the plane at ν and out of the plane at ν_z, and any component along the unstable direction grows like $e^{\lambda t}$.

Table 3: The linear exponents at the collinear points, in units of n (the e-folding time is $1/\lambda$ in time units).

	c_2	λ	e-folding time	ν (planar)	ν_z (vertical)
Earth–Moon L_1	5.1476	2.9321	1.48 d	2.3344	2.2688
Earth–Moon L_2	3.1904	2.1587	2.01 d	1.8626	1.7862
Sun–Earth L_1	4.0611	2.5327	22.95 d	2.0865	2.0152
Sun–Earth L_2	3.9405	2.4843	23.4 d	2.0570	1.9851

Unstable does not mean useless. The e-folding time at the Sun–Earth L_1 is 23 days. A spacecraft at L_1 must correct its orbit every few weeks (Aditya-L1 does, at spacings of 25 to 106 days), and the corrections are tiny, because the error is caught while it is small. The instability is also an asset: the *stable manifold* of the orbit is a trajectory that approaches it for free, and the unstable one a path away from it, the basis of low-energy transfers.

Sun–Earth L_1 has e-folding time 23 days. A navigation error of 1 cm/s in velocity: how big is it after three months?

Answer

The component along the unstable direction grows by $e^{t/\tau}$ with $\tau = 22.95$ d: after 91 days, $e^{3.97} = 53$. A 1 cm/s error becomes about 0.5 m/s in speed, and a corresponding position error of tens of thousands of km. That is why a halo orbit needs station-keeping, and why the Indian missions problem says so.

4.2 The triangular points: stable if $\mu < 0.0385$

At L_4 we have $r_1 = r_2 = 1$, $x + \mu = \frac{1}{2}$, $x - 1 + \mu = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$, and the second derivatives work out to

$$\Omega_{xx} = \tfrac{3}{4}, \qquad \Omega_{yy} = \tfrac{9}{4}, \qquad \Omega_{xy} = \pm\tfrac{3\sqrt{3}}{4}(1 - 2\mu),$$

with the sign + at L_4 and − at L_5 (and $\Omega_{zz} = -1$, so the vertical motion is a plain oscillation at frequency 1). Then $4-\Omega_{xx}-\Omega_{yy} = 1$ and $\Omega_{xx}\Omega_{yy}-\Omega_{xy}^2 = \frac{27}{16}[1-(1-2\mu)^2] = \frac{27}{4}\mu(1-\mu)$, so Equation 7 reads

$$\lambda^4 + \lambda^2 + \tfrac{27}{4}\mu(1-\mu) = 0, \qquad \lambda^2 = \frac{-1 \pm \sqrt{1 - 27\mu(1-\mu)}}{2}. \tag{9}$$

If $1 - 27\mu(1-\mu) \geq 0$ both values of λ^2 are real and negative, so all four λ are imaginary: **stable oscillations**. If it is negative, λ^2 is complex, and one of the four roots has a positive real part: **unstable**. The boundary is $27\mu(1-\mu) = 1$:

$$\boxed{\mu < \mu_{\text{Routh}} = \frac{1 - \sqrt{23/27}}{2} = 0.0385209} \tag{10}$$

