Challenge Problems

Devendra Ghate

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How to use these problems

These ten problems are for students who have worked through the questions that go with each note and want something harder. They are analytical. Each answer is a formula or a proof, and the numbers at the end of a problem only evaluate that formula.

Most of them take one assumption of the course and relax it. The course used a point-mass Earth that does not rotate, constant μ, impulsive burns and two bodies. Here μ changes, the engine is weak, the planet is a little oblate, the Earth turns, a third body is felt. The relaxed models are not meant to be realistic. They are chosen because they can be solved exactly, and because the way we solve them is the way the real problem is attacked. Three ideas come back again and again.

- A quantity that the course treats as conserved (energy, angular momentum, the circular speed) is followed as it changes, from its definition.
- A small effect is treated as a small error, and the influence coefficients or a Taylor expansion tell us what it does.
- A different frame (rotating with a planet, or with the Earth) turns a hard question into an easy one.

Every problem has a worked solution, folded under **Solution**. Try the problem on paper first, for a day or two if needed. Every number and identity in the solutions is checked by challenge-problems-verify.py, which also checks each key result independently, by integrating the equations of motion or by a brute-force search.

Data. $\mu_\oplus = 398\,600.4418$ km^3/s^2, $R_\oplus = 6378.137$ km, $\omega_\oplus = 7.2921151 \times 10^{-5}$ rad/s, $J_2 = 1.08263\times10^{-3}$, $\mu_\odot = 1.32712\times10^{11}$ km^3/s^2, 1 AU $= 1.495\,978\,707\times10^{8}$

km, $g_0 = 9.80665\ \mathrm{m/s^2}$. Each problem gives any other data it needs.

Problems

1. When the attracting mass changes

In the whole course μ has been a constant. Here we let it change and ask what the orbit does. This can happen in two ways. A solar sail that faces the Sun is pushed away from the Sun by sunlight, and that push falls off as $1/r^2$, exactly like gravity. So the sail feels the Sun's gravity with μ replaced by $(1-\beta)\mu$, where the **lightness number** β is the ratio of the push of the light to the pull of gravity. A star that loses mass, slowly in a wind or suddenly in an explosion, changes μ itself.

A spacecraft moves on a circular orbit of radius r about a body of gravitational parameter μ. At one instant μ drops to $(1-\beta)\mu$, with $0 < \beta < 1$, and stays there.

a. Explain why the position and the velocity of the spacecraft do not change at that instant. Show that the point where the change happens becomes the periapsis of the new orbit, and that the new orbit has

$$e' = \frac{\beta}{1-\beta}, \qquad a' = \frac{(1-\beta)\,r}{1-2\beta}, \qquad r'_a = \frac{r}{1-2\beta}.$$

Hence show that the spacecraft escapes if $\beta \geq \frac{1}{2}$.

b. Now the spacecraft is on an ellipse of eccentricity e when μ drops. Show that it escapes if $\beta \geq 1 - Q/2$, where $Q = v^2 r/\mu$ is evaluated at the point of the change. Where on the ellipse should the change happen to make escape easiest, and what is the threshold there? What is it at the other apsis?

c. Now let μ fall slowly, over many orbits. Show that the angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is still conserved, even though the energy is not. A slow change keeps a circular orbit circular (accept this without proof). Show that the radius then grows as $r \propto 1/\mu$, and that $v \propto \mu$ and $T \propto \mu^{-2}$. Compare with part (a) when the loss is small, $\beta \ll 1$.

d. A sail craft is on the Earth's orbit, taken as a circle of radius 1 AU, and opens its sail facing the Sun. What lightness number is needed for the orbit to just reach the orbit of Mars, 1.5237 AU? How long does it take to get there, and how does that compare with a Hohmann transfer?

Solution. The key idea is that a sudden change of μ is an impulse turned inside out. In an impulse the velocity changes and the field stays the same. Here the field changes and the velocity stays the same. Either way, the old state now belongs to a new conic, and we find that conic from the state, with the new μ.

(a) *Nothing jumps.* The change of μ changes the acceleration. The velocity is the integral of the acceleration, and the position is the integral of the velocity, so neither can jump when the acceleration jumps. Just after the change the spacecraft is still at radius r, moving with the circular speed $v_c = \sqrt{\mu/r}$ of the *old* field, at right angles to the radius.

