Complications to Impulsive Transfers

Solid motors that cannot be throttled, burns that last minutes, and engines so weak that the transfer is a spiral: a slow reading of Chobotov, chapter 6

Devendra Ghate

November 2, 2026

Table of contents

Where we are going **2**

1 How many impulses? **3**

2 Fixed-impulse transfers **4**
2.1 The problem: motors that cannot be shut down 4
2.2 Ways to use up the excess 5
2.3 The example: a Shuttle-delivered navigation satellite 6
2.4 Spending the excess on a plane change 7
2.5 How far apart can the parking orbit and the final orbit be? . 8
2.6 Non-Hohmann transfers: using up the excess in a different way 10
2.7 Launch window 12
2.8 The book's problems 6.1 and 6.3, worked 13

3 Finite-duration burns: gravity losses **14**
3.1 The equation of motion of a rocket in a gravity field 14
3.2 The simplest case: a vertical ascent 16
3.3 Gravity loss in orbit, and what drives it 16
3.4 The cure: several perigee burns 18

4 Very low thrust transfers **19**
4.1 Setting up 19
4.2 Derivation by a pseudo-velocity vector 20
4.3 Using the formula: propellant and time 21

Summary **23**

Exercises **24**

Where we are going

Up to this point in the course a transfer between two orbits has been a set of **instantaneous** velocity changes, ΔV, and the best transfer has been the one with the smallest total ΔV: the Hohmann transfer, the bi-elliptic transfer, the plane change done where the speed is lowest. That is a model. It is a very good one for planning, and every real mission is first sized with it. But a rocket engine that is *real* differs from the model in four ways, and Chobotov's chapter 6 (Chobotov 2002) looks at each in turn:

1. **How many impulses?** The best transfers so far used one, two or three. Would four or five ever do better?
2. **Fixed impulses.** A solid rocket motor cannot be throttled, stopped or restarted. It delivers one fixed ΔV, whatever the mission needs. What then?
3. **Finite-duration burns.** A real burn takes minutes, not zero seconds, and during those minutes gravity keeps acting. How much extra ΔV does that cost?
4. **Very low thrust.** An electric engine accelerates so gently that the transfer is a slow spiral lasting weeks, and the whole idea of "an impulse at a point" is gone. What does it cost?

The chapter is short (about fifteen pages) and very dense, because Chobotov leaves out nearly every intermediate step: the geometry of a burn that changes the plane is stated, not derived; a figure of allowed parking orbits is shown without saying how it was drawn; the gravity-loss equation is written down and then discussed in words. This note rebuilds all of it. Each equation is derived step by step, each figure is redrawn from the equations (the scripts are linked), and every number is reproduced by complications-verify.py, which also solves the three problems at the end of the chapter. Where Chobotov's number and ours differ slightly, the note says so and why.

> What you should be able to answer at the end
>
> By the end you should be able to answer: (1) When is a third impulse worth having, and what is the (rare) case for a fourth? (2) A two-motor solid upper stage has fixed ΔV_1 and ΔV_2; why does the transfer then have to include a plane change, and how large is it? (3) What does "the locus of parking orbits" mean, how is it computed, and why does it give a *launch window*? (4) What is the equation of motion of a rocket with gravity, where does the gravity loss come from, and why does splitting a burn into several perigee burns reduce it? (5) How is the ΔV of a continuous-thrust spiral computed, why is it larger than the Hohmann ΔV, and what is the transfer time?

The required background is the Hohmann transfer and plane changes from the manoeuvres note, the rocket equation, the angle between two orbit planes (inclination and node of the orbital-elements note), and the rotation of the Earth (time note). As in the other notes, a blue **Try it** box asks you to do a small calculation before reading on, a green **Check yourself** box asks a short question, and each has a hidden answer; a purple box lists what the book *assumes without saying*, which is where a student of the chapter usually gets stuck.

Notation. To let you read this note next to the book, it uses Chobotov's own symbols for the ΔV calculations: V_{c1}, V_{c2} for the circular speeds of the initial and final orbits, V_{PTR} and V_{ATR} for the speeds at the perigee and apogee of the transfer orbit, α_1, α_2 for the plane-change angles of the two burns, γ_2 for the flight-path-angle change at the second burn, and θ for the angle between the parking-orbit plane and the final-orbit plane. Speeds are in m/s when the book's numbers are used, otherwise km/s. We use $\mu_\oplus = 398\,600.4418$ km^3/s^2, $R_\oplus = 6378.137$ km, $g_0 = 9.80665$ m/s^2 and the Earth's rotation of 15.041°/h. The book rounds the Earth's radius to 6378 km and μ to 398 600; the differences are a few m/s.

1 How many impulses?

The question of Chobotov's 6.1 is a natural one. We know that

- one impulse is enough for some changes (a pure plane change, or changing the size of an orbit between two points that happen to lie on one ellipse);
- two impulses are best for the Hohmann transfer, and for changing the argument of perigee;
- three impulses are best for the **bi-elliptic transfer** of the manoeuvres note, when the ratio of the two radii is large.

More impulses mean more free parameters (when to burn, in which direction, how hard) and so, in principle, no worse an optimum. Is a fourth impulse ever *worth* the trouble? The answer, due to Edelbaum (Edelbaum 1967), is no except in rare cases: **the minimum-ΔV transfer between two orbits needs at most three impulses in all but exceptional situations.** (Chobotov quotes the result without proof, and so do we; the argument is in the cited paper.) That is why the earlier chapters stopped at three, and why this one turns to other complications.

To see how rare even the *third* impulse is, here is the coplanar transfer between two circular orbits with the radius ratio $R = r_2/r_1$, written in units of the first circular speed. The Hohmann ΔV is $(\sqrt{2R/(1+R)} - 1) + R^{-1/2}(1 - \sqrt{2/(1+R)})$, and the bi-elliptic transfer through an apoapsis at $r_b \to \infty$ costs

less than it when $R > 11.94$, by a tiny amount at first and by 8% at $R = 100$:

Table 1: $\Delta V / V_{c1}$ for coplanar circular-to-circular transfers (computed in the verify script). For $11.94 < R < 15.58$ the bi-elliptic transfer wins only if its apoapsis is far enough out; above 15.58 it wins for every $r_b > r_2$. The ratio 100 is a transfer from a 6600 km orbit to 660 000 km.

$R = r_2/r_1$	Hohmann (2 impulses)	bi-elliptic, $r_b \to \infty$ (3 impulses)
10	0.5298	0.5452
11.94	0.534 095	0.534 087
15.58	0.5363	0.5192
20	0.5347	0.5068
100	0.4931	0.4556

So the third impulse pays only for enormous radius ratios, and the fourth essentially never does. The rest of the chapter is about complications that *change the problem* rather than add impulses.

2 Fixed-impulse transfers

2.1 The problem: motors that cannot be shut down

Most satellites delivered to low orbit by the Space Shuttle then reached their working orbit with a **solid-propellant upper stage**: an inertial upper stage (IUS) or a payload assist module (PAM) (Chobotov 2002). A solid motor is a case full of propellant with a nozzle. It burns until the propellant is gone. It cannot be throttled; in practice it cannot be stopped or restarted. A transfer between two orbits usually needs two burns, so the upper stage is **two solid motors in a stack** (Figure 1): stage I fires first, stage II second.

Each motor changes the velocity by an amount fixed by the **rocket equation** of the first chapter. Let us derive it again, because it fixes the meaning of the symbols. A rocket of mass m ejects a small mass $-\mathrm{d}m > 0$ backwards at the exhaust speed u relative to itself. Momentum conservation for the rocket and the ejected gas (no external force) gives $m\,\mathrm{d}v = -u\,\mathrm{d}m$, so $\mathrm{d}v = -u\,\mathrm{d}m/m$. Integrating from the initial mass m_i to the final mass m_f (with u constant),

$$\Delta V = u \ln \frac{m_i}{m_f}, \qquad u = g_0 I_{sp}, \qquad \text{i.e.} \qquad \frac{W_i}{W_f} = \exp\Big(\frac{\Delta V}{g_0 I_{sp}}\Big), \tag{1}$$

where I_{sp} is the specific impulse (in seconds), $g_0 = 9.80665\,\mathrm{m/s^2}$ the standard gravity that *defines* I_{sp} in seconds, and $W = mg_0$ the weight at the Earth's

surface. The ratio W_i/W_f equals m_i/m_f, which is why the book can use weights or masses: only the ratio matters. This is Chobotov's (6.1).

A motor has $I_{sp} = 300$ s and burns from $W_i = 1896$ kg to $W_f = 998$ kg. What is its ΔV?

