Chase Manoeuvres

Devendra Ghate

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Table of contents

1 Where we are going **1**
1.1 Symbols and data 2

2 The chase problem **3**
2.1 Intercept and rendezvous 3
2.2 Two free parameters 4

3 Same orbit: phasing **5**
3.1 The phasing condition 5
3.2 Example 1: closing a 20° gap at 420 km 6
3.3 Cost against the time allowed 7
3.4 Phasing on an ellipse: from an angle to a time 8

4 Two circular orbits, inner chaser **9**
4.1 The Hohmann rendezvous and its lead angle 9
4.2 Example 2: from 300 km to 420 km 9
4.3 Waiting against paying 11

5 Two elliptic orbits **12**
5.1 The setting 12
5.2 Two apsis-to-apsis routes 12
5.3 Timing: no exact window 13
5.4 Example 3: the elliptic porkchop 14
5.4.1 The optimum arcs 14

6 The terminal phase **15**
6.1 Targeting with the cw state transition matrix 15
6.2 When the targeting fails 16
6.3 Example 4: 1 km below and 10 km behind 16

7 The outer chaser **18**
7.1 Time reversal 18
7.2 The asymmetry in numbers 18

8 Summary **19**

9 Exercises **20**

Further reading **23**

1 Where we are going

A **chaser** on one orbit must reach a **target** on another. A crew vehicle must reach the space station, a servicing spacecraft must reach a satellite, an impactor must

hit an asteroid. The Hohmann note taught us how to change an orbit. This note adds the missing ingredient, **timing**: it is not enough to arrive on the target's orbit, we must arrive there when the target does.

Two tools from earlier notes do most of the work.

- **Far away**, when the two spacecraft are thousands of kilometres apart, we need the conic that joins *where the chaser is now* to *where the target will be* in a given time. That is **Lambert's problem**, solved in the Lambert note by p-iteration.
- **Close in**, within a few tens of kilometres, the motion relative to the target obeys the linear **Clohessy–Wiltshire (cw) equations** of the cw note, and targeting becomes linear algebra.

History: learning to chase. Gemini 4's failed attempt to fly up to its own spent upper stage (June 1965) is told in the cw note; its lesson, that thrusting forward makes you fall behind, is the subject of Section 3. Six months later, on 15 December 1965, Walter Schirra and Thomas Stafford in Gemini 6A made the first space rendezvous, with Gemini 7, and held station as close as about 30 cm.

The plan is as follows.

1. **The chase problem** (Section 2). Intercept and rendezvous; why they share every transfer arc but not the best one; the two free parameters of every chase.
2. **Same orbit** (Section 3). The phasing manoeuvre, its cost against the time allowed, the atmosphere as a constraint, and the conversion of an angular gap to a time on an ellipse.
3. **Two circular orbits** (Section 4). The lead angle of the Hohmann rendezvous, the waiting time, and what it costs to go early.
4. **Two elliptic orbits** (Section 5). The windows are no longer periodic, and the porkchop plot is the design tool.
5. **The terminal phase** (Section 6). Targeting with the cw state transition matrix, its singular transfer times, and a check against Lambert.
6. **The outer chaser** (Section 7). Time reversal turns every inner-chaser result into an outer-chaser one, and shows what does *not* carry over.

As in the other notes, a blue **Try it** box asks you to do a small calculation before reading on, and a green **Check yourself** box asks a short question about what you have just read. Each has a hidden answer. When you finish, you should be able to answer the following without hand-waving.

1. Why do an intercept and a rendezvous with the same timing fly the same arc, and why do they still have different optima?
2. How do you catch a target a few degrees ahead on your own orbit, and why does waiting longer make it cheaper?
3. When must a chaser on a lower circular orbit fire, and how long must it wait for that moment?
4. Why is there no periodic launch window between two elliptic orbits?
5. How is a terminal approach targeted with the cw equations, and at which transfer times does that fail?
6. Which chase results carry over unchanged to a chaser *above* its target, and which do not?

Symbols and constants shared by all the notes are collected on the notation page; the study plan gives the order in which to read the notes.

1.1 Symbols and data

Every orbit is **coplanar** with the others and the motion is **prograde**. Subscript 1 always refers to the **chaser's** orbit and subscript 2 to the **target's**. The two burns of a transfer are labelled to match: the departure burn Δv_1 is made where the chaser leaves orbit 1, and the arrival burn Δv_2 where it joins orbit 2.

Table 1: Notation

Symbol	Meaning
μ	the Earth's gravitational parameter, 398 600.4418 km^3/s^2; $R_\oplus = 6378.137$ km
a_1, e_1, n_1, T_1	chaser's orbit: semi-major axis, eccentricity, mean motion $\sqrt{\mu/a_1^3}$, period
a_2, e_2, n_2, T_2	target's orbit, the same quantities
$\mathbf{r}_1(t), \mathbf{v}_1(t)$	chaser's state, before any burn
$\mathbf{r}_2(t), \mathbf{v}_2(t)$	target's state
t_0, Δt, $t_f = t_0 + \Delta t$	departure epoch, time of flight, meeting time
$\mathbf{v}_{\text{arc}}$	velocity on the transfer arc (the Lambert solution)
Δv_1, Δv_2	departure burn (on orbit 1) and arrival burn (on orbit 2)
θ	phase angle: how far the target leads the chaser (two circles)
T_{ph}, a_{ph}, Δv_{ph}	phasing orbit: period, semi-major axis, each of its two equal burns
a, e, n, T	in Section 3: the single orbit that chaser and target share
a_t, t_H	Hohmann transfer: semi-major axis and flight time
p_t, e_t	a Lambert transfer arc: semi-latus rectum and eccentricity
n, T	in Section 6: the target's mean motion and period

The recurring data are a target on a circular orbit at 420 km altitude ($r_2 = 6798.137$ km) and, in Section 4, a chaser on a circular orbit at 300 km ($r_1 = 6678.137$ km).

2 The chase problem

2.1 Intercept and rendezvous

There are two kinds of chase. At the meeting time t_f an **intercept** requires only that the two spacecraft be in the same place,

$$\mathbf{r}_1(t_f) = \mathbf{r}_2(t_f),$$

while a **rendezvous** requires them to be in the same place *with the same velocity*,

$$\mathbf{r}_1(t_f) = \mathbf{r}_2(t_f) \quad \text{and} \quad \mathbf{v}_1(t_f) = \mathbf{v}_2(t_f).$$

An intercept arrives with a **relative speed** $|\mathbf{v}_2 - \mathbf{v}_{\text{arc}}(t_f)|$. That is what a kinetic impactor wants, or a fly-by; for a crewed docking it would be a collision. A rendezvous cancels the relative speed with a second burn.

An intercept on purpose. NASA's DART spacecraft struck Dimorphos, the small moon of the asteroid Didymos, on 26 September 2022 at a relative speed of 6.14 km/s. The impact shortened Dimorphos's orbital period about Didymos by about 33 minutes.

Now fix the departure epoch t_0 and the time of flight Δt. The chaser leaves from $\mathbf{r}_1(t_0)$, and must be at $\mathbf{r}_2(t_0 + \Delta t)$ after Δt. Two positions and a time: that is exactly the data of Lambert's problem, and its solution is one conic arc (for zero

revolutions and the prograde sense). **Both kinds of chase must fly that arc.** The only difference is what happens at its end:

$$\Delta v_1 = |\mathbf{v}_{\text{arc}}(t_0) - \mathbf{v}_1(t_0)|, \qquad \Delta v_2 = |\mathbf{v}_2(t_f) - \mathbf{v}_{\text{arc}}(t_f)|,$$

and

$$\Delta v_{\text{intercept}} = \Delta v_1, \qquad \Delta v_{\text{rendezvous}} = \Delta v_1 + \underbrace{\Delta v_2}_{\text{intercept's arrival speed}}. \tag{1}$$

The braking burn of a rendezvous is exactly the arrival speed of the intercept on the same arc.

If they share every arc, why can the cheapest intercept and the cheapest rendezvous be on *different* arcs?

Answer

Because they minimise different functions of $(t_0, \Delta t)$: one minimises Δv_1, the other $\Delta v_1 + \Delta v_2$. An arc with a cheap departure and a fast, expensive arrival is the best intercept but may be a poor rendezvous. Section 5.4.1 shows a case where the two optima are four hours apart.

2.2 Two free parameters

Every chase is therefore a choice of two numbers:

- the **departure epoch** t_0: how long we wait before the first burn;
- the **time of flight** Δt: how long the transfer takes.

Given $(t_0, \Delta t)$, Kepler's equation places the chaser at $\mathbf{r}_1(t_0)$ and the target at $\mathbf{r}_2(t_0 + \Delta t)$, and one Lambert solution gives the arc and both burns. Plotting the cost over the $(t_0, \Delta t)$ plane gives a **porkchop plot**; every design question in this note is a walk on such a plot.

For **two circular orbits** the plot simplifies. A circle has no preferred point, so the geometry at departure depends only on the **phase angle** θ, the angle by which the target leads the chaser. With longitudes $\lambda_1 = \lambda_{1,0} + n_1 t$ and $\lambda_2 = \lambda_{2,0} + n_2 t$,

$$\theta(t) = \lambda_2 - \lambda_1 = \theta_0 - (n_1 - n_2)\, t.$$

For an inner chaser ($a_1 < a_2$, so $n_1 > n_2$) the phase angle **falls** steadily, and it returns to the same value modulo 2π after the **synodic period**

$$T_{\text{syn}} = \frac{2\pi}{|n_1 - n_2|}. \tag{2}$$

So on two circles the porkchop is periodic in t_0 with period T_{syn}, and it can be drawn over $(\theta, \Delta t)$ instead (Figure 6).

Short way and long way. Each point of the plot also fixes the **transfer angle** $\Delta\theta$: the angle, measured in the direction of motion, from the chaser's position at t_0 to the target's position at $t_0 + \Delta t$. On two circles the target moves on by $n_2 \Delta t$ during the flight, so $\Delta\theta = \theta + n_2 \Delta t \pmod{2\pi}$; on ellipses, $\Delta\theta$ is the difference of the two arguments of latitude, read from Kepler's equation. A prograde arc with $\Delta\theta < 180^\circ$ goes the **short way**, one with $\Delta\theta > 180^\circ$ the **long way**. At exactly 180° the two position vectors are collinear and the Lambert solution is singular (the Lambert note divides by $\sin\Delta\theta$): on every porkchop a line $\Delta\theta = 180^\circ$ separates the short-way arcs below from the long-way arcs above.