The new conic. The angular momentum is $h = r\,v_c = \sqrt{\mu r}$, and the new field has $\mu' = (1-\beta)\mu$. So the semi-latus rectum of the new orbit is

$$p' = \frac{h^2}{\mu'} = \frac{\mu r}{(1-\beta)\mu} = \frac{r}{1-\beta}.$$

The velocity has no radial component, so the point is an apsis of the new orbit ($v_r = (\mu'/h)\,e' \sin f = 0$). Which apsis? The orbit equation at that point reads $p'/r = 1 + e' \cos f$, and $p'/r = 1/(1-\beta)$ is larger than 1. So $\cos f = +1$. The point is the **periapsis**, and

$$e' = \frac{1}{1-\beta} - 1 = \frac{\beta}{1-\beta}.$$

This makes sense. The spacecraft keeps the speed that balanced the old, stronger pull. For the weaker pull it is now too fast, so it climbs, and the point where it was is the lowest point of the new orbit.

Size. The energy in the new field is

$$\varepsilon' = \frac{v_c^2}{2} - \frac{\mu'}{r} = \frac{\mu}{2r} - \frac{(1-\beta)\mu}{r} = -\frac{(1-2\beta)\,\mu}{2r},$$

so

$$a' = -\frac{\mu'}{2\varepsilon'} = \frac{(1-\beta)\,r}{1-2\beta}, \qquad r'_a = 2a' - r = \frac{2(1-\beta)-(1-2\beta)}{1-2\beta}\,r = \frac{r}{1-2\beta}.$$

Escape. The orbit is open when $\varepsilon' \geq 0$, which is $\beta \geq \frac{1}{2}$. The same condition is $e' \geq 1$. The reading is simple. On a circular orbit the kinetic energy per unit mass is $\mu/(2r)$, exactly half the depth μ/r of the well. If the well loses half its depth, the kinetic energy is enough to climb out.

(b) The position and velocity again stay the same, so the new energy is $\varepsilon' = \frac{1}{2}v^2 - (1-\beta)\mu/r$. It is not negative when

$$1 - \beta \leq \frac{v^2 r}{2\mu} = \frac{Q}{2} \iff \beta \geq 1 - \frac{Q}{2}.$$

To make escape easiest we want Q as large as possible. *Vis-viva*, $v^2 = \mu(2/r - 1/a)$, gives $Q = 2 - r/a$, which is largest where r is smallest, at periapsis. There $r_p = a(1-e)$, so $Q = 1+e$, and the threshold is

$$\beta_p = \frac{1-e}{2}.$$

At apoapsis $Q = 1-e$ and the threshold is $\beta_a = (1+e)/2$. For $e = 0$ both give $\frac{1}{2}$, as in (a). The more eccentric the orbit, the larger the gap. At periapsis the spacecraft carries the most kinetic energy compared with the depth of the well, so it needs the smallest reduction of the well to get out. This is the same reason that burns are cheapest at periapsis (the Oberth effect of the escape note).

History. A. Blaauw (1961) used this argument for the "runaway" stars of high speed. When one star of a close binary explodes as a supernova and the pair loses more than half of its total mass, the pair is unbound and the companion flies off with its orbital speed.

(c) *Angular momentum.* Differentiate $\mathbf{h} = \mathbf{r} \times \mathbf{v}$,

$$\dot{\mathbf{h}} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \left(-\frac{\mu(t)}{r^3}\,\mathbf{r}\right) = \mathbf{0}.$$

The proof uses only that the force is along $\mathbf{r}$, not that μ is constant. The energy is a different matter. The derivation of the energy integral uses $-\mu/r$ as a potential that does not change with time, and that is no longer true. So the energy changes, and we cannot use it. This is why $\mathbf{h}$ is the right tool here.

Circular orbit. On a circle $h = \sqrt{\mu r}$, so h constant means

$$\mu\, r = \text{const}, \qquad r \propto \frac{1}{\mu}, \qquad v = \frac{h}{r} \propto \mu, \qquad T = \frac{2\pi r}{v} \propto \mu^{-2}.$$

The orbit grows as the mass falls, and in relative terms the period grows twice as fast as the radius.