What you should find

$W_i/W_f = 1.8998$, $\ln 1.8998 = 0.6416$, and $\Delta V = 9.80665 \times 300 \times 0.6416 = 1888.0$ m/s. (This is the second motor of the example below.)

Because the motor cannot be stopped, the two ΔVs of the upper stage are **numbers given by the hardware**, ΔV_1 and ΔV_2. The transfer has to use *exactly* these. This is the **velocity-matching** (or **fixed-impulse**) problem (Chu, Lang, and Winn 1978). Two cases are trivial: if $\Delta V_1 + \Delta V_2$ is *less* than the total of the best transfer, nothing can be done, the satellite cannot reach its orbit. If it is *more*, the stage has **excess energy**, and the excess has to go somewhere. That is the complication.

Three things the book does not say. (i) The motors are *impulsive*: their burn time is neglected, so the ΔV of each is applied at one point; section 3 relaxes this. (ii) The two burns are at the *ends of the transfer orbit's major axis* (perigee and apogee, as in a Hohmann transfer), at least until the book relaxes it; the plane change at each burn is made by turning the thrust out of the orbit plane. (iii) The initial and final orbits are *circular*. Everything below depends on them.

2.2 Ways to use up the excess

A designer who has fixed motors and a mission whose optimal ΔV is *less* than the motors give can:

1. **Offload propellant**: remove some from the motor. This is costly, and for a large offload the motor may have to be requalified for flight.
2. **Add ballast**: put dead weight on the stage, which lowers its ΔV through the rocket equation. Once the second stage matches, the first may be left without enough, and bolting weights to the payload invalidates tests already made.
3. **Modify the trajectory**: change the *transfer* so that it needs more ΔV than the optimal one. The excess is used on an extra plane change, an extra altitude change, an extra flight-path-angle change. Usually there is not one solution but a range, which can be exploited, as we shall see, to get a *launch window*.

Chobotov develops the third. We follow him with the example of his text.

2.3 The example: a Shuttle-delivered navigation satellite

A constellation of satellites is in circular orbits with a 12-h period, at the height $h_2 = 20\,187$ km and the inclination $i_2 = 55°$. A satellite is delivered by the Shuttle to a circular parking orbit at $h_1 = 278$ km, at an inclination i_1 that we are free to choose within limits, and is then carried up by two solid motors. The motors are (Figure 1, with Chobotov's weights):

Table 2: Weights of the upper stage ($I_{sp} = 300$ s for both motors).

	structure	propellant
payload (satellite)		907 kg
motor II (second burn)	91 kg	898 kg
motor I (first burn)	181 kg	2174 kg

At the *second* burn the stage is payload plus motor II: $W_i = 907 + 91 + 898 = 1896$ kg before, $W_f = 907 + 91 = 998$ kg after (the propellant is gone, the structure stays). At the *first* burn the stage is everything: $W_i = 907 + 91 + 898 + 181 + 2174 = 4251$ kg before, and after burnout of motor I it is payload plus motor II plus motor I's empty case, $W_f = 907 + 91 + 898 + 181 = 2077$ kg. Then

$$\Delta V_2 = 9.80665{\times}300 \ln \frac{1896}{998} = 1888.0\,\mathrm{m/s}, \qquad \Delta V_1 = 9.80665{\times}300 \ln \frac{4251}{2077} = 2107.1\,\mathrm{m/s},$$

as in the book (1888 and 2107 m/s).

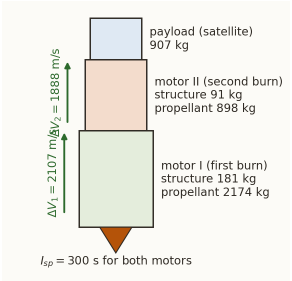


Figure 1: The upper stage: a payload on two solid motors. The stages are fired in turn, with the weights of Table 2. (Schematic.)

Plain Hohmann transfer. First what the transfer would cost with no plane change. The radii are $r_1 = 6378.137 + 278 = 6656.137$ km and $r_2 = 6378.137 + 20\,187 = 26\,565.137$ km. The circular speeds are $V_{c1} = \sqrt{\mu/r_1} = 7738.5$ m/s

and $V_{c2} = \sqrt{\mu/r_2} = 3873.6$ m/s. The transfer ellipse has $a_{TR} = (r_1 + r_2)/2 = 16\,610.6$ km, and by the vis-viva equation $V = \sqrt{\mu(2/r - 1/a)}$ the speeds at its two ends are

$$V_{PTR} = \sqrt{\mu\Big(\frac{2}{r_1} - \frac{1}{a_{TR}}\Big)} = 9786.3\,\text{m/s}, \qquad V_{ATR} = \sqrt{\mu\Big(\frac{2}{r_2} - \frac{1}{a_{TR}}\Big)} = 2452.1\,\text{m/s}.$$

A Hohmann transfer would cost $V_{PTR} - V_{c1} = 2047.8$ m/s and $V_{c2} - V_{ATR} = 1421.5$ m/s, a total of 3469.4 m/s. The motors supply $\Delta V_1 + \Delta V_2 = 3995.2$ m/s. The excess is 525.8 m/s, and it has to be spent.

2.4 Spending the excess on a plane change

The simplest way: do a **Hohmann-type transfer in which a plane change accompanies each burn**. Turning the velocity vector out of the orbit plane costs ΔV (Figure 2 (a)). At the first burn the velocity before is the circular velocity $\mathbf{V}_{c1}$ (length V_{c1}); after the burn it must be the transfer velocity at perigee, length V_{PTR}, in a direction rotated by the angle α_1 about the radius vector. The two vectors form a triangle with the angle α_1 between them, and the third side is the velocity change, so by the **law of cosines**

$$\Delta V_1^2 = V_{c1}^2 + V_{PTR}^2 - 2V_{c1}V_{PTR}\cos\alpha_1, \tag{2}$$

and in the same way at the apogee

$$\Delta V_2^2 = V_{c2}^2 + V_{ATR}^2 - 2V_{c2}V_{ATR}\cos\alpha_2. \tag{3}$$

(These are Chobotov's (6.3) and (6.2).) Here is the algebra behind the picture, which the book skips. Put the radius vector along the z axis. The two velocities are both horizontal (perpendicular to z): $\mathbf{V}_c = (V_c, 0, 0)$ before, and $\mathbf{V}_t = V_t(\cos\alpha, \sin\alpha, 0)$ after, the transfer-orbit velocity turned by α about the radius vector. Then $|\mathbf{V}_t - \mathbf{V}_c|^2 = V_t^2 + V_c^2 - 2V_cV_t\cos\alpha$. Note that α is *the angle by which the orbit plane tilts*: the orbit plane is the plane containing the radius vector and the velocity, and rotating the velocity about the radius vector by α rotates the plane by the same angle α about the same axis. (This is also checked numerically in the script.)

Equations (Equation 2) and (Equation 3) are now *solved for the angles*, given the fixed ΔVs:

$$\cos\alpha_1 = \frac{V_{c1}^2 + V_{PTR}^2 - \Delta V_1^2}{2V_{c1}V_{PTR}}, \qquad \cos\alpha_2 = \frac{V_{c2}^2 + V_{ATR}^2 - \Delta V_2^2}{2V_{c2}V_{ATR}}. \tag{4}$$

Inserting the numbers, with the speeds in km/s ($V_{c1} = 7.7385$, $V_{PTR} = 9.7863$, $\Delta V_1 = 2.1071$; $V_{c2} = 3.8736$, $V_{ATR} = 2.4521$, $\Delta V_2 = 1.8880$):

$$\cos\alpha_1 = \frac{59.883 + 95.772 - 4.440}{2(7.7385)(9.7863)} = 0.99838, \quad \alpha_1 = 3.27°; \qquad \cos\alpha_2 = \frac{15.005 + 6.013 - 3.565}{2(3.8736)(2.4521)} = 0.91$$

the book's 3.25° and 23.25°. So **the motors are exactly used up if the first burn tilts the plane by** 3.27° **and the second by** 23.26°: the second burn, where the speed is low, takes most of the plane change, as it should be, because a plane change is cheapest where the speed is lowest.

How good a choice is the split 3.27° / 23.26°, from the point of view of fuel? Suppose we wanted the total plane change 26.53° at the lowest total ΔV, and the motors were not fixed: how would the best split compare?