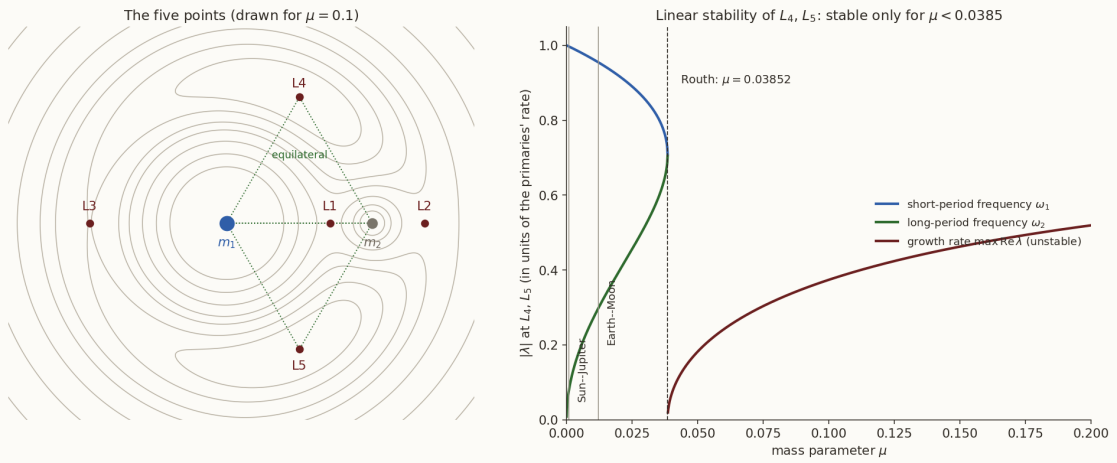


Figure 2: **Left:** the five equilibrium points, drawn for $\mu = 0.1$ with contours of 2Ω. L_1, L_2, L_3 are saddles, and L_4, L_5 are the maxima of the potential at the apexes of equilateral triangles. **Right:** the frequencies of the linear motion about L_4 and L_5 against μ. Below Routh's value $\mu = 0.0385$ they are real (stable); above, the growth rate is positive. *Script: fig-cr3bp.py.*

Both Sun–Jupiter ($\mu = 9.54 \times 10^{-4}$) and Earth–Moon ($\mu = 0.01215$) satisfy it. The Trojan asteroids, in the Sun–Jupiter L_4 and L_5 regions, are the standing evidence. The stable oscillations about L_4, L_5 have two frequencies $\omega_{1,2} = \sqrt{(1 \pm \sqrt{1 - 27\mu(1-\mu)})/2}$: for the Earth–Moon system 0.9545 and 0.2982, i.e. periods of 28.6 and 91.5 days. The stability found here is *linear*, and it holds for the three-body model only: in the real Sun–Earth–Moon system the Sun's perturbation destabilises the Earth–Moon L_4 and L_5, which is why no swarm of bodies is found there.

Verify that L_4 of Equation 4 satisfies $\Omega_x = \Omega_y = 0$ for any μ, and find where the discriminant of Equation 9 vanishes. What happens at $\mu = \frac{1}{2}$ (two equal masses)?

What you should find

At L_4, $r_1 = r_2 = 1$, so $\Omega_y/y = 1 - (1-\mu) - \mu = 0$, and with $x + \mu = \frac{1}{2}$ and $x - 1 + \mu = -\frac{1}{2}$, $\Omega_x = x - \frac{1-\mu}{2} + \frac{\mu}{2} = (\frac{1}{2} - \mu) - \frac{1}{2} + \mu = 0$. The discriminant vanishes at $27\mu(1-\mu) = 1$, i.e. $\mu = 0.03852$. At $\mu = \frac{1}{2}$, $27\mu(1-\mu) = 6.75 > 1$: strongly unstable, with $\lambda^2 = (-1 \pm i\sqrt{5.75})/2$.

5 Periodic orbits around the collinear points

A spacecraft cannot sit at L_1 exactly (it is unstable), but it can fly **around** it. The linear solution near a collinear point, with the unstable direction switched off, is a pair of independent oscillations:

$$\xi = -A_x \cos(\nu t + \phi), \qquad \eta = k A_x \sin(\nu t + \phi), \qquad \zeta = A_z \sin(\nu_z t + \psi),$$

where ν and ν_z are the frequencies of Table 3 and the ratio of the in-plane axes follows from substituting into $\ddot{\xi} - 2\dot{\eta} = (1 + 2c_2)\xi$:

$$k = \frac{\nu^2 + 1 + 2c_2}{2\nu}. \tag{11}$$

At the Sun–Earth L_1, $k = 3.229$: the in-plane motion is an ellipse three times longer along the Sun–Earth-perpendicular direction (y) than along the line to the Sun and Earth (x). ISRO's target for Aditya-L1 has $A_y/A_x = 663\,200/209\,200 = 3.17$.

5.1 Lyapunov orbits

With $A_z = 0$ this is the planar **Lyapunov orbit**, and its non-linear continuation exists for large amplitudes (Figure 3, left). Its period *grows* with the amplitude: from 11.70 d at small amplitude to 15.5 d for the largest drawn about the Earth–Moon L_1.