Comparison. For a small loss, the slow change gives $r' = r/(1-\beta) \approx r(1+\beta)$. The sudden change of (a) gives $a' = (1-\beta)r/(1-2\beta) \approx r(1+\beta)$ as well. So to first order in β the orbit grows by the same amount either way. The difference is in the shape. The slow loss leaves a circle. The sudden loss leaves an ellipse of eccentricity $e' \approx \beta$ whose periapsis marks the place where the change happened.

(d) The sail craft starts on a circle of radius $r = 1$ AU, so (a) applies with $\mu = \mu_\odot$. The new orbit reaches out to $r'_a = r/(1-2\beta)$. Setting this equal to 1.5237 AU,

$$\beta = \frac{1}{2}\Big(1 - \frac{1}{1.5237}\Big) = 0.172, \qquad e' = \frac{\beta}{1-\beta} = 0.208, \qquad a' = \frac{1-\beta}{1-2\beta}\ \text{AU} = 1.262\ \text{AU}.$$

The craft reaches the aphelion after half an orbit in the field $\mu' = (1-\beta)\mu_\odot$,

$$t = \pi\sqrt{\frac{a'^3}{(1-\beta)\mu_\odot}} = \frac{1\ \text{year}}{2}\,\frac{(a'/1\ \text{AU})^{3/2}}{\sqrt{1-\beta}} = 284\ \text{days}.$$

The Hohmann transfer to Mars takes 259 days. Its ellipse has the same perihelion and aphelion, 1 AU and 1.5237 AU, so it is *the same ellipse*. The sail flies it more slowly, by the factor $1/\sqrt{1-\beta}$, because the effective pull of the Sun is weaker.

What this means for a mission. The sail uses no propellant at all, and pays in time. It also cannot stop at Mars by itself in this simple setting, because a sail that keeps facing the Sun stays on this ellipse. Real sails are tilted, so that the push has a component along the velocity, and then they can spiral in or out. A lightness number of 0.17 also needs a very light sail. The value of the problem is the idea that a change of μ and an impulse are two sides of one thing.

2. A weak engine

The impulsive model replaces a burn by a jump in velocity. An electric thruster gives a thrust so weak that this picture fails completely, because the "burn" lasts for months. Here we treat the opposite limit, a small acceleration a_T along the velocity that acts all the time.

a. Show that the specific energy $\varepsilon = \frac{1}{2}v^2 - \mu/r$ changes at the rate $\dot\varepsilon = a_T v$ when the acceleration a_T acts along the velocity.

b. Suppose a_T is so small that the orbit is nearly circular at every moment, so that $\varepsilon = -\mu/(2r)$ and $v = \sqrt{\mu/r}$ with a slowly changing r. Show that

$$\frac{dv}{dt} = -a_T.$$

The engine pushes forward and the spacecraft slows down. Hence show that the total $\Delta v = \int a_T\, dt$ needed to go from a circle of radius r_1 to one of radius $r_2 > r_1$ is $v_{c_1} - v_{c_2}$, whatever the size of a_T.

c. Prove that this always costs more than the Hohmann transfer between the same circles. Use $x = \sqrt{r_1/r_2}$ and show that the difference is

$$\Delta v_{\text{spiral}} - \Delta v_{\text{H}} = v_{c_1}\,(1-x)\Big[2 - (1+x)\sqrt{\frac{2}{1+x^2}}\Big].$$

For a small raise, $r_2 = r_1(1+\epsilon)$, the difference is $v_{c_1}\epsilon^3/32$ to leading order (you may check this with a series). What does that say about small raises?

d. Atmospheric drag is a weak force *against* the velocity. A satellite on a low circular orbit feels a drag deceleration D. Show that its speed increases at the rate D. Where does the extra kinetic energy come from?

e. Compare the two routes from a circular orbit at 300 km altitude to the geostationary radius 42 164.17 km, in the same plane. How long does the spiral take with $a_T = 1.0 \times 10^{-7}$ km/s^2? Compare the propellant needed by a chemical stage with $I_{sp} = 320$ s flying the Hohmann transfer and by an electric thruster with $I_{sp} = 1600$ s flying the spiral.