Answer

Minimise $\Delta V_1 + \Delta V_2$ over α_1 with $\alpha_1 + \alpha_2 = 26.53°$ held fixed: the minimum is at $\alpha_1 = 3.17°$ and costs 3995.11 m/s. The motors, forced to 3.27°, cost 3995.16 m/s: 0.05 m/s more. The fixed motors happen to deliver almost exactly the best possible split. (Doing the whole 26.53° at the apogee burn would cost 4053.0 m/s, 58 m/s more.) The designer's motors were evidently chosen for this transfer.

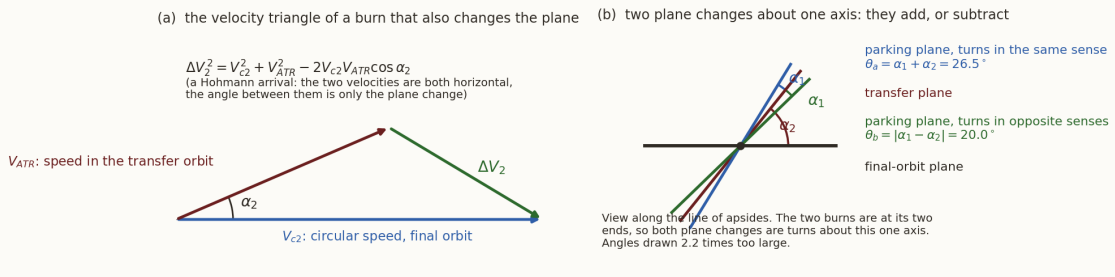


Figure 2: (**a**) A burn that changes the plane: the circular speed V_{c2} before, the transfer-orbit speed V_{ATR} after, at the angle α_2; the ΔV is the third side of the triangle. Drawn to scale for the second burn of the example. (**b**) The three orbit planes seen along the line of apsides. The first burn turns the plane by α_1, the second by α_2, and because the two burns are at opposite ends of the same line, both turns are about the same axis; they add if made in the same sense and subtract if made in opposite senses. *Script: fig-ci-geometry.py.*

2.5 How far apart can the parking orbit and the final orbit be?

What the motors fixed is α_1 and α_2. What we need to know is **which parking orbits** the Shuttle can be in. A parking orbit is characterised by its inclination i_1 and the right ascension of its ascending node Ω_1, and what matters is how far it is from the final orbit, in two numbers: the angle θ between the

two orbit planes, which this section computes, and the difference of nodes $\Delta\Omega = \Omega_1 - \Omega_2$, which the next one will relate to the launch time.

The total plane change. Both burns of a Hohmann-type transfer are at the two ends of the same straight line (the line of apsides). The first rotates the orbit plane by α_1 about this line; the second by α_2 about the same line. Two rotations about one axis simply add their angles when made in the same sense, and subtract when made in opposite senses (Figure 2 (b)). The total angle θ between the parking orbit and the final orbit is therefore

$$\theta_a = \alpha_1 + \alpha_2 = 26.53^\circ \text{ (same sense)}, \qquad \theta_b = |\alpha_1 - \alpha_2| = 19.99^\circ \text{ (opposite senses)}, \tag{5}$$

(Chobotov's 26.5° and 20°). The Shuttle's parking orbit may be at either of two angles from the final orbit; the transfer is made in the sense that gives the angle in question.

The relation of θ to the inclination and the node. Each orbit plane has a unit normal, $\hat{\mathbf{n}} = (\sin i \sin\Omega,\ -\sin i \cos\Omega,\ \cos i)$ (the direction of the angular momentum; check: for $i = 0$ it is the z axis). The angle between two planes is the angle between their normals, so $\cos\theta = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2$:

$$\boxed{\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos\Delta\Omega} \tag{6}$$

(Chobotov's (6.4); the product is expanded in the script by symbolic algebra.) This is the spherical law of cosines for the triangle formed by the pole and the two normals.

The special case $\Delta\Omega = 0$, parking and final orbits with the *same node*: then $\theta = |i_1 - i_2|$, and the plane change is purely an inclination change, so the parking inclinations are $i_1 = i_2 \pm \theta$. With $i_2 = 55^\circ$:

$$i_1 = 55^\circ \pm 26.53^\circ = 28.47^\circ \text{ or } 81.53^\circ, \qquad i_1 = 55^\circ \pm 19.99^\circ = 35.01^\circ \text{ or } 74.99^\circ,$$

the book's 28.5°, 81.5°, 35°, 75°. A Shuttle orbit at one of those four inclinations, with the right node, is served by the Hohmann transfer of the motors. In general the Shuttle orbit does not have to have the same node as the final orbit, and then (Equation 6) gives, for each i_1, the node difference $\Delta\Omega$ that makes the angle equal to θ_a (or θ_b): $\cos\Delta\Omega = (\cos\theta - \cos i_1 \cos i_2)/(\sin i_1 \sin i_2)$. Plotting $\Delta\Omega$ against i_1 for $\theta = \theta_a$ and for $\theta = \theta_b$ gives the two closed curves of Chobotov's Fig. 6.5, which are the **loci of parking orbits that match ΔV_1 and ΔV_2 with Hohmann transfers**; the outer one is the purple curve of Figure 3 (b) and the dashed green one inside it is θ_b.

2.6 Non-Hohmann transfers: using up the excess in a different way

There is nothing sacred about the Hohmann transfer. Suppose the **apogee of the transfer orbit is raised above the final orbit**, so that the transfer ellipse *crosses* the final circle at the second burn instead of touching it (Chobotov's Fig. 6.6). Three things change.

1. The perigee speed V_{PTR} rises (a larger ellipse). With ΔV_1 fixed, Equation 2 then allows *less* plane change: α_1 **decreases**, until at $V_{PTR} = V_{c1} + \Delta V_1$ it is zero (the whole ΔV_1 is needed for the speed change).
2. At the crossing the transfer velocity has a **flight-path angle** γ_2: it is not horizontal, it points outward (on the way up) or inward (on the way down). The circular velocity of the final orbit is horizontal, so the velocity change at the second burn must now remove the flight-path angle as well as the plane angle. The velocity after the burn is horizontal and in the final plane: $\mathbf{V}_c = (V_{c2}, 0, 0)$. The transfer velocity before the burn is $\mathbf{V}_t = V_t(\cos\gamma_2 \cos\alpha_2,\ \cos\gamma_2 \sin\alpha_2,\ \sin\gamma_2)$: horizontal components rotated by α_2, and a radial component $V_t \sin\gamma_2$. Then

$$\Delta V_2^2 = |\mathbf{V}_t - \mathbf{V}_c|^2 = V_{c2}^2 + V_t^2 - 2V_{c2}V_t \cos\gamma_2 \cos\alpha_2, \qquad (7)$$

   which reduces to Equation 3 for $\gamma_2 = 0$ (the proof is in the script, symbolically). Since $\cos\gamma_2 < 1$, the same ΔV_2 gives a *smaller* α_2: α_2 **decreases**.
3. The two burns are no longer at opposite ends of the line of apsides: the second is at the true anomaly $f_2 < 180°$ of the transfer orbit (the crossing on the way up) or at $360° - f_2$ (on the way down). The two plane changes are rotations about two *different* radius vectors, and the angles no longer simply add. Chobotov remarks that θ_a and θ_b "are no longer the simple sum and difference" and refers to the literature (Chu, Lang, and Winn 1978). Here is the result.

The composition of two plane changes about different axes. Let $\hat{\mathbf{n}}_t$ be the normal of the transfer plane. The final plane is the transfer plane turned about the radius vector $\hat{\mathbf{e}}_2$ of the second burn by α_2, and the parking plane is the transfer plane turned about the radius vector $\hat{\mathbf{e}}_1$ of the first burn by α_1. Both $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ lie in the transfer plane, with the angle f_2 between them (the true anomaly of the second burn, counted from the perigee, where the first is). A rotation of $\hat{\mathbf{n}}_t$ about an axis in the plane perpendicular to it by the angle α gives $\hat{\mathbf{n}} = \hat{\mathbf{n}}_t \cos\alpha \pm \hat{\mathbf{p}} \sin\alpha$, where $\hat{\mathbf{p}} = \hat{\mathbf{e}} \times \hat{\mathbf{n}}_t$ is a unit vector in the transfer plane, perpendicular to the axis, and the sign gives the sense. The vectors $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ are the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ both turned by 90° about $\hat{\mathbf{n}}_t$, so the angle between them is also f_2, and $\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2 = \cos f_2$. The angle between the parking and the final orbit planes is then given by

$$\boxed{\cos\theta = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \cos\alpha_1 \cos\alpha_2 \pm \sin\alpha_1 \sin\alpha_2 \cos f_2} \tag{8}$$

(the dot products $\hat{\mathbf{n}}_t \cdot \hat{\mathbf{p}}$ vanish; the script checks this on 600 random configurations). For $f_2 = 180°$ this is $\cos(\alpha_1 \mp \alpha_2)$: **the Hohmann result Equation 5 is the special case**. For $f_2 < 180°$ the two angles θ_a (the + sign) and θ_b (the − sign) come closer together. The ascending and the descending crossing have the same $\cos f_2$, so the same angles; they differ in the flight time, which is why the book calls the two end cases "max/min flight time" transfers.