5.2 Halo orbits

If both A_x and A_z are nonzero the linear solution is a quasi-periodic Lissajous curve, unless the two frequencies agree. At the Sun–Earth L_1 they differ slightly, $\nu = 2.0865$ against $\nu_z = 2.0152$ (periods 175.1 d and 181.2 d). But

the non-linear terms make the planar frequency depend on the amplitude, as the growing period of the Lyapunov family shows. At a large enough amplitude the periods meet, and the motion closes into a single periodic three-dimensional curve: a **halo orbit** (Figure 3, right), named for the way it looks from the Earth, a ring around the Sun. For Aditya-L1's halo, with $A_z = 120\,000$ km, the computed period is 177.87 d, between the two linear periods, and the half-axes are $A_x = 208\,270$ km and $A_y = 667\,997$ km.

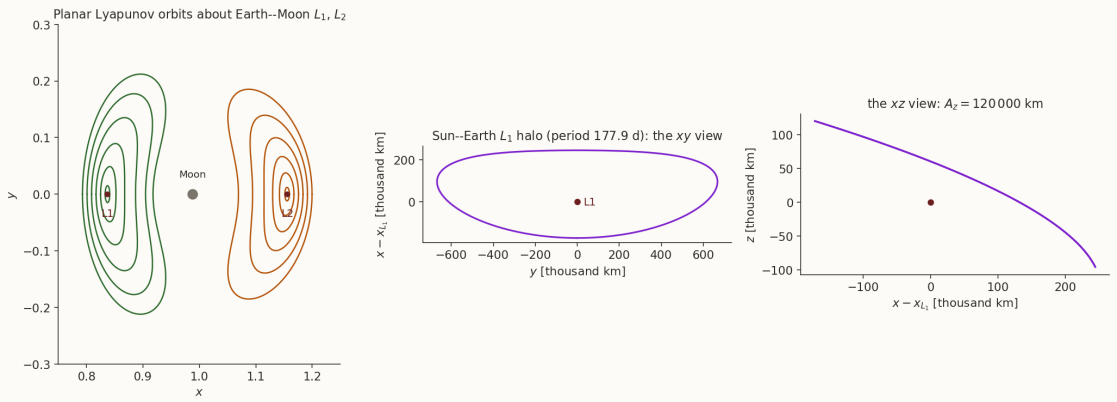


Figure 3: **Left:** planar Lyapunov orbits about the Earth–Moon L_1 and L_2 points (the Earth is off the left of the picture), at six amplitudes each. **Middle and right:** the xy and xz projections of the Sun–Earth L_1 halo orbit with $A_z = 120\,000$ km found by shooting, which is the orbit of Problem 6 of the Indian missions problems. Its period is 177.87 days. *Script: fig-cr3bp.py*.

How such orbits are computed. The symmetry $(x, y, z, t) \to (x, -y, z, -t)$ of Equation 1 means a periodic orbit that crosses the xz plane perpendicularly at one point ($\dot{x} = \dot{z} = 0, y = 0$) crosses it perpendicularly at the half period too. So only the initial state $(x_0, z_0, \dot{y}_0)$ needs to be found, with z_0 fixed by the chosen amplitude and the two conditions $\dot{x} = \dot{z} = 0$ at the next crossing. A Newton iteration on $(x_0, \dot{y}_0)$ (single shooting; the script uses a root-finder with finite differences) converges in a few steps from the linear guess. The Jacobi constant is conserved along the result to 10^{-14}, which is the first check of any integration.

For the Sun–Earth L_1 ($c_2 = 4.0611$) find λ, ν, ν_z and the axis ratio k from the formulas above. What are the in-plane and out-of-plane periods in days?

What you should find

The characteristic equation $\lambda^4 - 2.0611\lambda^2 - 27.92 = 0$ (since $1 + c_2 - 2c_2^2 = -27.92$) gives $\lambda^2 = 6.4146$ and -4.3535, so $\lambda = 2.5327$ and $\nu = 2.0865$. Also $\nu_z = \sqrt{4.0611} = 2.0152$ and $k = (4.3535 + 1 + 8.1222)/(2 \times 2.0865) = 3.229$. The periods are $2\pi/\nu = 3.0114$ and $2\pi/\nu_z = 3.1179$ time units, i.e. 175.1 and 181.2 days (one time unit is 58.132 d).