Solution. We want to know what a weak, steady push does to an orbit. The plan is to follow the energy, which the push changes at a known rate (a), and to tie the energy to the speed through the circular-orbit relations (b). The comparison with the Hohmann transfer (c) is then a piece of algebra, and drag (d) is the same calculation with the sign of the force reversed.

(**a**) Differentiate $\varepsilon = \frac{1}{2}\,\mathbf{v}\cdot\mathbf{v} - \mu/r$ in time,

$$\dot\varepsilon = \mathbf{v}\cdot\dot{\mathbf{v}} + \frac{\mu}{r^2}\,\dot r.$$

The equation of motion is now $\dot{\mathbf{v}} = -\mu\mathbf{r}/r^3 + a_T\,\hat{\mathbf{v}}$, and $\mathbf{v}\cdot\mathbf{r} = r\dot r$. So

$$\dot\varepsilon = -\frac{\mu}{r^3}\,r\dot r + a_T\,v + \frac{\mu}{r^2}\,\dot r = a_T\,v.$$

Gravity drops out, as it must, since it conserves energy. The engine adds energy at the rate force times speed, per unit mass.

(**b**) On a circle $\varepsilon = -\mu/(2r)$, so $\dot\varepsilon = \mu\dot r/(2r^2)$. Setting this equal to $a_T v$ gives the rate at which the circle grows,

$$\dot r = \frac{2a_T\,v\,r^2}{\mu}.$$

The speed is $v = \sqrt{\mu/r}$, so $\dot v = -\frac{1}{2}\sqrt{\mu}\,r^{-3/2}\dot r = -\dfrac{v}{2r}\,\dot r$. With $\dot r$ from above,

$$\dot v = -\frac{v}{2r}\cdot\frac{2a_T\,v\,r^2}{\mu} = -a_T\,\frac{v^2 r}{\mu} = -a_T,$$

because $v^2 r/\mu = 1$ on a circle.

The paradox explained. On a circle the kinetic energy is $\frac{1}{2}v^2 = \mu/(2r)$ and the potential energy is $-\mu/r$, so the kinetic energy is minus the total. When the engine adds energy, the total rises, so the kinetic energy *falls*. The engine's work and the kinetic energy lost both go into potential energy, which rises twice as fast as the total.

The total cost. Integrating, $\Delta v = \int a_T\,dt = -\int dv = v_{c_1} - v_{c_2}$. The size of a_T has dropped out, so the cost of a slow spiral does not depend on how slow it is, as long as the orbit stays nearly circular.

(**c**) We need the Hohmann cost in terms of $x = \sqrt{r_1/r_2}$. Write $R = r_2/r_1 = 1/x^2$. The two burns of the Hohmann transfer are

$$\frac{\Delta v_{\mathrm{H}}}{v_{c_1}} = \Big(\sqrt{\frac{2R}{1+R}} - 1\Big) + \frac{1}{\sqrt{R}}\Big(1 - \sqrt{\frac{2}{1+R}}\Big).$$

With $R = 1/x^2$ we have $2R/(1+R) = 2/(1+x^2)$. Call $\sigma = \sqrt{2/(1+x^2)}$. The last term is $x\sqrt{2x^2/(1+x^2)} = x^2\sigma$, so

$$\frac{\Delta v_{\mathrm{H}}}{v_{c_1}} = \sigma - 1 + x - x^2\sigma = \sigma\,(1-x^2) - (1-x).$$

The spiral costs $\Delta v_{\mathrm{spiral}}/v_{c_1} = 1 - v_{c_2}/v_{c_1} = 1 - x$. Subtracting, and using $1 - x^2 = (1-x)(1+x)$,

$$\frac{\Delta v_{\mathrm{spiral}} - \Delta v_{\mathrm{H}}}{v_{c_1}} = 2(1-x) - \sigma(1-x)(1+x) = (1-x)\big[2 - \sigma(1+x)\big].$$

For $r_2 > r_1$ we have $0 < x < 1$, so $1 - x > 0$. It remains to show that the bracket is positive, that is $2 > (1+x)\sqrt{2/(1+x^2)}$. Both sides are positive, so we may square,

$$4 > \frac{2(1+x)^2}{1+x^2} \iff 2(1+x^2) > (1+x)^2 \iff (1-x)^2 > 0,$$

which is true for every $x \neq 1$. So the spiral always costs more.