Numbers. Table 3 lists the family. As the apogee is raised, both α_1 and α_2 fall and the angles θ_a and θ_b close in on each other. The family stops where α_1 reaches zero, $V_{PTR} = V_{c1} + \Delta V_1 = 9845.7$ m/s, which is an apogee radius of 28 259 km, **only** 1694 **km above the Hohmann apogee** (Figure 3 (a)): a big increase of the apogee would need much more ΔV_1 than the motors have. At the end of the family the second burn removes a flight-path angle of $\gamma_2 = 22.95°$ and a plane change of $\alpha_2 = 13.06°$.

Table 3: The non-Hohmann family of the example: the angles when the transfer apogee is raised above the Hohmann value. The smallest θ_b along the family is 12.88° (at 1671 km).

apogee above the Hohmann one [km]	α_1	α_2	γ_2	f_2	θ_a	θ_b
0 (Hohmann)	3.27°	23.26°	0°	180.0°	26.53°	19.99°
100	3.17°	22.75°	6.05°	175.9°	25.91°	19.60°
500	2.73°	20.66°	13.23°	171.0°	23.35°	17.97°
1000	2.07°	17.79°	18.23°	167.4°	19.81°	15.79°
1694 (limit, $\alpha_1 = 0$)	0	13.06°	22.95°	163.9°	13.06°	13.06°

The region. For *each* member of the family there is a locus of parking orbits, from which the transfer can be flown with the exact ΔV_1 and ΔV_2: the curves $\theta = \theta_a$ and $\theta = \theta_b$ of that member, in the plane of $(\Delta\Omega, i_1)$ of Equation 6. All members together fill the region *between the outer curve* $\theta = 26.53°$ (the Hohmann transfer, with the largest total plane change) *and the inner curve* $\theta = 12.88°$ (the smallest θ of any member, near the end of the family); this is

Chobotov's Fig. 6.8, redrawn in Figure 3 (b). Points outside the outer curve are too far from the final orbit for the motors' ΔV to be enough; points inside the inner curve are too close, so that even the trajectory modification cannot absorb the excess energy. Inside the shaded region a parking orbit may have more than one transfer that uses the motors exactly.

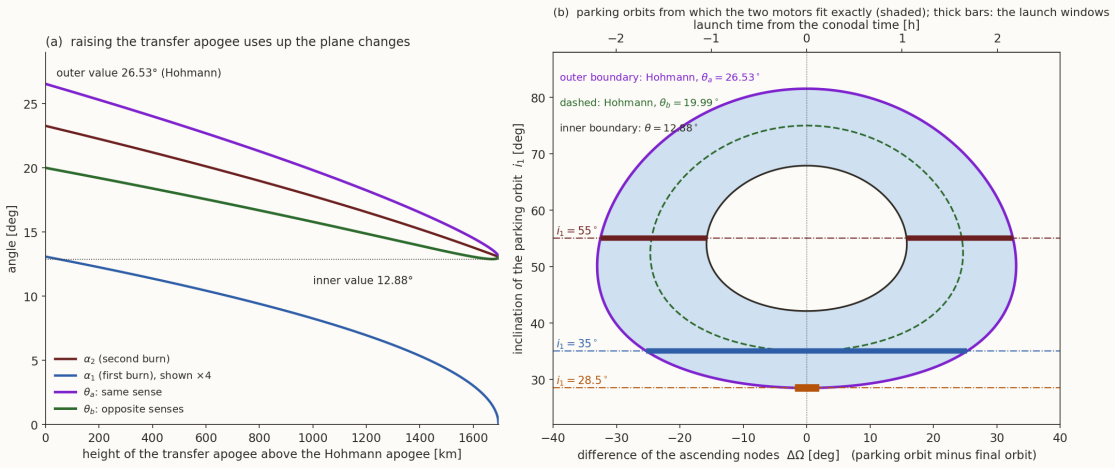


Figure 3: (**a**) The family of transfers of the example when the apogee is raised above the Hohmann apogee: the two plane changes (α_1 is drawn four times larger) and the total plane changes θ_a (purple) and θ_b (green) from Equation 8. The family ends where $\alpha_1 = 0$, after 1694 km. (**b**) The region of parking orbits (shaded) from which the final orbit ($i_2 = 55°$, node 0) can be reached with exactly $\Delta V_1 = 2107$ m/s and $\Delta V_2 = 1888$ m/s: the angle θ between the planes, Equation 6, must lie between 12.88° and 26.53°. The thick bars are the launch windows of the next section; the top axis converts the node difference to a launch time. *Script: fig-ci-region.py.*

2.7 Launch window

Chobotov observes that supplying excess energy *buys* something: a **launch window**, a stretch of time during which a ground launch to the Shuttle orbit is possible. The idea is simple once the region of Figure 3 (b) is known. A launch from the ground into an orbit of inclination i_1 puts the orbit's ascending node at a right ascension that depends on **when** the launch is made. The orbital plane of a launch is fixed *in inertial space* once the vehicle is in orbit, but the launch site is carried eastward by the Earth's rotation at 15.041° per hour, so a launch t hours *later* than the one that gives the target plane puts the node 15.041 t degrees *farther east*:

$$\Omega = \Omega_0 + 15\,t \quad \text{(degrees, } t \text{ in hours)}, \tag{9}$$

where Ω_0 is the right ascension of the node that the target plane would have for a launch at $t = 0$. The book writes 15; 360°/23.9345 h = 15.041°/h is the exact rotation against the stars. The launch time $t = 0$ at which the

node of the Shuttle orbit equals the node of the final orbit ($\Delta\Omega = 0$) is called the **conodal** launch time. So the node difference of the Shuttle orbit is $\Delta\Omega = 15.041\,t$, and the launch window for a Shuttle orbit of inclination i_1 is read off Figure 3 (b) along the horizontal line $i_1 = \text{const}$: the stretch of $\Delta\Omega$ inside the shaded region, divided by 15.041°/h.

- $i_1 = 55°$ (same inclination as the final orbit): the line cuts the shaded region in *two* stretches, $15.75° \le |\Delta\Omega| \le 32.53°$: **two windows,** 1.05 **to** 2.16 **h before and after the conodal time**, each about an hour long. Around $t = 0$ the Shuttle orbit is almost the final orbit (θ near 0), the motors have too much ΔV for any trajectory modification, and the window is closed. (The book reads 17° to 32°, 1.13 to 2.13 h, from his plot; the difference is the reading of the figure.)
- $i_1 = 35°$: the line goes through the region without a gap (the planes are 20° apart at $\Delta\Omega = 0$, inside the allowed band), $|\Delta\Omega| \le 25.27°$: **one window of** $\pm$1.68 **h around the conodal time** (the book: $\pm$1.73 h).
- $i_1 = 28.5°$ (the launch latitude: the lowest inclination one can reach): the line touches the region only at its lowest point, $|\Delta\Omega| \le 1.9°$, $\pm$7.7 minutes: the book calls it "an instant", the moment when the final orbit plane passes over the launch site.

Whenever the Shuttle launch falls inside the window, a transfer exists that consumes exactly the motors' ΔV. The *width* of the window is a direct measure of how much room the excess energy gives.

Why is there no launch window *close* to the conodal time for the 55° Shuttle orbit, even though that is the time when the two planes coincide and one might expect the best transfer?

Answer

When the two planes coincide, the best transfer is the coplanar Hohmann transfer, which costs only 3469 m/s. The motors carry 3995 m/s. The surplus of 526 m/s cannot be spent: a plane change is the way to spend it, and when the planes already coincide there is nothing to change; raising the apogee does not help either (Table 3: the smallest total plane change of the family is 12.88°). The motors would put the satellite in the wrong orbit. The excess energy can be absorbed only if the parking orbit is *tilted* away from the final orbit by at least 12.88°: that is why the window has a hole at the centre.