Reading a porkchop

- **Axes.** Across: when to leave (t_0, or on circles the phase θ at departure). Up: how long to fly (Δt). Every point is one Lambert arc.
- **Colour and contours.** The total cost of that arc: both burns for a rendezvous, the first burn only for an intercept. Contours join arcs of equal cost.
- **Valleys and the star.** Cheap transfers lie in valleys. The star marks the cheapest grid point; an optimiser then refines it, and the refined optimum is a little off the star.
- **The name.** In interplanetary plots the closed cost contours look like a pork chop.
- **Why intercept valleys are wider.** An intercept pays one burn, so a timing error costs it once; a rendezvous pays at both ends.
- **What is not drawn.** Only zero-revolution arcs ($N = 0$). The multi-revolution arcs of the Lambert note would add one more sheet for each N; phasing (Section 3) is where this note uses them.

3 Same orbit: phasing

3.1 The phasing condition

Chaser and target share a circular orbit of radius a and period T, and the target **leads by** θ. The phasing note derives the remedy, which is counter-intuitive: burn **backward**, onto a smaller ellipse whose period T_{ph} is shorter. The chaser flies k revolutions of it and comes back to the burn point (the apoapsis of the phasing ellipse) **early**, just as the target gets there. In the time kT_{ph} the target must cover k revolutions less the lead θ:

$$k\,T_{ph} = kT - \frac{\theta}{2\pi}\,T \qquad \Longrightarrow \qquad \boxed{T_{ph} = T\Big(1 - \frac{\theta}{2\pi k}\Big).} \tag{3}$$

If instead the target **trails** by θ, the chaser must arrive *late*: replace $-\theta$ by $+\theta$, and the phasing ellipse is larger, with the burn point at its periapsis.

Kepler's third law turns the period into a size, $a_{ph} = a\,(T_{ph}/T)^{2/3}$, and vis-viva at the burn point $r = a$ gives each of the two equal burns:

$$\Delta v_{ph} = \left|\sqrt{\frac{\mu}{a}} - \sqrt{\mu\Big(\frac{2}{a} - \frac{1}{a_{ph}}\Big)}\right|. \tag{4}$$

A **rendezvous** costs $2\Delta v_{ph}$: one burn onto the phasing ellipse, one back off it. An **intercept** costs only the first, Δv_{ph}.

The intercepting chaser meets the target at the burn point, without the second burn. Show that the relative speed at the meeting is exactly Δv_{ph}.

Answer

On the phasing ellipse the chaser returns to the same radius a, so by vis-viva it has the same speed $\sqrt{\mu(2/a - 1/a_{ph})}$ as just after the first burn. The burn point is an apsis of the phasing ellipse, so its velocity there is horizontal, as is the target's circular velocity $\sqrt{\mu/a}$. Two parallel velocities differ by the difference of their speeds, which is Equation 4. This is Equation 1 in its simplest form.

3.2 Example 1: closing a 20° gap at 420 km

At $a = 6798.137$ km the period is $T = 2\pi\sqrt{a^3/\mu} = 5578.22$ s and the circular speed is $v_c = 7.657269$ km/s. The target leads by 20° (the chaser drops and arrives early) or trails by 20° (the chaser rises and arrives late).

Work out the $k = 3$ row for the target 20° **ahead**: T_{ph}, a_{ph}, the altitude of the phasing ellipse's perigee, and the rendezvous cost.

What you should find

$T_{ph} = 5578.22\,(1 - 20/1080) = 5474.92$ s and $a_{ph} = 6798.137\,(53/54)^{2/3} = 6713.95$ km. The burn point is the apogee, so the perigee radius is $2a_{ph} - a = 6629.76$ km: altitude 251.6 km. On the ellipse at $r = a$ the speed is $\sqrt{\mu(2/a - 1/a_{ph})} = 7.609109$ km/s, so $\Delta v_{ph} = 48.16$ m/s and the rendezvous costs 96.32 m/s. The whole manoeuvre takes $3T_{ph} = 4.562$ h.

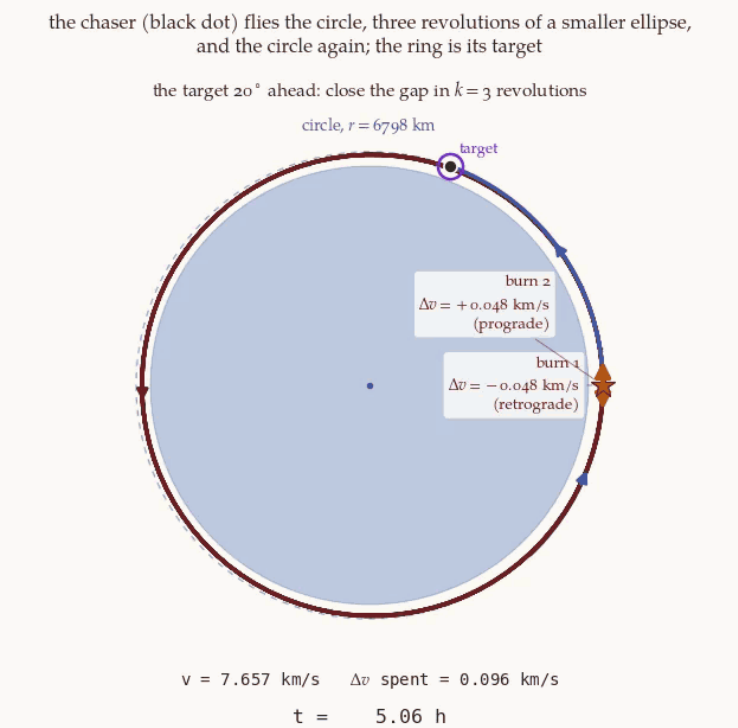


Figure 1: The task above, to scale: the target (ring) is 20° ahead. Burn 1 (48.16 m/s, retrograde) drops the chaser (black dot) onto the phasing ellipse, whose apogee is the burn point and whose perigee, at 6629.76 km, is 251.6 km up. The chaser flies three revolutions of it, $3T_{ph} = 4.562$ h, each a little shorter than the circle's 5578 s; the target, which keeps to the circle, is by then back at the burn point, and burn 2 (48.16 m/s, prograde) restores the circle. The cost is 96.32 m/s. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-chase.py, with the engine _manim.py; the burns are computed from the orbits, and the ring is checked to be at the meeting point when the chaser arrives.*

Table 2: Phasing at 420 km to close a 20° gap in k revolutions. The far apsis is the perigee when the target is ahead and the apogee when it is behind.

target	k	a_{ph} [km]	far apsis altitude [km]	$2\Delta v_{ph}$ [m/s]	time [h]
ahead	1	6543.96	**−88.3**	300.36	1.463

target	k	a_{ph} [km]	far apsis altitude [km]	$2\Delta v_{ph}$ [m/s]	time [h]
	2	6671.66	167.0	145.86	3.013
	3	6713.95	251.6	**96.32**	4.562
	6	6756.11	335.9	47.71	9.211
behind	1	7047.64	919.0	268.73	1.636
	3	6881.81	587.3	92.82	4.735

With one revolution the phasing ellipse **hits the Earth**: its perigee is 88 km below the surface. With two, its perigee at 167 km is too deep in the atmosphere for a spacecraft to fly a full revolution through it. Three revolutions is the practical minimum when the target is ahead.

3.3 Cost against the time allowed

Table 2 shows the pattern of Figure 2: **the cost falls essentially as** $1/k$. The reason is that the period change per revolution, $T - T_{ph} = \theta T/2\pi k$, is k times smaller, and for small changes Δv_{ph} is proportional to it. (Differentiate $T \propto a^{3/2}$ and vis-viva at fixed r: $\delta T/T = \frac{3}{2}\,\delta a/a$ and $\delta v = \mu\,\delta a/(2va^2)$, so $\delta v = \frac{1}{3}v_c\,\delta T/T$ at $r = a$.) In phasing, **time buys propellant**.

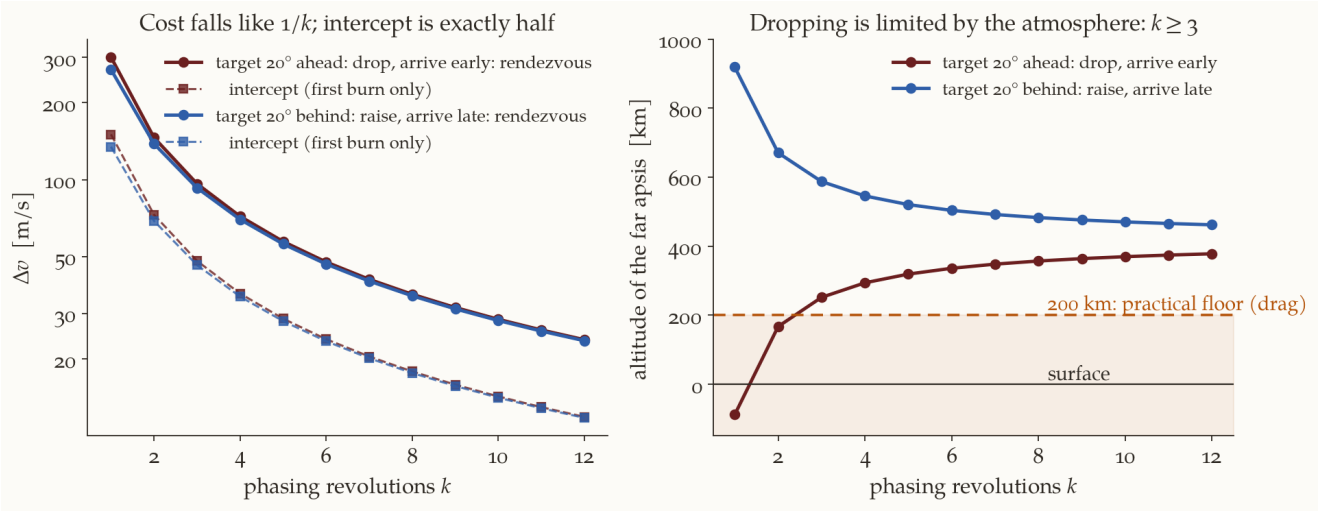


Figure 2: **Left:** the phasing cost against the number of revolutions k; it falls essentially as $1/k$, and the intercept costs exactly half the rendezvous. **Right:** the altitude of the phasing ellipse's far apsis. Dropping to catch a leading target is limited by the atmosphere; rising to wait for a trailing one is not. *Script: fig-chase-coorbital.py.*

The two cases are not symmetric. A trailing target is cheaper to wait for (92.82 against 96.32 m/s at $k = 3$) and has no atmosphere to worry about, but the chaser arrives later (4.735 against 4.562 h), because its phasing orbit is the longer one.

Using the $1/k$ rule, roughly what would the 20° correction cost with $k = 12$, and how long would it take?

Answer

About a quarter of the $k = 3$ cost, so about 24 m/s, and about four times as long, about 18 h. (Exactly: 23.74 m/s and 18.51 h.) (The right-hand panel of Figure 2 shows the perigee then above 370 km.)

3.4 Phasing on an ellipse: from an angle to a time

On an elliptic orbit the gap is naturally stated as an angle, a difference of **true anomaly**. But Equation 3 needs a **time**, and on an ellipse equal angles do not take equal times. Kepler's equation makes the conversion.