6 What the restricted problem is used for

- **Sun–Earth L_1.** SOHO (since 1996) and Aditya-L1 (2023–24) fly halo orbits about the Sun–Earth L_1, where the Sun is always in view and the line to the Earth is short. The Indian missions problems work out why Aditya-L1's cruise took about 110 days and what it cost: insertion Δv against transfer time, the stable manifold, and the instability.
- **Sun–Earth L_2.** The James Webb Space Telescope flies in an orbit about the Sun–Earth L_2 point, 1.5077×10^6 km from the Earth on the far side from the Sun, where the Earth, the Moon and the Sun are all in nearly the same direction, so one sunshield protects the telescope.
- **Earth–Moon L_2.** A relay satellite in a halo orbit about the Earth–Moon L_2 point can see both the far side of the Moon and the Earth, which a lunar orbiter cannot: the Chinese relay Queqiao (2018) did this for Chang'e 4.
- **Low-energy transfers.** The unstable and stable manifolds of the periodic orbits are tubes in phase space that carry a spacecraft with little Δv between regions; the Japanese Hiten (1991) used a version of it for its transfer to the Moon.
- **Natural bodies.** The Trojan asteroids at Jupiter's L_4 and L_5 points are the natural counterpart of the model. (Faint dust clouds near the Earth–Moon L_4 and L_5, first reported by Kordylewski in 1961, remain a matter of dispute.)

What the model leaves out. (i) The primaries are on *circles*: the real Earth–Moon orbit has eccentricity 0.055 and the Earth's about the Sun 0.017, and the *elliptic* restricted problem adds a time-dependent term. (ii) For Earth–Moon, the Sun is a fourth body, large enough to destroy the stability of L_4, L_5 and to shift the periodic orbits. (iii) Solar radiation pressure, the Earth's J_2, and the lumping of Earth and Moon in the Sun–Earth problem. Real mission design therefore uses the CR3BP to *find* the orbits and manifolds, and then converges them in an ephemeris model with all the real forces.

7 The Hill sphere and the sphere of influence

Equation 6 says L_1 and L_2 sit at the **Hill radius** $r_{\mathrm{H}} = a\,(m/3M)^{1/3}$ of the small body (of mass m, on a circular orbit of radius a about the large one of mass M). Inside it, the small body's gravity wins against the combined tidal and centrifugal terms of the rotating frame, so a particle *can* orbit it. For the Earth: $r_{\mathrm{H}} = 1.4966 \times 10^6$ km. The sphere of influence of the patched-conics note, $r_{\mathrm{SOI}} = a\,(m/M)^{2/5} = 9.25 \times 10^5$ km, is smaller by a factor 1.6. They answer different questions. The Hill sphere is where a body *can hold on* to a satellite in the rotating frame (and so where the equilibrium points are); the sphere of influence is where treating the body as the *central* one and the

other as the disturbance gives the smaller **relative** perturbation, the criterion that decides where to *switch* the central body in a patched-conic calculation. The Moon, at 384 400 km, is at 0.26 of the Earth's Hill radius: well inside, as it must be.

In one paragraph. Two primaries on circles and a massless third body, in the rotating frame and in units where the primaries are 1 apart and rotate at 1, give the equations Equation 1 with the single parameter $\mu = m_2/(m_1 + m_2)$. The Jacobi constant $C = 2\Omega - v^2$ is conserved and forbids the spacecraft the regions where $2\Omega < C$; as C falls the walls at L_1, L_2, L_3 open in turn. There are five equilibrium points: three collinear ones, from a quintic, and two equilateral ones at $(\frac{1}{2} - \mu, \pm\frac{\sqrt{3}}{2})$. The collinear points are unstable saddles with a centre; the triangular ones are stable only for $\mu < 0.0385$. About the collinear points there are periodic orbits, planar (Lyapunov) and three-dimensional (halo), found by shooting, and the Sun–Earth ones are the working orbits of SOHO, JWST and Aditya-L1.