Small raises. Put $x = 1/\sqrt{1+\epsilon}$ and expand. The two costs agree in the first-order term $\frac{1}{2}\epsilon$ and in the second-order term $-\frac{3}{8}\epsilon^2$, and the difference starts at $\epsilon^3/32$. For a small raise the spiral loses almost nothing. This is why a weak engine is excellent for station keeping and small corrections, and loses only for large changes of orbit.

(d) Drag is the case $a_T = -D$. Part (b) then gives $\dot{v} = +D$. Drag makes the satellite go *faster*. Its total energy falls at the rate Dv, so the orbit shrinks. As it shrinks the satellite falls deeper into the well. The potential energy falls at twice the rate the total falls, $2Dv$. Half of that pays for the drag losses, and the other half becomes kinetic energy. This is the "satellite drag paradox". It is why a decaying satellite speeds up as it comes down.

(e) The circular speeds are $v_{c_1} = \sqrt{\mu/6678.137} = 7.726$ km/s and $v_{c_2} = \sqrt{\mu/42\,164.17} = 3.075$ km/s. So

$$\Delta v_{\mathrm{spiral}} = 7.726 - 3.075 = 4.651 \text{ km/s}, \qquad \Delta v_{\mathrm{H}} = 3.893 \text{ km/s},$$

and the spiral needs 19.5% more. At $a_T = 10^{-7}$ km/s^2 it takes $4.651/10^{-7}$ s $= 538$ days.

The propellant fraction follows from the rocket equation, $m_p/m_0 = 1 - e^{-\Delta v/(I_{sp} g_0)}$. The chemical stage on the Hohmann transfer burns $1 - e^{-3.893/3.138} = 71.1\%$ of its starting mass. The electric thruster on the spiral burns $1 - e^{-4.651/15.69} = 25.7\%$.

The trade-off. The spiral needs more Δv, but its exhaust is five times faster, so it needs about a third of the propellant. The price is time, about a year and a half instead of the five hours of the Hohmann coast (half the period of its ellipse, $\pi\sqrt{a_t^3/\mu}$ with $a_t = (r_1 + r_2)/2$), much of it spent crossing the radiation belts, which degrades solar cells and electronics. The choice between the two is a choice between mass and time.

3. A Hohmann transfer round a toy oblate planet

Problem 21 of the two-body questions showed that an extra attraction δ/r^3 makes the apse line of an orbit turn. The Earth's equatorial bulge does something similar to an orbit in the equatorial plane, since it pulls a little harder close in. We have not studied the bulge, so we use a toy planet whose acceleration is

$$\ddot{\mathbf{r}} = -\Big(\frac{\mu}{r^2} + \frac{\lambda}{r^3}\Big)\hat{\mathbf{r}}, \qquad \lambda > 0 \text{ constant.}$$

a. Show that a circular orbit of radius r has the speed $v_c = \sqrt{\mu/r + \lambda/r^2}$, and that $\varepsilon = \frac{1}{2}v^2 - \mu/r - \lambda/(2r^2)$ is conserved.
b. A transfer orbit touches the circle r_1 and the circle $r_2 > r_1$ tangentially, as in the Hohmann transfer. Show that its angular momentum h_t satisfies

$$h_t^2 - \lambda = \frac{2\mu r_1 r_2}{r_1 + r_2},$$

and that the two burns are $\Delta v_1 = (h_t - h_{c_1})/r_1$ and $\Delta v_2 = (h_{c_2} - h_t)/r_2$, where $h_c = \sqrt{\mu r + \lambda}$ is the angular momentum of the circle of radius r.
c. Show, with Binet's equation, that the transfer orbit is $r = p'/(1 + e\cos k\theta)$ with $k^2 = 1 - \lambda/h_t^2$. Hence show that the spacecraft reaches r_2 after turning through π/k, more than 180°.
d. Does the extra attraction make the transfer cheaper or dearer? Find $d\Delta v/d\lambda$ at $\lambda = 0$ and settle its sign without numbers.
e. For a transfer from 300 km altitude to the geostationary radius 42 164.17 km, take $\lambda = 3J_2\mu R_\oplus^2/p_t$ with $p_t = 2r_1r_2/(r_1 + r_2)$. This choice makes the apse line of the transfer ellipse turn at the rate $3\pi J_2(R_\oplus/p)^2$ per orbit, which is the rate for an equatorial orbit about the real Earth (from the rates of Problem 19 of the orbital-elements questions). Find the transfer angle and the change in the total Δv.