Given: $a = 10\,000$ km, $e = 0.3$, so $n = \sqrt{\mu/a^3} = 6.313481 \times 10^{-4}$ rad/s and $T = 9952.01$ s. The chaser is at periapsis, and the target is 30° **behind**, at $f = -30°$. The chaser must arrive late by the time the target needs to go from $f = -30°$ to $f = 0$. By the symmetry of the ellipse about its apse line, that is the time from $f = 0$ to $f = 30°$. With the half-angle relation between true and eccentric anomaly and Kepler's equation,

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{30°}{2} \;\Rightarrow\; E = 0.388289, \qquad M = E - e\sin E = 0.274707,$$

$$\Delta t = \frac{M}{n} = \frac{0.274707}{6.313481 \times 10^{-4}} = 435.11 \text{ s}.$$

So $T_{ph} = T + \Delta t = 10\,387.13$ s and $a_{ph} = a\,(T_{ph}/T)^{2/3} = 10\,289.39$ km. At the periapsis $r_p = 7000$ km the speed rises from $\sqrt{\mu(2/r_p - 1/a)}$ to $\sqrt{\mu(2/r_p - 1/a_{ph})}$, a burn of 64.904 m/s, and the rendezvous costs $2 \times 64.904 =$ **129.81 m/s**.

Repeat the calculation with the chaser at **apoapsis** and the target again 30° behind it, at $f = 150°$. Find Δt and the rendezvous cost.

What you should find

Δt is the time from $f = 150°$ to $f = 180°$, which is $T/2$ minus the time from 0 to 150°: $\Delta t = 1415.28$ s, 3.3 times longer than at periapsis. Then $T_{ph} = 11\,367.30$ s, $a_{ph} = 10\,926.91$ km, and at $r_a = 13\,000$ km the rendezvous costs 703.17 m/s.

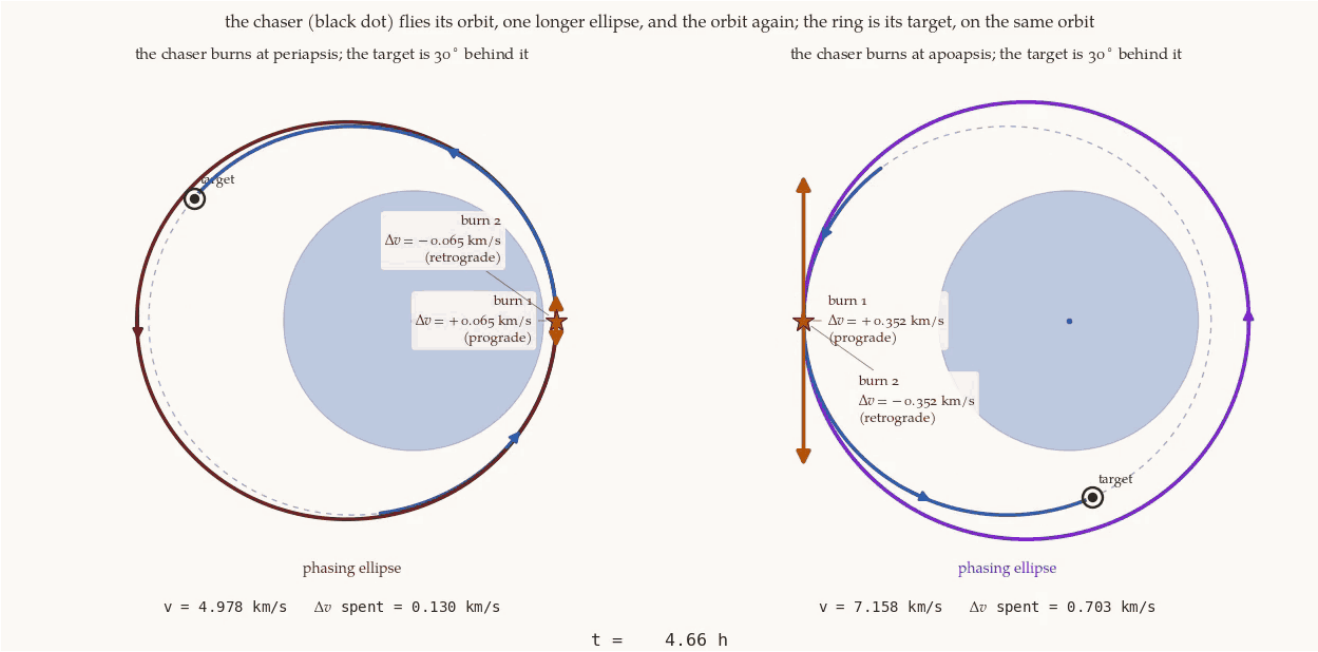


Figure 3: The example and the task above, on one clock: the chaser (black dot) burns at periapsis (left) or at apoapsis (right), with the target (ring) 30° behind it on the same orbit. The same 30° is a time of 435.11 s at periapsis and 1415.28 s at apoapsis. One longer orbit, $T_{ph} = 10\,387.13$ s or 11 367.30 s, makes the chaser arrive exactly as the target does, for $2 \times 64.904 = 129.81$ m/s at periapsis and 703.17 m/s at apoapsis. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-chase.py, with the engine _manim.py; the burns are computed from the orbits, and the ring is checked to be at the meeting point when the chaser arrives.*

The same 30° is a time gap 3.3 times larger at apoapsis, where the target moves slowly, and the burn there is also less effective, because $\delta v = \mu\,\delta a/(2va^2)$ is larger where the speed v is small (phasing note, "Phasing on an ellipse"). **A fixed Δf is not a fixed Δt**: the burn point changes the required timing as well as the price.

4 Two circular orbits, inner chaser

4.1 The Hohmann rendezvous and its lead angle

Now the chaser is on a circle of radius r_1 and the target on a larger circle $r_2 > r_1$. The cheapest transfer is the Hohmann half-ellipse, with

$$a_t = \tfrac{1}{2}\,(r_1 + r_2), \qquad t_H = \pi\sqrt{\frac{a_t^3}{\mu}}.$$

The chaser leaves at the periapsis of the transfer ellipse and arrives 180° further on, at its apoapsis. For a rendezvous the target must arrive there at the same moment. During t_H the target sweeps the angle $n_2 t_H$, so at the first burn it must already be ahead of the chaser by $\pi - n_2 t_H$. This is the **lead angle**:

$$\boxed{\theta^* = \pi - n_2\, t_H.} \tag{5}$$

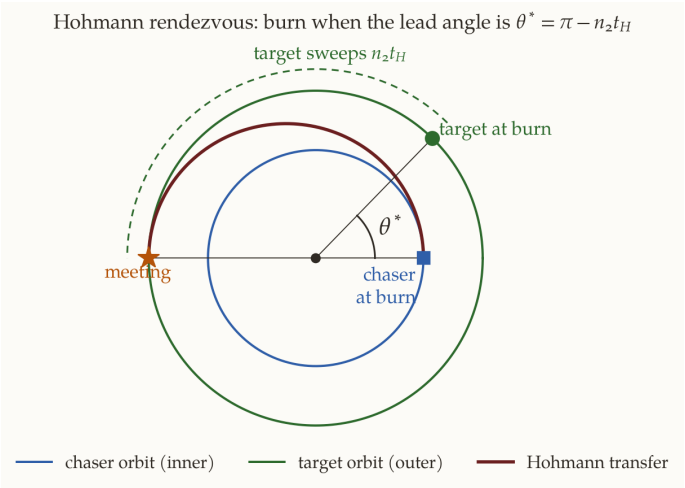


Figure 4: The chaser burns when the target leads by θ^*. During the transfer time t_H the target sweeps $n_2 t_H$ and arrives at the transfer's apoapsis together with the chaser. Radii not to scale. *Script: fig-chase-geom.py.*

4.2 Example 2: from 300 km to 420 km

The two mean motions and the transfer are

$$n_1 = 1.156874 \times 10^{-3}, \qquad n_2 = 1.126378 \times 10^{-3}\ \text{rad/s},$$

$$a_t = 6738.137\ \text{km}, \qquad t_H = 2752.27\ \text{s} = 45.87\ \text{min},$$

and the two Hohmann burns are

$$\Delta v_1 = 34.321, \qquad \Delta v_2 = 34.168, \qquad \Delta v_H = \mathbf{68.489}\ \text{m/s}.$$

Find the lead angle θ^* and the synodic period T_{syn} for these two orbits.

What you should find

$n_2 t_H = 1.126378 \times 10^{-3} \times 2752.27 = 3.10014$ rad $= 177.622°$, so $\theta^* = 180° - 177.622° = \mathbf{2.378}°$: the target must be only just ahead. The phase closes at $n_1 - n_2 = 3.0496 \times 10^{-5}$ rad/s $= 6.290°/\text{h}$, and Equation 2 gives $T_{\text{syn}} = 57.23$ h $= 2.38$ days.

The wait. The chaser cannot choose θ; it can only wait for it. If the target now leads by θ_0, the phase falls at 6.290° per hour, so the window comes after

$$t_w = \frac{\theta_0 - \theta^*}{n_1 - n_2} \quad \text{(plus a multiple of } T_{\text{syn}}\text{)}. \tag{6}$$

For $\theta_0 = 40°$:

$$t_w = \frac{40° - 2.378°}{6.290°/\text{h}} = \frac{37.622°}{6.290°/\text{h}} = \mathbf{5.98\ h}.$$

Suppose the target now leads by $\theta_0 = 0°$: the window, at 2.378°, has just been missed. How long is the wait now?

Answer

The phase must fall all the way round, from $0° \equiv 360°$ to 2.378°: $t_w = (360° - 2.378°)/(6.290°/\text{h}) = 56.85$ h, almost a full synodic period. Two orbits that are close together have a long synodic period, so a missed window is expensive in time.