2.8 The book's problems 6.1 and 6.3, worked

Problem 6.1 (a heavier satellite). The payload mass of the example is raised from 907 kg to 990 kg on the same upper stage. (a) The ΔVs are now $\Delta V_2 = 9.80665 \times 300 \ln \frac{990+91+898}{990+91} = 1779.0$ m/s and $\Delta V_1 = 9.80665 \times 300 \ln \frac{990+91+898+181+2174}{990+91+898+181} = 2048.8$ m/s. (b) The plain Hohmann transfer

needs $V_{PTR} - V_{c1} = 2047.8$ m/s at the first burn: the motor delivers 2048.8 m/s, only 0.9 m/s more, so $\cos\alpha_1 = (V_{c1}^2 + V_{PTR}^2 - \Delta V_1^2)/(2V_{c1}V_{PTR})$ gives $\alpha_1 = 0.41°$ (the book: ≈ 0). And $\alpha_2 = 19.99°$ (book: 19.97°): the second motor takes all the plane change. (c) The loci are two curves, for $\theta_a = 20.39°$ and $\theta_b = 19.58°$, almost the single circle $\theta = 19.99°$ (Equation 6, as before). (d) Non-Hohmann transfers: the first burn has 0.9 m/s of slack and, as we found, a higher apogee needs more ΔV_1: the apogee can rise by only 25 km. So the loci are essentially unchanged. The allowed locus does *not* grow. (e) For a Shuttle orbit at $i_1 = 55°$: $\theta = 20.39°$ or 19.58° gives $\Delta\Omega = 24.96°$ or 23.96°, i.e. launch at 1.66 h or 1.59 h before or after the conodal time. The two loci are 1.0° apart, so the window is **two pairs of windows about four minutes long**: nearly instants. Less excess energy, narrower windows.

Problem 6.3 (two identical motors, a 296 km orbit to a 741 km orbit, highest possible inclination). The payload is 1452 kg; each motor has 1044 kg of propellant, 136 kg of structure and $I_{sp} = 285$ s. The highest Shuttle-orbit inclination allowed by range safety is 57°; the experimenters want an orbit as steeply inclined as possible, preferably above 72°.

(a) Second burn: $\Delta V_2 = 9.80665 \times 285 \ln\frac{1452+136+1044}{1452+136} = 1412.2$ m/s. First burn: $\Delta V_1 = 9.80665 \times 285 \ln\frac{1452+136+1044+136+1044}{1452+136+1044+136} = 894.4$ m/s. (The book: 894, 1412.)

(b) The Hohmann transfer between $r_1 = 6674$ km and $r_2 = 7119$ km (the book's Earth radius 6378 km) has $V_{c1} = 7728.2$, $V_{PTR} = 7851.8$, $V_{ATR} = 7361.0$ and $V_{c2} = 7482.7$ m/s, so a tangential burn would need only 123.7 m/s at the first burn and 121.7 m/s at the second. The motors have 894.4 and 1412.2 m/s: almost all of it goes into plane change. Equation 4 gives $\alpha_1 = 6.52°$ and $\alpha_2 = 10.88°$.

(c) The destination orbit can be inclined from the Shuttle orbit by $\alpha_1 + \alpha_2 = 17.40°$ at most (both changes in the same sense), so $i_{2,\max} = 57° + 17.40° = 74.40° > 72°$: the goal is met.

(d) The transfer orbit is tilted by α_1 from the Shuttle orbit: $i_{tr} = 57° + 6.52° = 63.52°$. Both plane changes are pure inclination changes (the node does not change), so they are made at the **nodes**: the burns are at the equator, at latitude 0°, the first at one node and the second at the opposite one (180° later).

3 Finite-duration burns: gravity losses

3.1 The equation of motion of a rocket in a gravity field

Every ΔV so far was applied in no time at all. A real engine has a thrust T that is finite, and the **thrust-to-weight ratio** T/W (here and below W_0 is

the weight at ignition) is in practice between 10^{-5} and 10. A burn of a few minutes covers a good arc of the orbit, and during it gravity keeps changing the velocity. To see by how much, we write the equation of motion of the rocket along its flight path, with the thrust along the velocity.

Take a rocket of mass m and speed V (we write the speed as V in this section; the book writes v). The exhaust leaves with the speed u relative to the rocket. The momentum balance for a rocket under an external force F along its direction of motion follows from comparing the momentum at t and at $t + \mathrm{d}t$. At t the rocket has mass m and speed V: momentum mV. At $t + \mathrm{d}t$ the rocket has mass $m + \mathrm{d}m$ (with $\mathrm{d}m < 0$) and speed $V + \mathrm{d}V$, and the gas $-\mathrm{d}m > 0$ that it ejected has the speed $V - u$ (the exhaust speed u is relative to the rocket, backwards). The change of the total momentum is the impulse of the external force:

$$(m + \mathrm{d}m)(V + \mathrm{d}V) + (-\mathrm{d}m)(V - u) - mV = F\,\mathrm{d}t.$$

Expanding, $m\,\mathrm{d}V + V\,\mathrm{d}m + \mathrm{d}m\,\mathrm{d}V - V\,\mathrm{d}m + u\,\mathrm{d}m = F\,\mathrm{d}t$; the terms $V\,\mathrm{d}m$ cancel, and the product $\mathrm{d}m\,\mathrm{d}V$ is of second order and is dropped, so $m\,\mathrm{d}V + u\,\mathrm{d}m = F\,\mathrm{d}t$, that is

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t} + F.$$

The first term is the thrust $T = -u\,\dot{m} = g_0 I_{sp}\,|\dot{m}|$ (positive because $\dot{m} < 0$). The external forces are, along the velocity, the drag D (opposing the motion) and the component of the weight mg along the flight path. If γ is the angle of the flight path above the local horizontal, the weight points toward the centre of the Earth, which is at the angle $90^\circ + \gamma$ from the velocity, so its component along the velocity is $-mg\sin\gamma$. Hence $F = -D - mg\sin\gamma$:

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t} - D - mg\sin\gamma.$$

Multiply by $\mathrm{d}t/m$:

$$\mathrm{d}V = -u\frac{\mathrm{d}m}{m} - \frac{D}{m}\,\mathrm{d}t - g\sin\gamma\,\mathrm{d}t.$$

Integrate over the burn, from the initial mass m_i to the final mass m_f (the first term is $u\ln(m_i/m_f)$, as in Equation 1):

$$\boxed{\Delta V = g_0 I_{sp}\ln\frac{m_i}{m_f} - \int\frac{D}{m}\,\mathrm{d}t - \int g\sin\gamma\,\mathrm{d}t} \tag{10}$$

(Chobotov's (6.5).) The left side is the **change of speed that the burn actually produces**. The first term on the right is the ideal rocket-equation ΔV. The other two are **losses**: the drag loss (zero above the atmosphere, which we assume from now on) and the **gravity loss** $\int g \sin\gamma \, \mathrm{d}t$. The sum $\int g \sin\gamma \, \mathrm{d}t$ is positive when the vehicle is climbing ($\gamma > 0$): gravity is then slowing it down, and the engine has to make up for that. When the vehicle descends ($\gamma < 0$) the integrand is negative, and gravity is helping.

Four things in (Equation 10). (i) The thrust is along the velocity (a "tangential" burn): otherwise there would be a steering term. (ii) ΔV is a *scalar speed change along the path*, not a vector change: a burn also turns the velocity, and the turning is not counted. (iii) g is the local gravitational acceleration, which changes as the vehicle climbs. (iv) For a burn from the ground γ is the angle between the flight path and the horizontal; for a burn in orbit it is the same angle, the flight-path angle of the orbit's velocity.

3.2 The simplest case: a vertical ascent

If the burn is vertical, $\gamma = 90°$, and with constant g (a short burn) the gravity loss is exactly $g\,t_b$ with t_b the burn time. The burn time follows from the mass flow: constant thrust T gives $\dot{m} = T/u$, so the mass falls from m_0 to $m_f = m_0/R$ in $t_b = (m_0 - m_f)u/T = (1 - 1/R)\,u/(T/m_0)$. With $T/m_0 = (T/W_0)\,g$,

$$\text{loss} = g\,t_b = \frac{u\,(1 - 1/R)}{T/W_0}\,, \qquad \Delta V = u\Big[\ln R - \frac{1 - 1/R}{T/W_0}\Big], \qquad R = \frac{m_0}{m_f}. \tag{11}$$

So the loss is **inversely proportional to the thrust-to-weight ratio**: double the thrust, halve the loss (Figure 4 (a)). With $I_{sp} = 300$ s and $R = 3$ ($m_f = m_0/3$): the ideal ΔV is $u \ln R = 3232$ m/s. At $T/W_0 = 1.5$ the loss is 1308 m/s and the vehicle gains 1925 m/s; at $T/W_0 = 3$ the loss is 654 m/s. Integrating the equation of motion numerically reproduces this to the last digit (the script). This is why launch vehicles have T/W_0 of order 1.2 to 1.5 at lift-off: it must exceed 1 for the vehicle to leave the pad at all, a value close to 1 wastes ΔV on gravity, and a much higher value needs a heavier engine and structure.