Solution. The Hohmann argument used three facts. Energy is conserved, angular momentum is conserved, and the transfer orbit has its apsides on the two circles. All three survive the extra force, because it is still central and still comes from a potential. What changes is the shape of the transfer orbit. It is no longer an ellipse with apsides 180° apart. So we redo the Hohmann calculation with the new energy integral (a, b), find the new shape from Binet's equation (c), and then ask about the cost (d).

(a) *Circular speed.* On a circle the acceleration is v^2/r towards the centre, so $v_c^2/r = \mu/r^2 + \lambda/r^3$, and $v_c^2 = \mu/r + \lambda/r^2$.

Energy. The acceleration is minus the gradient of the potential $U = -\mu/r - \lambda/(2r^2)$, since $-dU/dr = -\mu/r^2 - \lambda/r^3$. The argument for gravity then goes through unchanged. The rate of change of the kinetic energy is $\mathbf{v}\cdot\ddot{\mathbf{r}} = -\dot{r}\,dU/dr = -\dot{U}$, so $\frac{1}{2}v^2 + U$ is constant.

(b) At an apsis the velocity is transverse, $v = h/r$. The transfer orbit has its apsides at r_1 and r_2, so the energy at both must agree,

$$\frac{h_t^2}{2r_1^2} - \frac{\mu}{r_1} - \frac{\lambda}{2r_1^2} = \frac{h_t^2}{2r_2^2} - \frac{\mu}{r_2} - \frac{\lambda}{2r_2^2}.$$

Collect the terms. The quantity $h_t^2 - \lambda$ appears together,

$$\frac{h_t^2 - \lambda}{2}\Big(\frac{1}{r_1^2} - \frac{1}{r_2^2}\Big) = \mu\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big).$$

Divide by $\frac{1}{r_1}-\frac{1}{r_2}$, and use $\frac{1}{r_1^2}-\frac{1}{r_2^2}=(\frac{1}{r_1}-\frac{1}{r_2})(\frac{1}{r_1}+\frac{1}{r_2})$,

$$\frac{h_t^2-\lambda}{2}\cdot\frac{r_1+r_2}{r_1r_2}=\mu \quad\Longrightarrow\quad h_t^2-\lambda=\frac{2\mu r_1r_2}{r_1+r_2}.$$

For $\lambda=0$ this is the Hohmann value $h_0^2=2\mu r_1r_2/(r_1+r_2)$. The extra force simply adds λ to h^2.

The burns. All velocities at the burn points are transverse, so each speed is an angular momentum divided by the radius. On the circle $h_c=r\,v_c=\sqrt{\mu r+\lambda}$. So $\Delta v_1=(h_t-h_{c_1})/r_1$ and $\Delta v_2=(h_{c_2}-h_t)/r_2$. Notice that λ is added to every h^2 in the problem: $h_t^2=h_0^2+\lambda$ and $h_c^2=\mu r+\lambda$.

(c) With $u=1/r$ and the angular momentum h, Binet's equation for a central acceleration $f(u)$ towards the centre is $u''+u=f/(h^2u^2)$, where the prime is $d/d\theta$. Here $f=\mu u^2+\lambda u^3$, so

$$u''+u=\frac{\mu}{h^2}+\frac{\lambda}{h^2}\,u \quad\Longrightarrow\quad u''+k^2u=\frac{\mu}{h^2},\qquad k^2=1-\frac{\lambda}{h^2}.$$

This is the equation of a harmonic oscillator in θ with frequency k. Its solution is $u=\mu/(h^2k^2)+C\cos k\theta$, which is $r=p'/(1+e\cos k\theta)$ with $p'=h^2k^2/\mu$ and $e=Cp'$. The apsides are where $u'=0$, at $k\theta=0$ and $k\theta=\pi$. So the transfer from the inner apsis to the outer one turns through

$$\Delta\theta=\frac{\pi}{k}=\frac{\pi}{\sqrt{1-\lambda/h_t^2}}>\pi.$$

For small λ, $\Delta\theta\approx\pi(1+\lambda/(2h_t^2))$. The extra pull bends the path a little more, so the spacecraft has to go further round before it starts to come back in.