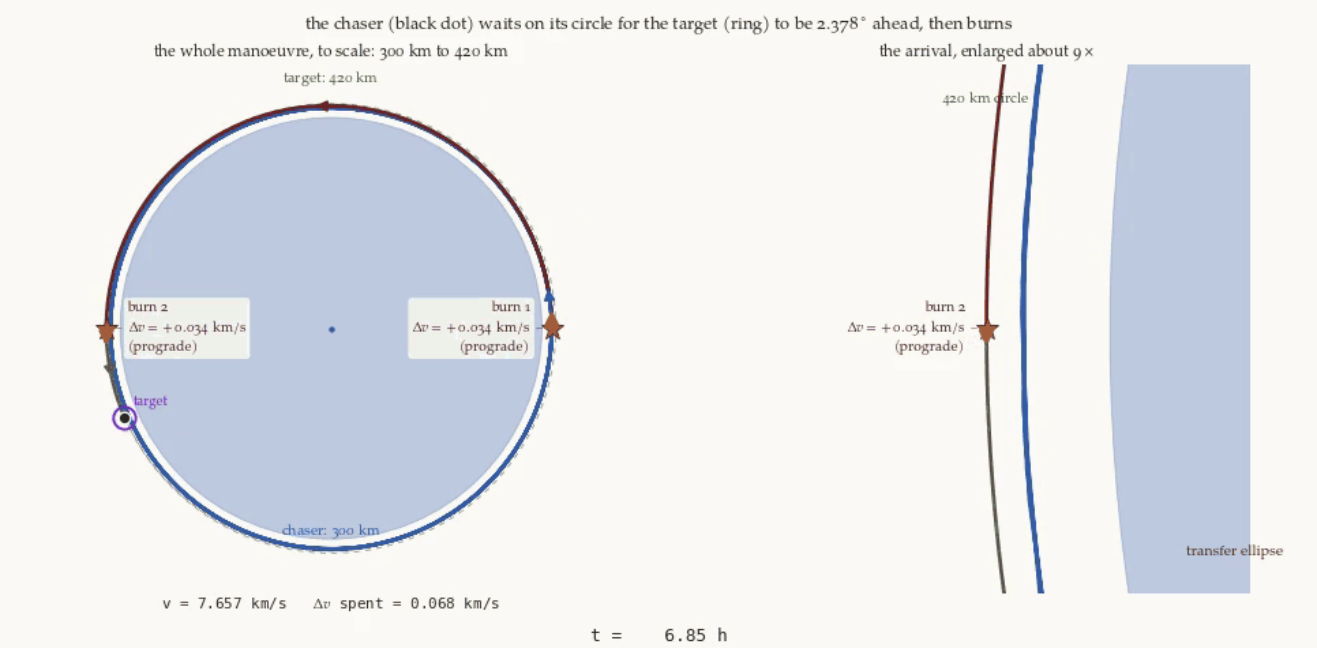


Figure 5: Example 2 from start to finish: **left,** to scale; **right,** enlarged about 9× around the arrival. The chaser (black dot) waits on the 300 km circle while the target (ring) draws ahead of it by less and less: the phase angle falls from $\theta_0 = 40°$ to the lead angle $\theta^* = 2.378°$ in 5.98 h. Burn 1 (34.321 m/s) starts the 2752.27 s transfer; chaser and target reach the transfer's apoapsis together, and burn 2 (34.168 m/s) circularises, for 68.489 m/s in all. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-chase.py, with the engine _manim.py; the burns are computed from the orbits, and the ring is checked to be at the meeting point when the chaser arrives.*

4.3 Waiting against paying

Can the chaser leave at once instead, on a faster or slower Lambert arc? Figure 6 answers this. The upper panels are the porkchop over $(\theta, \Delta t)$ for zero-revolution arcs, with any arc that dips below 100 km altitude excluded as infeasible. The lower panel is the cheapest cost at each departure phase, with a second axis giving the wait needed to reach that phase from $\theta_0 = 40°$.

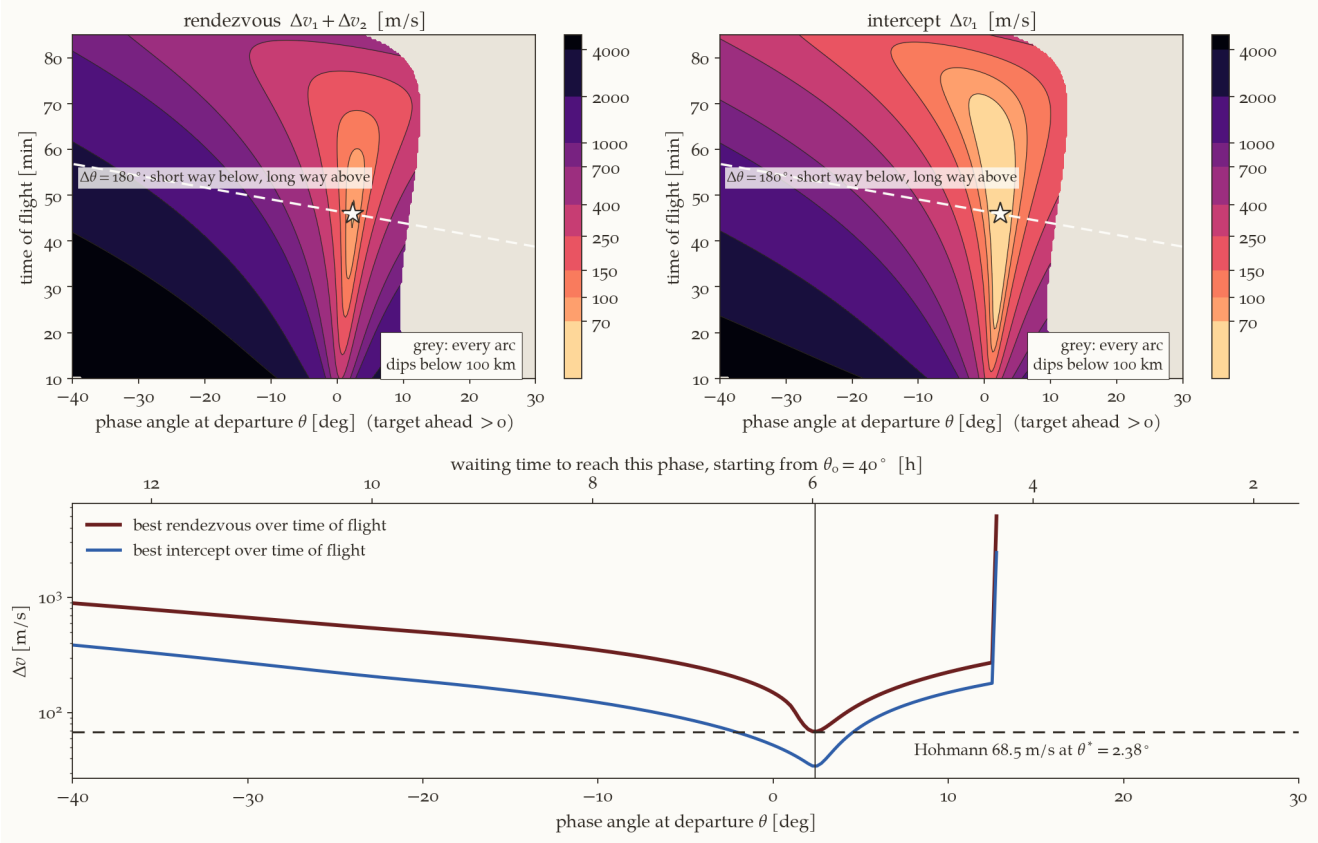


Figure 6: **Top:** porkchop over the phase angle at departure and the time of flight, for rendezvous (left) and intercept (right). Only a narrow wedge around the Hohmann point (star) is cheap. Grey means that the arc dips below 100 km. Dashed: the transfer angle $\Delta\theta = 180°$, with short-way arcs below it and long-way arcs above. **Bottom:** the cheapest cost at each departure phase, and (top axis) the waiting time needed to reach that phase from $\theta_0 = 40°$. Grid: 0.25° in phase, 0.5 min in time of flight from 10 to 85 min. *Script: fig-chase-porkchop-circ.py, grids in _chase_grids.py.*

Table 3: The cheapest zero-revolution transfer at each departure phase, minimised over the time of flight.

phase θ at departure	−20°	−10°	0°	**2.378°**	5°	10°
best rendezvous [m/s]	499.6	348.3	149.4	**68.5**	119.6	224.5
best intercept [m/s]	187.5	122.7	52.4	**34.3**	75.9	149.7

Five things can be read off.

- **Leaving now is impossible.** From $\theta_0 = 40°$, every zero-revolution arc passes below 100 km. The straight chord from the departure point to the target's arrival point already passes through the Earth, and slower arcs do not help. In fact no zero-revolution arc longer than half an hour is feasible for any θ above 12.66° (the arcs exist, but they dip into the atmosphere). Above that, only arcs of less than ten minutes survive, and they cost several km/s (5.26 km/s at 13°). The chaser *must* wait, or phase over several

revolutions: that is, use the multi-revolution ($N \geq 1$) arcs of the Lambert note.

- **Waiting is cheap in propellant and dear in time.** Each degree of phase costs $60/6.290 = 9.54$ minutes of waiting. Leaving one hour early, at $\theta \approx 8.7°$, costs 198.2 m/s: nearly three times the Hohmann bill.
- **The best intercept is the Hohmann first burn**, 34.32 m/s, arriving at 34.17 m/s. A tangential burn at r_1 is the cheapest way to reach r_2 at all.
- **The intercept valley is wider.** The rendezvous cost stays below twice its minimum for $0.41° < \theta < 5.77°$, a band of 5.4°; the intercept cost does so for $-2.18° < \theta < 4.60°$, a band of 6.8°. A timing error costs a rendezvous twice, once at each end.
- **In practice** the wait is spent in a lower **phasing orbit**, with small height adjustments to tune the arrival: phasing and Hohmann combined.

How fast in practice. A Soyuz or Progress launched to the space station takes the place of the chaser here, and its launch time sets the phase. Fast-track profiles of four orbits took about six hours from launch to docking. On 9–10 July 2018 the cargo ship Progress MS-09 docked 3 h 40 min after launch, after only two orbits. Making that possible required a reboost of the station some two weeks earlier, to put it in the right place at launch: the phase was set by moving the target.

5 Two elliptic orbits

5.1 The setting

Chaser: $a_1 = 6800$ km, $e_1 = 0.02$, so $r_{p,1} = 6664$ km, $r_{a,1} = 6936$ km and $T_1 = 1.5501$ h. **Target:** $a_2 = 10\,000$ km, $e_2 = 0.3$, so $r_{p,2} = 7000$ km, $r_{a,2} = 13\,000$ km and $T_2 = 2.7644$ h. The apse lines are **aligned** (both periapsides in the same direction) and the orbits do not cross ($r_{a,1} < r_{p,2}$). At $t = 0$ the chaser is at periapsis and the target at mean anomaly $M_2 = 90°$.

Where is the target at time t? Kepler's equation, used forward:

$$M_2(t) = M_2(0) + n_2 t \;\to\; E_2 \text{ (Newton's method on } M_2 = E_2 - e_2 \sin E_2) \;\to\; f_2 \;\to\; \mathbf{r}_2(t),\ \mathbf{v}_2(t),$$

and likewise for the chaser. Now nothing is rotationally symmetric. A circle looks the same from every point on it, but an ellipse does not, so the cost depends on the epoch t_0 itself and not on a single phase angle.

5.2 Two apsis-to-apsis routes

With aligned apsides there are two half-ellipses that leave one orbit at an apsis tangentially and join the other at the opposite apsis tangentially (the Hohmann note studies this pair):

- **route (i):** from the chaser's periapsis (6664 km) to the target's apoapsis (13 000 km);
- **route (ii):** from the chaser's apoapsis (6936 km) to the target's periapsis (7000 km).