3.3 Gravity loss in orbit, and what drives it

For an orbital burn the same equation holds, with γ now the flight-path angle of the orbit. Chobotov lists three factors that raise the loss: **a long burn** (low thrust-to-weight ratio), **high** g (a burn at low altitude), and **a large** γ (a burn over a large central angle). The first two are visible in (Equation 10):

the integral grows with the duration $\mathrm{d}t$ and the factor g. The third needs an explanation.

A Hohmann burn is impulsive and at perigee, where $\gamma = 0$. A finite burn surrounds the perigee: it starts *before* it, with the vehicle descending ($\gamma < 0$), and ends *after* it, with the vehicle climbing ($\gamma > 0$), and if the thrust is along the velocity the angle γ sweeps a range of about the size of the central angle covered. The book writes (its Fig. 6.11) that if the burn keeps the impulsive direction, the values of γ at the start and the end are about minus and plus the central angles ϕ_1, ϕ_2 swept before and after the impulsive point: near the middle of the burn $\gamma \approx 0$ and the contribution is nothing, but at the ends the $\sin\gamma$ term is significant. This is a heuristic. To make it concrete, here is an exact numerical example, a burn of the type the next section uses. A vehicle on a 278 km orbit has already raised its apogee halfway to the geosynchronous radius (equal shares of the perigee-speed gain); its second perigee burn has $T/W_0 = 0.1$ and $I_{sp} = 300$ s and is started half its duration before perigee, so as to be centred on it; the thrust is along the velocity, and it ends when the apogee reaches 42 164 km. The numerical integration gives

- burn duration 11.2 min; γ runs from $-6.17°$ at the start to $+11.13°$ at the end (it crosses zero at the lowest point of the path, about 0.4 of the way through);
- ideal $\Delta V = u \ln(m_i/m_f) = 1231.9$ m/s; gravity integral $\int g \sin\gamma \, \mathrm{d}t = 152.6$ m/s; the speed actually gained is 1079.3 m/s, and $1231.9 - 152.6 = 1079.3$: **Equation 10 holds** (to 10^{-9} m/s in the script).

So for a burn of only 11 minutes at $T/W_0 = 0.1$, 12% of the ideal ΔV is spent climbing. And the integral is not zero even though γ changes sign: the vehicle is *higher* at the end than at the start (6934.8 km to 7130.2 km), and the net effect of descending in the first part and climbing in the second is a net climb.

What is the effect on what we are after, the orbit's energy? The energy of the orbit changes only because of the thrust, $\mathrm{d}\varepsilon = \mathbf{V}\cdot\mathbf{a}_T \, \mathrm{d}t = V \, \mathrm{d}(\Delta V_{\text{ideal}})$: **every metre per second delivered at speed V raises the energy by V**, and the faster, the better (the Oberth effect). An impulsive burn at perigee delivers all its ΔV at the highest speed of the orbit. A finite burn delivers part of it after the vehicle has climbed and slowed down, at a lower speed, so each metre per second buys less energy. In the example the energy gained is 11.541 $\mathrm{km}^2/\mathrm{s}^2$ (the integral of $V \, \mathrm{d}\Delta V$ equals the energy gain: checked in the script), 1.0% *less* than the 11.664 $\mathrm{km}^2/\mathrm{s}^2$ that the same ideal ΔV would give at the perigee speed. The shortfall of 0.123 $\mathrm{km}^2/\mathrm{s}^2$ divided by the perigee speed (8.85 km/s) is about 14 m/s: the engine has to supply that much additionally. (The two ways of counting the loss, 153 m/s of speed lost to climbing and 14 m/s of energy shortfall, measure different things: the first is the bookkeeping of the *speed* in Equation 10; the second is what the orbit

energy gains. The same burn has both.)

3.4 The cure: several perigee burns

The obvious cure is a stronger engine. There is another one: **break one long burn into several shorter ones, made at successive perigee passages**. The vehicle is left in an orbit that is a little larger than before; after one revolution it is back at (nearly) the same perigee, and burns again. Each burn covers a smaller central angle, so the range of γ is smaller, the climb is smaller, and the loss is smaller: the second term in (Equation 10) is reduced. Chobotov's Fig. 6.12 and 6.13 show the effect for a transfer from LEO (278 km, 28.5°) to GEO; the ΔV approaches the impulsive value (4237 m/s) for high thrust, and the saving from splitting is "substantial", paid for by transfer time.

To see the numbers ourselves, here is a clean experiment of our own (not Chobotov's data, which are only in his plots). The model: a 278 km circular orbit; the goal is an orbit with apogee at the geosynchronous radius ($r = 42\,164$ km); the thrust is along the velocity and constant, $I_{sp} = 300$ s; the burn is divided into N burns of equal perigee-speed increments, each burn starts half its duration before the perigee so that it is centred on it, and each burn ends when the osculating apogee reaches its target. The engine ΔV is the sum of $u \ln(m_i/m_f)$ over the burns. To measure the **loss**, we compare with the cheapest *impulsive* transfer to the *same final orbit* (a long burn also raises the perigee, so the final orbit is not exactly the Hohmann ellipse): a Hohmann-type apogee raise at 278 km followed by a perigee raise at the apogee.

Table 4: Loss against the best impulsive transfer to the same orbit, and the elapsed time, for N perigee burns. For a single burn the loss *rises* as the thrust falls: 1.2 m/s at $T/W_0 = 1$, 13.6 m/s at 0.3, 107 m/s at 0.1 and 727 m/s at 0.003, while the final perigee rises from 6662 km to 21 624 km.

T/W_0	$N = 1$	$N = 2$	$N = 4$	$N = 8$
0.1	107.0 m/s, 0.5 h	30.9 m/s, 3.1 h	7.9 m/s, 9.5 h	2.0 m/s, 23.5 h
0.03	499.8 m/s, 1.9 h	278.9 m/s, 4.0 h	84.2 m/s, 10.0 h	21.8 m/s, 23.8 h
0.01	640.3 m/s, 6.0 h	547.9 m/s, 7.9 h	387.6 m/s, 13.1 h	200.2 m/s, 25.7 h

The picture (Figure 4 (b), (c)) is the one of the book: **more burns, less loss, more time.** Compare the two ways of reducing the loss at $T/W_0 = 0.03$: either multiply the thrust by 3.3 ($T/W_0 = 0.1$, one burn: 107 m/s) or split

into eight burns (21.8 m/s, but 24 h instead of 1.9 h). The tradeoff is the designer's to make.



Figure 4: (**a**) Vertical ascent, mass ratio 3, $I_{sp} = 300$ s: the speed actually gained against T/W_0; the shaded part is the gravity loss, $g\,t_b$, Equation 11. (**b**) Orbit raising from a 278 km circle to the geosynchronous radius by N perigee burns: loss against the best impulsive transfer to the same final orbit. (**c**) The elapsed time. The numbers are those of Table 4. *Script: fig-ci-gravity.py.*

Three regimes. Chobotov divides the problem by thrust level.

- **High thrust**, $T/W_0 \gtrsim 0.5$ to 1: the burn is short, the loss is a few m/s or tens of m/s (our 1.2 m/s at $T/W_0 = 1$), and the impulsive model is adequate, or one can correct it with Robbins's analysis of the impulsive approximation (cited in the book).
- **Low thrust**, $T/W_0 \approx 0.01$ to 0.1: thrust and gravity are comparable, estimates are unreliable, and "the best method of solution is to run an integrated trajectory computer program". That is exactly what the experiment above is.
- **Very low thrust**, $T/W_0 \approx 10^{-5}$: the thrust is a *perturbation* of the orbit, and the transfer is many revolutions of a slowly growing spiral. The next section treats it analytically.

4 Very low thrust transfers

4.1 Setting up

Electric propulsion accelerates at 10^{-5} of the weight or so. Its specific impulse is high (thousands of seconds), so it uses very little propellant, but it takes weeks to do what a chemical stage does in minutes. The orbit is almost circular at every instant: each revolution is a slightly larger circle (and, if a plane change is wanted, a slightly different plane). Edelbaum (Edelbaum 1961) found a first-order analytic solution for the transfer between **two circular orbits**, with a change of plane of α (the angle between the planes, in radians), which Chobotov reports:

$$\boxed{\Delta V = \sqrt{V_0^2 - 2V_0V_1\cos\left(\frac{\pi}{2}\alpha\right) + V_1^2}} \tag{12}$$

where V_0 and V_1 are the circular speeds of the initial and the final orbit. The structure is the law of cosines again, but with the angle $\frac{\pi}{2}\alpha$ instead of α. We now derive it, since the book only quotes it.