Summary

The numbers of this note are checked by cr3bp-verify.py, using the helper module _cr3bp.py (equations of motion, equilibrium points, linearisation, shooting for Lyapunov and halo orbits); fig-cr3bp.py draws the figures.

Constants used. $\mu_\oplus = 398\,600.4418$ km^3/s^2, $\mu_{\text{Moon}} = 4902.800066$ km^3/s^2, $\mu_\odot = 1.32712440018 \times 10^{11}$ km^3/s^2, 1 AU $= 1.495978707 \times 10^8$ km, Moon distance 384 400 km (the full list is on the notation page).

Further reading. V. Szebehely, *Theory of Orbits: The Restricted Problem of Three Bodies* (Szebehely 1967), is the classical reference; W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, *Dynamical Systems, the Three-Body Problem and Space Mission Design* (Koon et al. 2011), treats halo orbits, manifolds and low-energy transfers from the viewpoint of the mission designer; C. D. Murray and S. F. Dermott, *Solar System Dynamics* (Murray and Dermott 1999), chapter 3, is the astronomer's account; H. D. Curtis, *Orbital Mechanics for Engineering Students*, has the Lagrange points in the chapter on the restricted problem. D. L. Richardson's analytic construction of halo orbits is in *Celestial Mechanics* 22 (1980) 241–253.

Exercises

1. **Units.** Check Table 1: from $G(m_1 + m_2)$ and the distance, compute the time and speed units of the Earth–Moon system. (Answers: $T = 375\,190$ s $= 4.3425$ d; 1.0245 km/s.)

2. **Equilateral points.** Show that for $\mu = \frac{1}{2}$ the triangular points of Equation 4 are at $(0, \pm\frac{\sqrt{3}}{2})$, and that the collinear ones are at $x = 0$ (between the equal masses) and symmetric about it. Is the problem then stable at L_4? (No: $27 \cdot \frac{1}{4} > 1$.)
3. **Collinear point by Newton.** Find the Earth–Moon L_2 from Equation 5 starting at $x = 1 + \gamma_{\text{Hill}}$, and the Sun–Earth L_2 distance from the Earth in km. (Answers: $x = 1.155682$; 1.5077×10^6 km.)
4. **Jacobi constant of a lunar orbit.** Find C for a prograde circular orbit 100 km above the Moon's surface (radius 1837.4 km), at four points of the orbit. (Answer: $C = 5.5088$ at all four points, which differ only in the fifth decimal. The orbit is deep inside the Moon's region, and $C \gg C_{L_1} = 3.188$: to leave it, a burn must lower C by 2.3.)
5. **Linear analysis.** For the Earth–Moon L_1 ($c_2 = 5.1476$) compute λ, ν, ν_z and the e-folding time in days. (Answers: 2.9321, 2.3344, 2.2688, 1.48 d.)
6. **Routh.** Show that $\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$ has purely imaginary roots exactly for $\mu \leq 0.0385$, and find the periods of the two motions for Sun–Jupiter ($\mu = 9.54 \times 10^{-4}$) in units of Jupiter's orbital period. (Answers: $\omega_1 = 0.99676$ and $\omega_2 = 0.08046$, so periods of 1.003 and 12.43 Jupiter years, i.e. about 147 years for the slow libration of the Trojans.)
7. **Hill and SOI.** For the Moon about the Earth ($m/M = 0.01215/0.98785$, $a = 384\,400$ km), compare the Hill radius with the sphere of influence. (Answers: $r_{\text{H}} = 61\,500$ km and $r_{\text{SOI}} = 66\,200$ km; for a mass ratio this large the SOI is the *larger* of the two, the opposite of the Earth's case.)

Hill, George William. 1878. "Researches in the Lunar Theory." *American Journal of Mathematics* 1 (1).

Koon, Wang Sang, Martin W. Lo, Jerrold E. Marsden, and Shane D. Ross. 2011. *Dynamical Systems, the Three-Body Problem and Space Mission Design*. Marsden Books.

Murray, Carl D., and Stanley F. Dermott. 1999. *Solar System Dynamics*. Cambridge: Cambridge University Press.

Szebehely, Victor. 1967. *Theory of Orbits: The Restricted Problem of Three Bodies*. New York: Academic Press.