Table 4: The two apsis-to-apsis routes.

route	t_H [h]	Δv_1 [m/s]	Δv_2 [m/s]	rendezvous [m/s]	intercept [m/s]
(i) peri → apo	1.3475	1082.17	74.10	**1156.27**	1082.17
(ii) apo → peri	0.8040	93.58	1075.12	1168.70	**93.58**

The two rendezvous totals nearly tie, but the **intercept** costs differ by a factor of 11.6. Route (ii) makes its cheap burn first, where the two orbits nearly touch (6936 against 7000 km), and leaves the expensive velocity matching to the braking burn, which an intercept never pays.

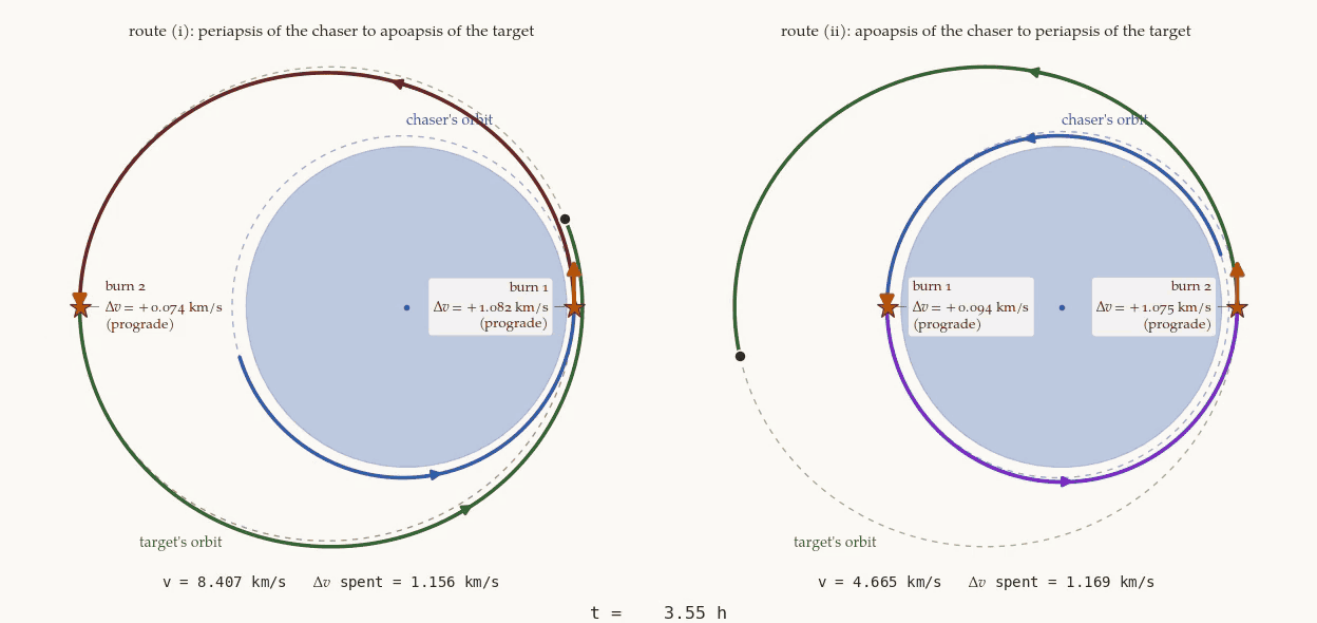


Figure 7: The two apsis-to-apsis routes of the table, on one clock. **(i)** Periapsis of the chaser's orbit to apoapsis of the target's: 1082.17 + 74.10 = 1156.27 m/s in 1.3475 h. **(ii)** Apoapsis to periapsis: 93.58 + 1075.12 = 1168.70 m/s in 0.8040 h. The target is not drawn: when it must be at the arrival apsis is the subject of the next section. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-chase.py, with the engine _manim.py; the burns are computed from the orbits.*

5.3 Timing: no exact window

Route (i) must depart at the chaser's periapsis, $t_0 = kT_1$, and route (ii) at its apoapsis, $t_0 = (k + \frac{1}{2})T_1$, for an integer k. In both cases the target must be at the arrival apsis t_H later. For route (i) that is the condition

$$M_2(0) + n_2(kT_1 + t_H) = \pi \pmod{2\pi}.$$

The left side moves by $n_2 T_1 = 2\pi\, T_1/T_2$ from one k to the next. The ratio T_1/T_2 is irrational for practical purposes, so the condition is never met exactly, only approximately at some k. **There is no periodic launch window.** Within 24 h the closest calls are route (ii) departing at 17.83 h, with the chaser arriving 106 s before the target, and route (i) departing at 18.60 h, 335 s early.

Check the route (ii) closest call. It departs at $t_0 = 11.5\,T_1 = 17.827$ h and arrives $t_H = 0.8040$ h later. Find the target's mean anomaly at arrival, and from it how early the chaser is.

What you should find

The arrival is at 18.631 h, which is 18.631/2.7644 = 6.7395 target periods. So $M_2 = 90° + 6.7395 \times 360° = 2516.2° \equiv 356.2°$. The target is 3.8° of mean anomaly short of periapsis, which is $(3.8/360)\,T_2 \approx 105$ s. The verify script, without rounding, gives 106 s.

5.4 Example 3: the elliptic porkchop

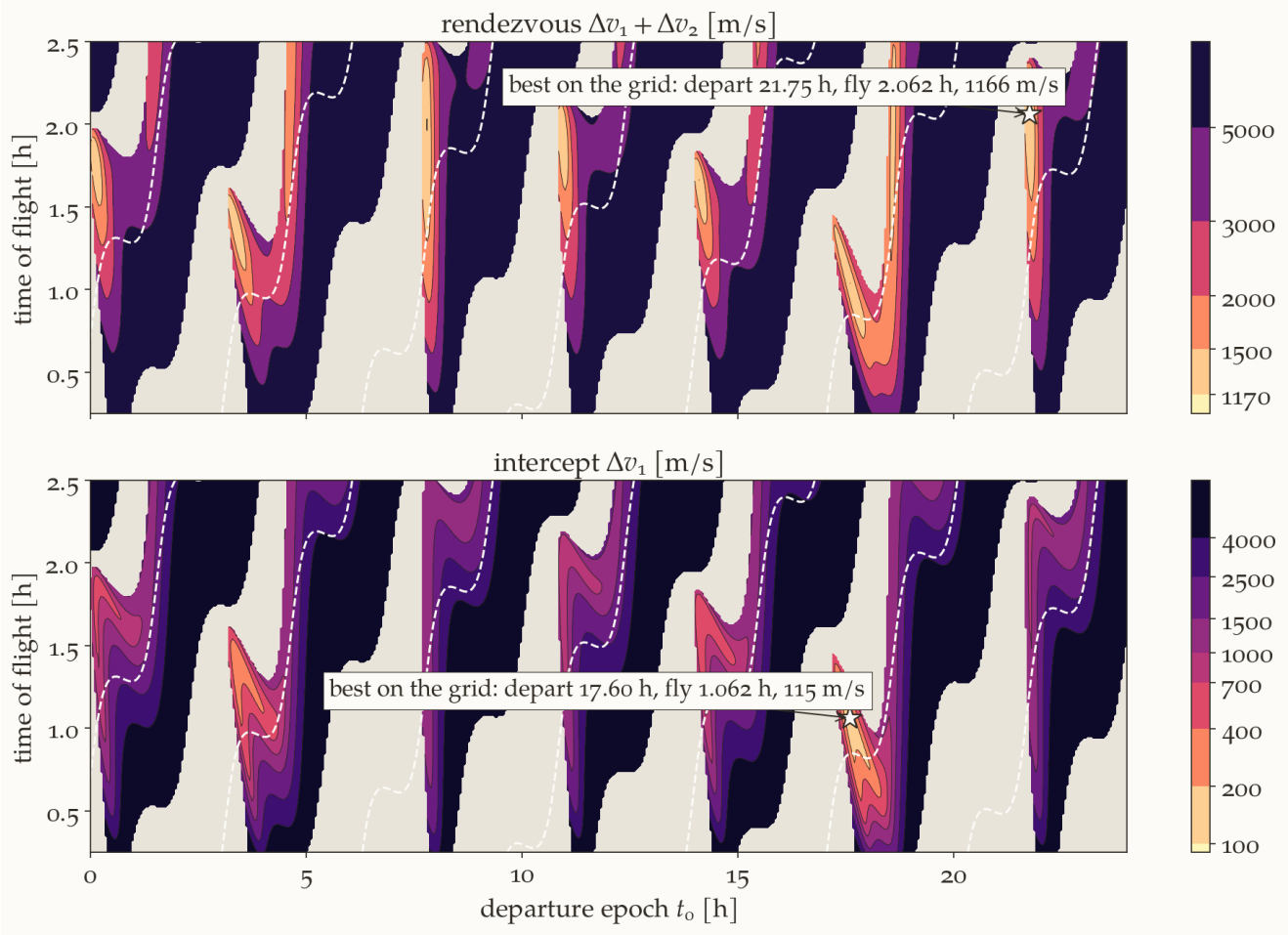


Figure 8: Porkchop for the elliptic chase over 24 h of departure epochs; top rendezvous, bottom intercept. The cheap islands recur roughly once per *synodic* period of the two orbits, $2\pi/(n_1 - n_2) = 3.53$ h (seven in 24 h), not once per chaser period $T_1 = 1.55$ h, because a good departure needs both bodies in the right places. Their depth changes from one island to the next because the pattern is only quasi-periodic: T_1/T_2 is not a ratio of small integers. Grey: the arc dips below 100 km altitude (a zero-revolution arc exists for every cell, as the Lambert note shows). Dashed: $\Delta\theta = 180°$, with short-way arcs below each dashed line and long-way arcs above it. The stars mark the grid optima (rendezvous 21.75 h, 2.062 h, 1166 m/s), which the text refines (to 21.757 h, 2.058 h, 1164.40 m/s). *Script: fig-chase-porkchop-ell.py*.

5.4.1 The optimum arcs

Refining the grid optima by a two-variable minimisation over $(t_0, \Delta t)$ gives the following.

Best rendezvous:

$$t_0 = 21.757 \text{ h}, \qquad \Delta t = 2.058 \text{ h},$$

$$\Delta v_1 = 952.50, \quad \Delta v_2 = 211.90, \quad \boxed{\Delta v = 1164.40 \text{ m/s}}$$

on a transfer arc with $p_t = 8557.10$ km and $e_t = 0.2835$ that sweeps 265.7°. It is **not** apsis to apsis, yet it costs only 0.7% more than route (i), whose timing never quite works.

Best intercept:

$$t_0 = 17.623 \text{ h}, \qquad \Delta t = 1.048 \text{ h}, \qquad \boxed{\Delta v_1 = 114.46 \text{ m/s}}$$

arriving at a relative speed of 1.087 km/s. It lies near the route (ii) closest call (17.83 h) and pays 21 m/s above route (ii)'s ideal 93.58 m/s to correct the timing.