4.2 Derivation by a pseudo-velocity vector

Let the thrust acceleration have the constant magnitude $a = T/m$ (to first order; mass changes are accounted for afterwards). Let it make the angle β with the velocity, in the plane that contains the velocity and the orbit normal, so that the *tangential* component is $a\cos\beta$ and the *normal* one $a\sin\beta$. Averaged over one revolution (this is where "very low thrust" is used: the orbit changes little in a revolution), the circular orbit evolves as follows.

- **Speed.** For a circular orbit $V = \sqrt{\mu/r}$; a tangential acceleration a_t changes the energy at the rate $\dot\varepsilon = Va_t$, and $\varepsilon = -\mu/(2r) = -V^2/2$, so $\dot\varepsilon = -V\dot V$, which gives $\dot V = -a_t$. The speed *decreases* when the vehicle is accelerated tangentially: the orbit grows and its circular speed falls (the paradox of low-thrust spirals). So $\dot V = -a\cos\beta$.
- **Inclination.** A normal acceleration a_n changes the inclination at the rate $\dot i = (a_n/V)\cos u$ with u the argument of latitude (a result of the Gauss equations). If the normal thrust changes sign at the nodes, so that it is always on the side that raises i, the average of $|\cos u|$ over a revolution is $2/\pi$, so $\dot i = \dfrac{2}{\pi}\dfrac{a\sin\beta}{V}$.

Define the **pseudo-velocity vector** in a plane:

$$\mathbf{u} = V\,(\cos\varphi,\ \sin\varphi), \qquad \varphi = \frac{\pi}{2}\,(i - i_0) \quad\Longrightarrow\quad \dot\varphi = \frac{\pi}{2}\dot i = \frac{a\sin\beta}{V}.$$

($\mathbf{u}$ is not a real velocity; it is a construct whose length is the circular speed and whose polar angle is $\pi/2$ times the inclination change.) Differentiate:

$$\dot{\mathbf{u}} = (\dot V\cos\varphi - V\dot\varphi\sin\varphi,\ \dot V\sin\varphi + V\dot\varphi\cos\varphi) = a(-\cos\beta\cos\varphi - \sin\beta\sin\varphi,\ -\cos\beta\sin\varphi + \sin\beta\cos\varphi),$$

and with the difference formulas for the cosine and sine,

$$\dot{\mathbf{u}} = a(-\cos(\beta - \varphi),\ \sin(\beta - \varphi)).$$

The pseudo-velocity vector moves with the constant speed $|\dot{\mathbf{u}}| = a$, whatever the steering angle β. (The script derives it symbolically.) Now the argument is the one of a ship that has to get from one point to another at fixed speed: it is fastest on a **straight line**. The initial and final pseudo-velocities are

$$\mathbf{u}_0 = (V_0, 0), \qquad \mathbf{u}_1 = V_1(\cos\tfrac{\pi}{2}\alpha,\ \sin\tfrac{\pi}{2}\alpha)$$

(Figure 5 (a)); the distance is $|\mathbf{u}_1 - \mathbf{u}_0|$, the time is that distance over a, and the total $\Delta V = \int a\,\mathrm{d}t = |\mathbf{u}_1 - \mathbf{u}_0|$. By the law of cosines in the triangle $0, \mathbf{u}_0, \mathbf{u}_1$,

$$|\mathbf{u}_1 - \mathbf{u}_0|^2 = V_0^2 + V_1^2 - 2V_0V_1\cos\frac{\pi\alpha}{2},$$

which is (Equation 12). The optimal steering keeps the direction of $\dot{\mathbf{u}}$ constant: $\beta - \varphi$ = constant. As a check, the script integrates the averaged equations $\dot{V} = -a\cos\beta$, $\dot{\varphi} = a\sin\beta/V$ with this steering, from LEO to GEO radius with a plane change of 28.5°, and finds the final speed and inclination change right to nine digits.

Two limiting cases show the formula is sensible.

- **No plane change** ($\alpha = 0$): $\Delta V = |V_0 - V_1|$. A low-thrust **spiral between coplanar circles costs the difference of the circular speeds**. For the example of the book, the 278 km orbit to the 20 187 km orbit: $7738.5 - 3873.6 = 3864.9$ m/s. The Hohmann transfer costs 3469.4 m/s: *the spiral costs* 395.6 *m/s more*. This is the gravity loss of a continuous-thrust transfer. (The book's Problem 6.2 (c) asks exactly this, and gives 395 m/s.)
- **No change of size** ($V_0 = V_1 = V$): $\Delta V = 2V\sin(\pi\alpha/4) \approx \frac{\pi}{2}V\alpha$ for small α. Compare an impulsive plane change, $2V\sin(\alpha/2) \approx V\alpha$: **a continuous-thrust plane change costs $\pi/2 = 1.571$ times as much** as an impulsive one (the $\pi/2$ is the factor that appears in $\varphi = \frac{\pi}{2}i$). For 10° at $V = 7.7385$ km/s: 2115 m/s against 1349 m/s.

4.3 Using the formula: propellant and time

Once ΔV is known, the rest is the rocket equation and the mass flow. The weight in the initial orbit is $W_0 = W_1\exp(\Delta V/(g_0 I_{sp}))$ for the delivered payload weight W_1 (Equation 1); the propellant is $\Delta W = W_0 - W_1$; and the transfer time is $\Delta t = \Delta W/\dot{W}$, where $\dot{W} = T/I_{sp}$ is the weight flow rate (the thrust in newtons divided by I_{sp} in seconds gives newtons of weight per second): $\dot{W} = \dot{m}\,g_0$ with $\dot{m} = T/(g_0 I_{sp})$.

Problem 6.2 (the ion engine). Instead of the solid motors: thrust 4.45 N, $I_{sp} = 3000$ s, a payload (satellite, engine and empty tanks) of weight 1361 kg (as mass $\times g_0$) to be delivered into the same 20 187 km, 55° orbit from the Shuttle orbit at 278 km.

(a) The least plane change: if the Shuttle's parking orbit is *also* at 55° and has the right node, **none**: $\alpha = 0$. (b) $\Delta V = V_0 - V_1 = 3864.9$ m/s. (c) Against the Hohmann 3469.4 m/s the loss is 395.6 m/s. (d) $W_0 = 1361 \exp(3864.9/(9.80665 \times 3000)) = 1361 \times 1.1404 = 1552.1$ kg, so the propellant is 191.1 kg. (e) The mass flow is $\dot{m} = 4.45/(9.80665 \times 3000) = 1.5126 \times 10^{-4}$ kg/s, so $\Delta t = 191.1/1.5126 \times 10^{-4} = 1.262 \times 10^6$ s = 14.6 days (the book: 14.7).

Why anyone uses it. Consider the same trip, LEO (278 km) to GEO (35 786 km), including the 28.5° plane change. With a chemical stage that does the best two-impulse transfer (the plane change split between the burns, $\alpha_1 = 2.19°$ at perigee) the cost is 4239 m/s (the book's 4237), against 4264 m/s if the whole plane change is made at apogee. With continuous low thrust, Equation 12 gives 5963 m/s: 1724 **m/s more**, and in the book's Fig. 6.12 the high-thrust limit is indeed the impulsive 4237 m/s. But what matters for the payload is the *propellant*, and that depends on I_{sp} through the rocket equation: the chemical stage ($I_{sp} = 300$ s) needs $W_0/W_1 = \exp(4239/2942) = 4.22$, so only 24% of the initial weight reaches GEO; the ion stage ($I_{sp} = 3000$ s) needs $\exp(5963/29\,420) = 1.22$, and 82% reaches GEO. Low thrust is bought with *time* (weeks to months), and it saves mass.

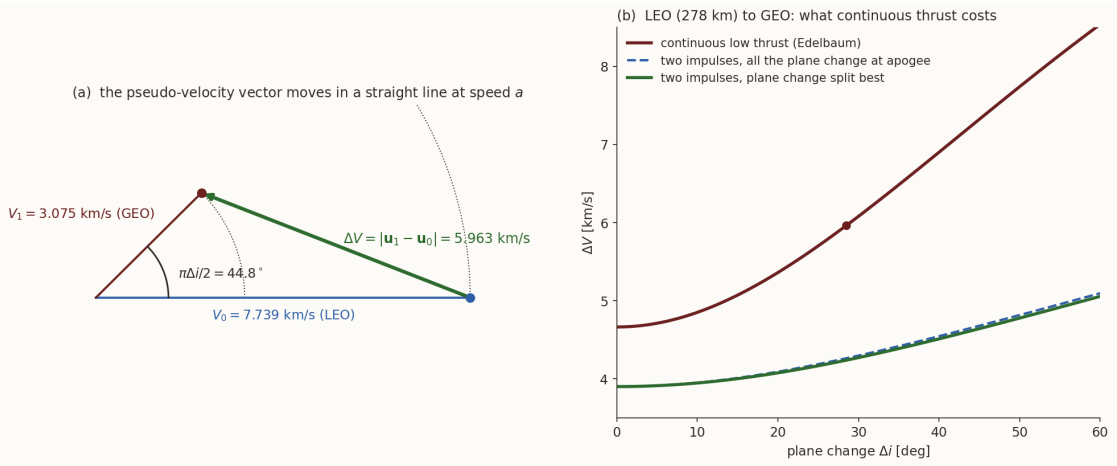


Figure 5: (**a**) The pseudo-velocity vector $\mathbf{u} = V(\cos\frac{\pi}{2}\Delta i, \sin\frac{\pi}{2}\Delta i)$ moves at the constant speed a (the thrust acceleration), so the least ΔV is the straight line from $\mathbf{u}_0$ (LEO) to $\mathbf{u}_1$ (GEO), drawn for a plane change of 28.5°. (**b**) ΔV against the plane change for LEO (278 km) to GEO: continuous low thrust (maroon), the best two-impulse transfer (green), and the two-impulse transfer with the whole plane change at apogee (dashed blue). *Script: fig-ci-edelbaum.py.*

Summary

In one paragraph. The best transfers need at most three impulses, three only for very large radius ratios ($R > 11.94$ at the earliest), so extra impulses are not the issue. **Fixed-impulse stages**: the motors' $\Delta V_{1,2} = g_0 I_{sp} \ln(W_i/W_f)$ are fixed; if their sum exceeds the optimal ΔV, the excess is spent on plane change, $\cos\alpha = (V_c^2 + V_t^2 - \Delta V^2)/(2V_c V_t)$ at each burn. For a Hohmann transfer the two turns are about one axis and the parking orbit is at $\theta = \alpha_1 \pm \alpha_2$ from the final orbit; raising the apogee uses up the plane changes (a flight-path-angle change appears, Equation 7) and composes them as $\cos\theta = \cos\alpha_1 \cos\alpha_2 \pm \sin\alpha_1 \sin\alpha_2 \cos f_2$. The parking orbits from which the motors fit form a ring, $12.88° \le \theta \le 26.53°$, in the $(\Delta\Omega, i_1)$ plane, and since $\Delta\Omega = 15.04\,t$ the ring is a **launch window**. **Finite burns**: $\Delta V = g_0 I_{sp} \ln(m_i/m_f) - \int D/m\,\mathrm{d}t - \int g \sin\gamma\,\mathrm{d}t$; the gravity loss is $g\,t_b \propto 1/(T/W_0)$ in a vertical ascent, and in orbit it grows with burn duration, with low altitude and with the central angle of the burn; splitting a burn into N perigee burns cuts the loss and lengthens the transfer. **Very low thrust**: the circular-to-circular transfer costs $\sqrt{V_0^2 - 2V_0V_1\cos(\pi\alpha/2) + V_1^2}$ (the pseudo-velocity moves in a straight line), 395.6 m/s more than Hohmann for the example, but with so high an I_{sp} that much less propellant is needed.

The numbers of this note are checked by complications-verify.py (symbolic algebra, explicit vectors, numerical integration of the equations of motion, the worked examples and the three problems of the book), using the helper module _complic.py. The figure scripts are fig-ci-geometry.py, fig-ci-region.py, fig-ci-gravity.py and fig-ci-edelbaum.py.

What we left out. (i) *Optimal control.* The argument behind Edelbaum's theorem on the number of impulses, and the optimal thrust steering of a finite burn, are beyond this note. (ii) *Non-circular orbits and J_2* in the low-thrust spiral: Edelbaum's formula assumes circular orbits and no perturbations, and a real spiral crosses the Van Allen belts, the Earth's shadow, and is perturbed by J_2 (the book's Chapter 14 treats low thrust in detail). (iii) *The drag term* of Equation 10, essential for launch vehicles. (iv) *The steering loss* of a burn along a fixed inertial direction, which our numerical burns avoid by steering along the velocity. (v) The *Shuttle-era setting*: the logic of the fixed-impulse problem is that of any upper stage whose motors cannot be re-ignited, whatever delivers it to the parking orbit.

Further reading. Chobotov, chapter 6 (Chobotov 2002), the book from which this note is made; T. N. Edelbaum (Edelbaum 1967) (how many impulses) and (Edelbaum 1961) (continuous low thrust); S. T. Chu, T. J. Lang and B. E. Winn (Chu, Lang, and Winn 1978) on the velocity-matching technique, the source of the region figures; the manoeuvres note for the Hohmann, bi-elliptic and plane-change transfers used here.

Exercises

All the answers are checked in complications-verify.py. The three problems of the book are worked in the text (6.1 and 6.3 in Section 2.8, 6.2 in Section 4.3).

1. **A bigger payload.** The payload of the example is raised to 1200 kg on the same two-motor stage. Find ΔV_1 and ΔV_2 and say whether the transfer to the 20 187 km orbit is possible. (Answer: $\Delta V_1 = 1915.0$ m/s and $\Delta V_2 = 1553.5$ m/s; their sum, 3468.5 m/s, is 0.9 m/s *below* the Hohmann total 3469.4 m/s, and ΔV_1 is below the 2047.8 m/s that even a tangential first burn needs. There is no solution: the stage cannot do the job.)
2. **The composition formula.** Show that for $f_2 = 180°$, Equation 8 reduces to $\cos(\alpha_1 \mp \alpha_2)$, and that for $\alpha_1 = 0$ it gives $\theta = \alpha_2$ whatever f_2 is. (The second is obvious: with no first plane change the parking plane is the transfer plane.)
3. **The excess.** How much excess ΔV does the stage of the example carry? Could the whole plane change $\theta_a = 26.53°$ be made at the second burn alone? (Answer: 525.8 m/s of excess. With the first burn tangential (2047.8 m/s) and the whole 26.53° at the apogee burn, Equation 3 gives $\Delta V_2 = 2005.2$ m/s and a total of 4053.0 m/s, 57.9 m/s *more* than the stage has (3995.2 m/s): so the first burn has to take part, which is why $\alpha_1 = 3.27°$.)
4. **Vertical ascent.** A rocket with $I_{sp} = 300$ s and $m_0/m_f = 3$ has $T/W_0 = 1.5$. What is the gravity loss, the burn time, and the final speed? (Answer: loss 1307.6 m/s, burn time 133.3 s, speed gained 1924.6 m/s.)
5. **Low-thrust plane change.** What does a continuous-thrust plane change of 10° at the 278 km orbit cost, compared with an impulsive one? (Answer: 2114.9 m/s against 1348.9 m/s; the ratio tends to $\pi/2$ for small angles.)
6. **Mass of a transfer.** Compare the fraction of the initial weight that reaches GEO for the best chemical transfer ($I_{sp} = 300$ s, 4239 m/s) with the continuous-thrust transfer ($I_{sp} = 3000$ s, 5963 m/s). (Answer: 24% and 82%.)
7. **Three impulses.** At what radius ratio does a bi-elliptic transfer with its apoapsis at infinity cost the same as a Hohmann transfer? Is the bi-elliptic cheaper for every r_b above that? (Answer: $R = 11.939$; no: for $11.94 < R < 15.58$ it needs r_b large enough, and for $R > 15.58$ any $r_b > R$ works; the second threshold is where the slope of the bi-elliptic cost with respect to r_b at $r_b = R$ (where it equals the Hohmann cost) changes sign, $R = 15.582$. Both are computed in the script.)

Chobotov, Vladimir A., ed. 2002. *Orbital Mechanics*. 3rd ed. AIAA Education Series. Reston, VA: American Institute of Aeronautics; Astronautics.

Chu, S. T., T. J. Lang, and B. E. Winn. 1978. "A Velocity Matching Technique

for Three Dimensional Orbit Transfer in Conceptual Mission Design." *Journal of the Astronautical Sciences* 26: 343–68.

Edelbaum, Theodore N. 1961. "Propulsion Requirements for Controllable Satellites." *ARS Journal* 31 (8): 1079–89.

———. 1967. "How Many Impulses?" *Aeronautics and Astronautics*.