Table 5: Intercept against rendezvous optima in the three settings of this note.

setting	best rendezvous	best intercept	same departure?
two circles, 300 → 420 km (Section 4.2)	68.49 m/s at $\theta = 2.378°$, $\Delta t = 45.9$ min	34.32 m/s, the same Hohmann arc	yes
two ellipses (Section 5.4.1)	1164.40 m/s, $t_0 = 21.757$ h, $\Delta t = 2.058$ h	114.46 m/s, $t_0 = 17.623$ h, $\Delta t = 1.048$ h	no: four hours apart
outer chaser, 420 → 300 km (Section 7.2)	68.49 m/s at $\theta = -2.431°$	34.17 m/s, the same arc reversed	yes

The two optima are **four hours apart**, on **different arcs**. "Chase" is not one problem: decide whether the velocity must be matched before you look for the window.

6 The terminal phase

6.1 Targeting with the CW state transition matrix

Once the chaser is within a few tens of kilometres of the target, Lambert's problem still works, but its answer is the small difference of two large velocities. Better to work in the target's local-vertical, local-horizontal frame: x radially outward, y along the direction of motion, origin on the target. The cw note solves the linearised equations (Clohessy and Wiltshire 1960) for the in-plane state $(x, y, \dot{x}, \dot{y})$ in the form $\mathbf{X}(t) = \Phi(t)\,\mathbf{X}(0)$. Split the 4×4 state transition matrix into position and velocity blocks,

$$\begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} \Phi_{rr}(t) & \Phi_{rv}(t) \\ \Phi_{vr}(t) & \Phi_{vv}(t) \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix},$$

where, with $\tau = nt$, $s = \sin\tau$ and $c = \cos\tau$,

$$\Phi_{rr} = \begin{bmatrix} 4 - 3c & 0 \\ 6(s - \tau) & 1 \end{bmatrix}, \qquad \Phi_{rv} = \frac{1}{n}\begin{bmatrix} s & 2(1 - c) \\ -2(1 - c) & 4s - 3\tau \end{bmatrix},$$

$$\Phi_{vr} = \begin{bmatrix} 3ns & 0 \\ -6n(1 - c) & 0 \end{bmatrix}, \qquad \Phi_{vv} = \begin{bmatrix} c & 2s \\ -2s & 4c - 3 \end{bmatrix}.$$

The chaser is at $\mathbf{r}_0 = (x_0, y_0)$ with velocity $\mathbf{v}_0^-$ just before the first burn, and we choose a transfer time t_f.

Intercept: reach the origin. We need $\mathbf{r}(t_f) = \mathbf{0}$. The position equation is linear in the unknown post-burn velocity $\mathbf{v}_0^+$:

$$\mathbf{0} = \Phi_{rr}(t_f)\,\mathbf{r}_0 + \Phi_{rv}(t_f)\,\mathbf{v}_0^+,$$

so

$$\boxed{\mathbf{v}_0^+ = -\Phi_{rv}^{-1}(t_f)\,\Phi_{rr}(t_f)\,\mathbf{r}_0, \qquad \Delta\mathbf{v}_1 = \mathbf{v}_0^+ - \mathbf{v}_0^-.} \tag{7}$$

The 2×2 inverse is explicit. With $d = 8(1-c) - 3\tau s$ (which is $n^2 \det_{rv}$, see Equation 9),

$$\underset{rv}{}^{-1} = \frac{n}{d}\begin{bmatrix} 4s - 3\tau & -2(1-c) \\ 2(1-c) & s \end{bmatrix}, \tag{8}$$

so the targeting can be done by hand, and it fails exactly where $d = 0$.

Rendezvous: also stop there. The arc is the same (Equation 1 again). Its velocity on arrival is

$$\mathbf{v}_f = {}_{vr}(t_f)\,\mathbf{r}_0 + {}_{vv}(t_f)\,\mathbf{v}_0^+,$$

and the target is at rest in its own frame, so the braking burn is $\Delta\mathbf{v}_2 = -\mathbf{v}_f$.

6.2 When the targeting fails

The intercept formula Equation 7 needs ${}_{rv}$ to be invertible. Its determinant is

$$\begin{aligned} n^2 \det{}_{rv} &= s\,(4s - 3\tau) + 4(1-c)^2 \\ &= 4s^2 - 3\tau s + 4 - 8c + 4c^2 \\ &= 8(1 - \cos\tau) - 3\tau\sin\tau, \end{aligned} \tag{9}$$

using $s^2 + c^2 = 1$ in the last step. It vanishes at

- $\tau = 2\pi k$, after whole periods. There $s = 0$ and $c = 1$, so the first row of ${}_{rv}$ vanishes and $x(t_f) = (4 - 3c)\,x_0 = x_0$ whatever the velocity: no burn can change where the chaser is radially after a whole number of periods, and a chaser that starts above or below the target cannot reach it;
- one further root between $2\pi k$ and $(2k+1)\pi$ for each $k \geq 1$: $\tau = 8.8387$ ($t_f = 1.4067\,T$), then 15.3643, 21.7471, and so on.

At those transfer times no two-impulse solution exists for a general $\mathbf{r}_0$, and close to them the cost grows without bound (Figure 9, right).

For a very short transfer, $\tau \ll 1$, expand Equation 9. What does $\det{}_{rv}$ become, and why does the result make physical sense?

Answer

$8(1-\cos\tau) = 4\tau^2 - \tau^4/3 + \ldots$ and $3\tau\sin\tau = 3\tau^2 - \tau^4/2 + \ldots$, so $n^2 \det{}_{rv} = \tau^2 + \tau^4/6 + \ldots$ and $\det{}_{rv} \approx t_f^2$. For a short time gravity has no time to act: ${}_{rv} \approx t_f\,\mathbf{1}$, the chaser flies in a straight line, and the intercept velocity is just $\mathbf{v}_0^+ = -\mathbf{r}_0/t_f$.

6.3 Example 4: 1 km below and 10 km behind

Given: the target at 420 km ($n = 1.126378 \times 10^{-3}$ rad/s, $T = 5578.22$ s), the chaser at $\mathbf{r}_0 = (-1,\ -10)$ km on a **co-circular** orbit, i.e. a circular orbit 1 km lower. In the cw frame a co-circular orbit at height x_0 drifts along the track at $\dot{y} = -\frac{3}{2}nx_0$ (cw note), so

$$\mathbf{v}_0^- = (0,\ -\tfrac{3}{2}nx_0) = (0,\ 1.690)\ \text{m/s}.$$

Left alone, the chaser closes the gap at $1.690 \times T = 9.42$ km per orbit, passing below the target.

Table 6: cw targeting from $(-1, -10)$ km.

t_f	$T/4$	$T/2$	$3T/4$	$5T/4$	$3T/2$
minutes	23.2	46.5	69.7	116.2	139.5
intercept Δv_1 [m/s]	6.401	2.171	1.222	1.158	0.872
arrival speed $= \Delta v_2$ [m/s]	7.154	2.171	0.618	0.580	0.872
rendezvous [m/s]	13.555	4.342	**1.840**	**1.738**	1.744

Short transfers are expensive, as on the porkchop of Section 4.3. The rendezvous cost falls to about 1.7 m/s between the singular times, and within two periods its smallest value is 0.87 m/s, at $t_f \approx 1.63\,T$.

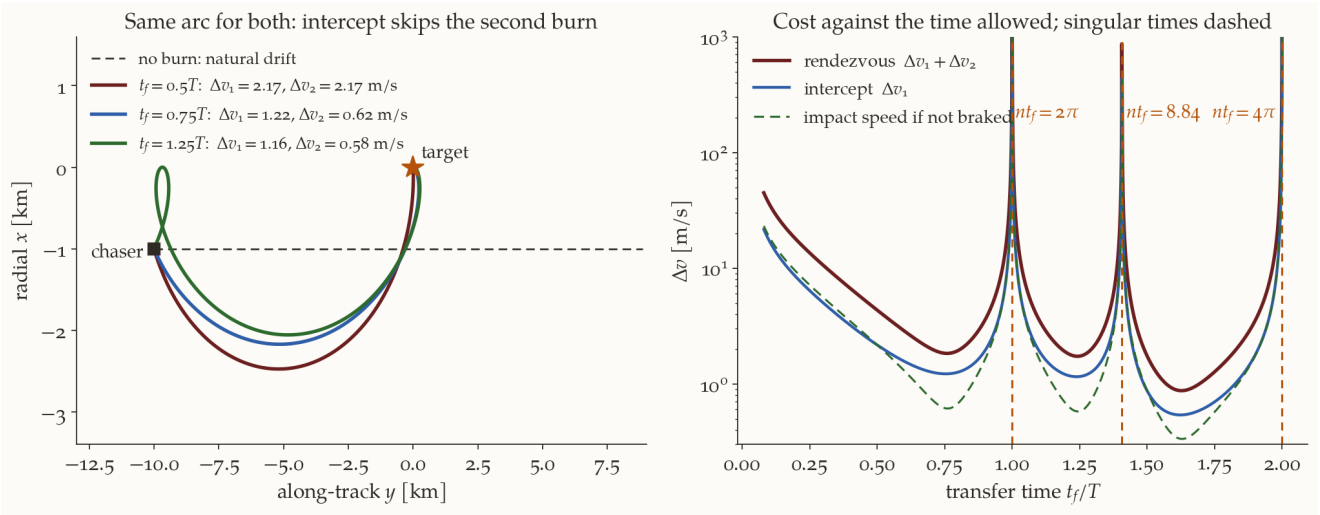


Figure 9: **Left:** cw transfers from $(-1, -10)$ km for three transfer times. Intercept and rendezvous fly the same arc and differ only in the braking burn at the origin. The dashed line is the natural drift. **Right:** cost against the transfer time, with the singular times $nt_f = 2\pi$, 8.84 and 4π dashed. *Script: fig-chase-cw.py.*

A check against Lambert. Put the chaser at the inertial position $\mathbf{R} = (a + x_0)\,\hat{\mathbf{i}} + y_0\,\hat{\mathbf{j}}$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ are the target's LVLH axes at $t = 0$ and the cw coordinates are read as straight-line offsets. Its velocity is the target's, plus $\omega \times \rho$ for the rotating frame, plus $\mathbf{v}_0^-$. Solving the same rendezvous exactly by p-iteration gives $(\Delta v_1, \Delta v_2) = (6.401,\ 7.156)$, $(2.171,\ 2.174)$ and $(1.219,\ 0.618)$ m/s at $T/4$, $T/2$ and $3T/4$: agreement with Table 6 to better than 0.3%. The $T/2$ arc sweeps 180.08°, right at the pinch where p-iteration struggles (Lambert note). cw has no difficulty there; its singularities are at other transfer times.

Reproduce the $T/2$ column by hand. At $\tau = \pi$ write down ${}_{rr}$ and ${}_{rv}$, solve Equation 7 for $\mathbf{v}_0^+$, and find Δv_1. In which direction is the burn mostly pointed?

What you should find

With $s = 0$, $c = -1$: ${}_{rr}\mathbf{r}_0 = (7x_0,\ -6\pi x_0 + y_0) = (-7,\ 8.8496)$ km and ${}_{rv} = \frac{1}{n}\begin{bmatrix} 0 & 4 \\ -4 & -3\pi \end{bmatrix}$. The first row gives $4\dot{y}_0^+/n = 7$, so $\dot{y}_0^+ = \frac{7}{4}n = 1.9712$ m/s; the second gives $\dot{x}_0^+ = -1.91095\,n = -2.1525$ m/s. Then $\Delta\mathbf{v}_1 = (-2.1525,\ 0.2816)$ m/s and $\Delta v_1 = 2.171$ m/s, as in Table 6. The burn is mostly **radially inward**: from below and behind, the chaser closes by dropping further first, and the cw motion carries it forward and up to the target.

7 The outer chaser

7.1 Time reversal

So far the chaser was on the inner orbit. When it is on the **outer** one ($a_1 > a_2$), nothing needs to be computed again, because of a symmetry of the two-body equation $\ddot{\mathbf{r}} = -\mu\,\mathbf{r}/r^3$.

Reversal. If $\mathbf{r}(t)$ is a solution, so is $\mathbf{q}(t) = \mathbf{r}(-t)$: differentiating twice gives $\ddot{\mathbf{q}}(t) = \ddot{\mathbf{r}}(-t) = -\mu\,\mathbf{q}/q^3$. The path is the same, run backwards, with every velocity reversed: $\dot{\mathbf{q}}(t) = -\dot{\mathbf{r}}(-t)$. (We write $\mathbf{q}$ rather than ρ, which is the CW relative position.)

Reflection. Reversed motion is retrograde, since $\mathbf{q} \times \dot{\mathbf{q}} = -\mathbf{h}$. Reflect it in a line through the centre, in the plane: $\mathbf{S}$, a constant matrix with $\mathbf{S}^\mathsf{T}\mathbf{S} = \mathbf{1}$ and $\det \mathbf{S} = -1$. Gravity is isotropic, so $\mathbf{S}\mathbf{q}(t)$ is again a solution, and a reflection reverses the sense of rotation, so it is **prograde**.

A prograde circle maps to itself under these two operations. Now apply them to a transfer from orbit 1 to orbit 2 that leaves at t_0 and arrives at t_f:

- the arc becomes a transfer from orbit 2 to orbit 1, with the **same time of flight**;
- the new departure velocity on the arc is $-\mathbf{S}\,\mathbf{v}_{\text{arc}}(t_f)$ and that of the circle there is $-\mathbf{S}\,\mathbf{v}_2(t_f)$, so the new departure burn is $|\mathbf{S}(\mathbf{v}_2 - \mathbf{v}_{\text{arc}})|$ = the old **arrival** burn; likewise the new arrival burn is the old departure burn.

So:

- **Rendezvous** maps to rendezvous with $\Delta v_1 \leftrightarrow \Delta v_2$: **the same total**. Every rendezvous result for $a_1 < a_2$ carries over to $a_1 > a_2$ unchanged.
- **Intercept does not.** The reversed intercept pays at departure what the forward one arrived with:

$$\Delta v^{2\to1}_{\text{intercept}} = \text{arrival speed}^{\,1\to2}, \qquad \text{arrival speed}^{\,2\to1} = \Delta v^{1\to2}_{\text{intercept}}.$$

7.2 The asymmetry in numbers

For Hohmann intercepts between circles:

Table 7: Hohmann intercepts in both directions. GEO is the circle of radius 42 164.137 km.

orbits	inner → outer: Δv_1	its arrival speed	outer → inner: Δv_1	its arrival speed
300 ↔ 420 km	34.32 m/s	34.17 m/s	34.17 m/s	34.32 m/s
300 km ↔ GEO	2425.7 m/s	1466.8 m/s	1466.8 m/s	2425.7 m/s

Nearby orbits hide the asymmetry (0.4%); widely separated ones expose it, a factor of 1.65.

The window reverses too. Equation 5, $\theta^* = \pi - n_2 t_H$, holds unchanged, but for an outer chaser the target is on the inner orbit and moves faster, so $n_2 t_H > \pi$ and θ^* is negative.

Find the lead angle for a chaser at 420 km and a target at 300 km.

What you should find

$t_H = 2752.27$ s as before (the same ellipse), and now $n_2 = 1.156874 \times 10^{-3}$ rad/s, so $n_2 t_H = 3.18402$ rad $= 182.431°$ and $\theta^* = -2.431°$: the target must **trail** the

chaser by 2.431° at the first burn. It is moving faster and will catch up with the arrival point during the transfer.

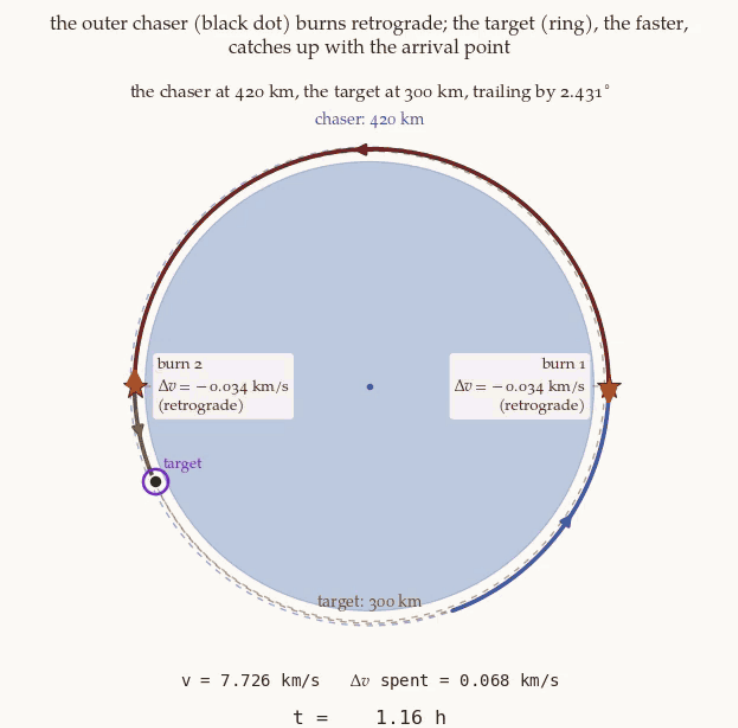


Figure 10: The outer chaser, the same ellipse flown the other way. The chaser (black dot) at 420 km burns retrograde (34.17 m/s) when the target (ring) at 300 km *trails* it by 2.431°; the target is the faster, and arrives at the transfer's periapsis together with the chaser, where burn 2 (34.32 m/s) circularises. The total is 68.49 m/s, that of the inner chaser. The craft moves at its true speed, only slowed down, and the clock stops during a burn. *Script: anim-chase.py, with the engine _manim.py; the burns are computed from the orbits, and the ring is checked to be at the meeting point when the chaser arrives.*

8 Summary

1. **Intercept** matches position; **rendezvous** matches position and velocity. For given $(t_0, \Delta t)$ they fly the same Lambert arc, and the rendezvous adds the intercept's arrival speed as a braking burn. Their optima are on different arcs.
2. **Same orbit:** phasing with $T_{ph} = T(1 \mp \theta/2\pi k)$. The cost falls as $1/k$, so time buys propellant, but the atmosphere limits how far the chaser may drop. On an ellipse, convert the angular gap Δf to a time with Kepler's equation; the burn point changes both.
3. **Two circles:** burn at the lead angle $\theta^* = \pi - n_2 t_H$, which recurs every synodic period. Leaving off the window is expensive or impossible; **waiting is cheap**.
4. **Two ellipses:** the windows are not periodic. The porkchop over $(t_0, \Delta t)$ is the design tool, and the intercept and rendezvous optima can be hours apart.
5. **Close in:** cw targeting, $\mathbf{v}_0^+ = -_{rv}^{-1}{}_{rr}\mathbf{r}_0$, agrees with Lambert and has no difficulty with a 180° transfer, but fails at its own singular transfer times, where $8(1 - \cos nt_f) = 3nt_f \sin nt_f$.
6. **Outer chaser:** time reversal plus a reflection preserves rendezvous costs and **swaps** the intercept cost with the arrival speed.

Everything in this note can be reproduced with chase-lambert-verify.py: the determinant of $_{rv}$ and its roots, every table, the porkchop minima (minimised over the time of flight, not read off a grid), the elliptic optima, the cw and Lambert targeting, and the time-reversal swap.

9 Exercises

1. **Phasing in GEO.** A geostationary satellite must drift 5° west in 3 days. Design the phasing orbit. Which way is the first burn, and what is the total cost? Take the geostationary radius as 42 164 km.

Solution. Seeing it as a chase. The satellite has to reach the point of the geostationary orbit that sits above the new longitude. That point turns with the Earth, so it moves exactly like a target on the same circle. It is 5° to the west, which is **behind** the satellite, because the satellite moves east. So this is the case of Equation 3 with a trailing target. The satellite must arrive late, so it needs a **longer** period, a larger orbit, and the first burn is **prograde**. It slows down relative to the ground because a higher orbit has a longer period, and so it drifts west.

The phasing orbit. The period on the circle is $T = 2\pi\sqrt{a^3/\mu} = 86\,163.6\text{ s} = 23.934$ h, one sidereal day. Three days is about three revolutions, so $k = 3$, and the satellite must lose $5°/3$ on the target in each. With $+\theta$ in place of $-\theta$,

$$T_{ph} = T\Big(1 + \frac{\theta}{2\pi k}\Big) = 86\,163.6\Big(1 + \frac{5}{1080}\Big) = 86\,562.5\text{ s},$$

which is 6.65 min longer than the sidereal day. The three revolutions take $3T_{ph} = 3.006$ days. Then

$$a_{ph} = a\Big(\frac{T_{ph}}{T}\Big)^{2/3} = 42\,294.0\text{ km}.$$

The burn point is the perigee of the phasing ellipse, and its apogee is at $2a_{ph} - a = 42\,424.1$ km, 260 km above the geostationary circle.

The cost. By Equation 4, the speed at $r = a$ rises from $\sqrt{\mu/a} = 3.07467$ km/s to $\sqrt{\mu(2/a - 1/a_{ph})}$, a burn of $\Delta v_{ph} = 4.723$ m/s. The second burn, retrograde and of the same size, stops the drift at the new longitude. The total is $2{\times}4.723 = 9.45$ m/s. As a check, the small-change rule of Section 3.3 gives $\Delta v_{ph} \approx \frac{1}{3} v_c\,(T_{ph} - T)/T = 4.745$ m/s.

Reading the result. Moving a geostationary satellite is cheap if there is time. Over three days, 5° costs under 10 m/s, and by the $1/k$ rule the same move over fifteen days would cost about 1.9 m/s. Operators relocate satellites in exactly this way, with a small burn to start a drift and an equal one to stop it. The phasing note works the eastward case.

2. **The other window.** In Example 2, the chaser decides to use a phasing orbit at 250 km for its wait, starting from $\theta_0 = 40°$. How long is the wait now, and how much does the descent to 250 km and the climb back cost?

Solution. The idea. On the 300 km circle the phase closes at 6.290°/h, so reaching the window takes 5.98 h. A lower circle has a shorter period, so on it the phase closes faster. The price is two extra transfers. We take the simplest plan. A Hohmann transfer takes the chaser down to a circle at 250 km, it waits there, and a Hohmann transfer then takes it straight up to the target at 420 km. There is no need to climb back to 300 km first.

Step 1, the descent. The transfer from $r_1 = 6678.137$ km down to $r_3 = 6628.137$ km takes $t_d = 2700.4$ s, and its two burns, both retrograde, are 14.529 and 14.556 m/s, a total of 29.085 m/s. During the descent the chaser sweeps 180° and the target sweeps $n_2 t_d = 174.27°$. So the lead falls by $180° - 174.27°$, and at the end of the descent it is

$$\theta = 40° + 174.27° - 180° = 34.27°.$$

Step 2, the new window. The climb from 250 to 420 km is a Hohmann transfer with $a_t = (r_3 + r_2)/2$ and $t_H = 2737.0$ s. By Equation 5 it needs the lead

$$\theta^* = \pi - n_2 t_H = 3.365°,$$

a little larger than before, because the climb is longer and the target travels farther meanwhile.

Step 3, the wait. On the 250 km circle, $n_3 = \sqrt{\mu/r_3^3}$, and the phase closes at $n_3 - n_2 = 8.995°/\text{h}$. By Equation 6 the wait is

$$t_w = \frac{34.27° - 3.365°}{8.995°/\text{h}} = 3.44 \text{ h}.$$

Step 4, the climb. Its two burns are 48.940 and 48.632 m/s, a total of 97.572 m/s.

Comparison. From the moment of decision to the rendezvous, the new plan takes $t_d + t_w + t_H = 0.75 + 3.44 + 0.76 = 4.95$ h. Example 2 took 5.98 h of waiting plus the 45.87 min transfer, 6.75 h in all. The phasing orbit saves 1.80 h. It costs $29.085 + 97.572 = 126.66$ m/s against 68.489 m/s, so the descent and the climb back cost 58.2 m/s more. That is almost exactly twice the descent, as we would expect, because climbing the extra 50 km costs about what descending it did.

Reading the result. Buying 1.8 h for 58 m/s is about 32 m/s per hour saved. The Lambert arc of Section 4.3 that leaves one hour early costs 198.2 m/s, which is 130 m/s for one hour. So a lower phasing orbit is a far cheaper way to buy time. That is why real rendezvous missions spend their wait in such an orbit. The limit is the atmosphere. At 250 km the drag is already significant, which is why the phasing orbits used in practice are seldom much lower than this.

3. **Elliptic phasing, both ways.** On the orbit of Section 3.4, the target is 30° *ahead* of the chaser at periapsis. Find the phasing period, the new apoapsis and the rendezvous cost. Why is there no need to check the perigee?

Solution. The time gap. The target is at $f = +30°$ and the chaser at periapsis. The chaser must get back to periapsis just as the target does. The target needs T less the time from $f = 0$ to $f = 30°$, and Section 3.4 found that time, $\Delta t = 435.11$ s. So the chaser must arrive **early** by Δt, and

$$T_{ph} = T - \Delta t = 9952.01 - 435.11 = 9516.90 \text{ s}.$$

The phasing orbit. Kepler's third law gives $a_{ph} = a\,(T_{ph}/T)^{2/3} = 9706.36$ km. A shorter period needs a smaller orbit, so the burn at periapsis is **retrograde**. The burn point stays an apsis, and the new apoapsis is at

$$r_{a,ph} = 2a_{ph} - r_p = 2 \times 9706.36 - 7000 = 12\,413 \text{ km},$$

lower than the 13 000 km of the original orbit.

The cost. At $r_p = 7000$ km, vis-viva gives the speed $\sqrt{\mu(2/r_p - 1/a)} = 8.60382$ km/s on the original orbit and $\sqrt{\mu(2/r_p - 1/a_{ph})} = 8.53346$ km/s on the phasing orbit. Each burn is 70.36 m/s, and the rendezvous costs $2 \times 70.36 = 140.73$ m/s.

Why the perigee needs no check. In the circular Example 1 the chaser dropped onto an ellipse whose far apsis, its perigee, came close to the atmosphere. Here the burn is made **at** periapsis and is retrograde. It lowers the opposite apsis, the apoapsis, while the burn point remains the periapsis of the new orbit, as long as the new apoapsis stays above 7000 km, which it does. So the lowest point of the phasing orbit is still 7000 km, 622 km above the surface.

Reading the result. The target ahead costs 140.73 m/s, the target behind 129.81 m/s. The two are close, as in Example 1, and catching a target ahead is again the dearer of the two.

4. **cw targeting by hand.** For a chaser at $\mathbf{r}_0 = (0, -5)$ km with $\mathbf{v}_0^- = \mathbf{0}$ behind a target in a 90-minute orbit, find the intercept burn for $t_f = T/2$ using Equation 7.

Solution. Setting up. The mean motion is $n = 2\pi/T = 2\pi/5400 = 1.16355 \times 10^{-3}$ rad/s. At $t_f = T/2$ we have $\tau = \pi$, so $s = 0$ and $c = -1$. The chaser is on the target's orbit, 5 km behind it, and at rest in the cw frame.

The two blocks. With $s = 0$ and $c = -1$,

$$_{rr}(t_f) = \begin{bmatrix} 7 & 0 \\ -6\pi & 1 \end{bmatrix}, \qquad _{rv}(t_f) = \frac{1}{n}\begin{bmatrix} 0 & 4 \\ -4 & -3\pi \end{bmatrix}.$$

Since $x_0 = 0$, $_{rr}\mathbf{r}_0 = (0,\ y_0) = (0,\ -5)$ km.

Solving. The intercept condition $_{rv}\mathbf{v}_0^+ = -_{rr}\mathbf{r}_0 = (0,\ 5)$ km is two equations. The first row gives $4\dot{y}_0^+/n = 0$, so $\dot{y}_0^+ = 0$. The second gives $(-4\dot{x}_0^+ - 3\pi\dot{y}_0^+)/n = 5$ km, so

$$\dot{x}_0^+ = -\frac{5}{4}\, n \text{ km} = -1.454 \text{ m/s}.$$

The burn is $\Delta\mathbf{v}_1 = \mathbf{v}_0^+ - \mathbf{0} = (-1.454,\ 0)$ m/s. It is 1.454 m/s, pointed radially **inward**.

The path. With these values the cw solution reads $x(t) = -\frac{5}{4}\sin\tau$ km and $y(t) = -5 + \frac{5}{2}(1 - \cos\tau)$ km. The chaser dips to 1.25 km below the target's orbit at the quarter period, and it rises back to meet the target at $\tau = \pi$. If the chaser is also to stop there, the arrival velocity is $_{vv}\mathbf{v}_0^+ = (c\,\dot{x}_0^+,\ -2s\,\dot{x}_0^+) = (+1.454,\ 0)$ m/s, and the rendezvous costs $2 \times 1.454 = 2.909$ m/s.

Reading the result. A purely radial burn moves the chaser along the track. Dropping below the target makes the chaser faster than the target on average, so it gains on it, and the motion brings it back up after half an orbit. This "radial hop" closes a gap of 5 km in 45 min for under 1.5 m/s. Its path is a closed loop. If the second burn is not made, the chaser does not drift away, but comes back to its starting point after one full period. Hops of this kind are used in real approaches to move between holding points along the target's track, where that safety matters.

5. **Time reversal on an ellipse.** Apply the argument of Section 7.1 to the elliptic example of Section 5, taking the reflection line along the common apse line. Show that each ellipse maps onto itself. What happens to the epochs, and to the target's starting mean anomaly $M_2(0) = 90°$?

Solution. The plan. We apply the two operations of Section 7.1 one at a time to a body on one of the ellipses, and follow its true anomaly and its mean anomaly.

Each ellipse maps onto itself. Put the common apse line along the x axis. The reflection **S** in that line sends (x, y) to $(x, -y)$, which turns a point at true anomaly f into the point at $-f$. Each ellipse is $r = p/(1 + e\cos f)$, which is unchanged when f is replaced by $-f$. So the reflection maps each ellipse onto itself. This works only because the apse lines are aligned. A line that is an axis of symmetry of both ellipses at once must be their common apse line.

The time reversal does not move the path at all. It only runs it backwards, which is retrograde. The reflection then reverses the sense of rotation again. So a body that moved prograde on an ellipse moves prograde on the same ellipse after both operations.

The epochs. The new motion at time t is the old motion at time $-t$, reflected. A transfer that left orbit 1 at t_0 and reached orbit 2 at t_f becomes a transfer that leaves orbit 2 at $-t_f$ and reaches orbit 1 at $-t_0$. Its time of flight, $t_f - t_0$, is unchanged, but the departure and arrival epochs are swapped and their signs reversed.

The mean anomalies. The mean anomaly is an odd function of the true anomaly, so the reflection sends M to $-M$. A body with $M(t) = M(0) + nt$ therefore becomes a body with

$$\tilde{M}(t) = -M(-t) = -M(0) + nt.$$

It still advances at the rate n, as a prograde body must, but it starts from $-M(0)$. The chaser was at periapsis, $M_1(0) = 0$, and it stays there. The target started at $M_2(0) = 90°$, and it now starts at $\tilde{M}_2(0) = -90°$, that is, 270°. It is a quarter of a period **before** its periapsis instead of after it.

What this gives us. The reversed problem has the chaser on the outer ellipse and the target on the inner one, with the target at periapsis at $t = 0$ and the chaser at $M = 270°$. Every result of Section 5 carries over. A rendezvous keeps its total, with Δv_1 and Δv_2 swapped. Route (i), from the inner periapsis to the outer apoapsis for 1156.27 m/s, becomes a route from the outer apoapsis to the inner periapsis for the same total. An intercept does not keep its cost. The reversed route (ii) pays 1075.12 m/s at departure, the old braking burn, instead of 93.58 m/s.

Further reading

Curtis (2014) and Vallado (2013) treat phasing, rendezvous and the cw targeting problem; Clohessy and Wiltshire (1960) is the original cw paper.

Clohessy, W. H., and R. S. Wiltshire. 1960. "Terminal Guidance System for Satellite Rendezvous." *Journal of the Aerospace Sciences* 27 (9): 653–58. https://doi.org/10.2514/8.8704.

Curtis, Howard D. 2014. *Orbital Mechanics for Engineering Students*. 3rd ed. Oxford: Butterworth-Heinemann.

Vallado, David A. 2013. *Fundamentals of Astrodynamics and Applications*. 4th ed. Hawthorne, CA: Microcosm Press.