(d) We differentiate $\Delta v=\Delta v_1+\Delta v_2$ with respect to λ. Each h depends on λ through $h^2=(\ldots)+\lambda$, so $dh/d\lambda=1/(2h)$. At $\lambda=0$, with $h_{c_1}=\sqrt{\mu r_1}$ and $h_{c_2}=\sqrt{\mu r_2}$,

$$\left.\frac{d\Delta v}{d\lambda}\right|_{\lambda=0}=\frac{1}{2r_1}\Big(\frac{1}{h_0}-\frac{1}{h_{c_1}}\Big)+\frac{1}{2r_2}\Big(\frac{1}{h_{c_2}}-\frac{1}{h_0}\Big).$$

Now $h_0^2=\mu\cdot 2r_1r_2/(r_1+r_2)$, and $2r_1r_2/(r_1+r_2)$ is the harmonic mean of r_1 and r_2, which lies between them. So $h_{c_1}<h_0<h_{c_2}$. The first bracket is then negative $(1/h_0<1/h_{c_1})$, and so is the second $(1/h_{c_2}<1/h_0)$. So $d\Delta v/d\lambda<0$. **The extra attraction makes the transfer cheaper.**

Why. Adding the same λ to every h^2 raises a small h more than a large one, since $dh=\lambda/(2h)$. At the first burn the circle has the smaller h, so it gains more than the transfer orbit, and the gap $h_t-h_{c_1}$ shrinks. At the second burn the transfer orbit has the smaller h, so it gains more than the outer circle, and that gap shrinks too.

(e) With $r_1=6678.137$ km and $r_2=42\,164.17$ km, $p_t=11\,530$ km and $\lambda=3J_2\mu R_\oplus^2/p_t=4.568\times10^6$ km^4/s^2. Then $k=\sqrt{1-\lambda/h_t^2}$ gives

$$\Delta\theta=180.089^\circ,\qquad \Delta v=3.8906\ \text{km/s}\ \ (\text{against } 3.8926\ \text{km/s for } \lambda=0).$$

So the transfer is 1.96 m/s cheaper (the slope of (d) predicts 1.97 m/s), and the apogee is reached 0.089° beyond the point opposite the first burn. At the geostationary radius that is 66 km further along the orbit.

What this means for a mission. The change in cost is negligible. The change in geometry is not. A spacecraft that fires its apogee motor 180° after perigee would fire 66 km early, at the wrong point. In practice the apogee burn is timed from the tracked orbit and not from a Kepler ellipse, which takes care of this and of everything else the point-mass model leaves out.

4. Moving the node by waiting

One launch often places several satellites in the same orbit, while the operator wants them in different planes, with the same inclination but different right ascensions of the node Ω. Problem 5 of the plane-changes questions shows that one impulse moving the node by $\Delta\Omega$ costs $\Delta v = 2v \sin i \sin(\Delta\Omega/2)$. Here we use a slow turning of the node instead. We have not studied what causes it (the Earth's oblateness), so we take the result as given. The node of a circular orbit of radius a and inclination i turns at

$$\dot\Omega = -\frac{K\cos i}{a^{7/2}}, \qquad K = \tfrac{3}{2}J_2 R_\oplus^2\sqrt{\mu} \ \text{(a constant)}.$$

a. Show that $d\dot\Omega/da = -\frac{7}{2}\,\dot\Omega/a$. One satellite moves from radius r to $r+\delta r$ by a Hohmann transfer, waits there for a time T, and comes back, while its companion stays at r. Show that, to first order in $\delta r/r$ and neglecting the time spent on the transfers, the cost is $\Delta v = v\,\delta r/r$ and the node moves relative to the companion by $\Delta\Omega = \frac{7}{2}|\dot\Omega|\,(\delta r/r)\,T$.

b. Hence show that
$$\Delta v \cdot T = \frac{2}{7}\,\frac{v\,\Delta\Omega}{|\dot\Omega|},$$
independent of δr. What does this say?

c. For $r = R_\oplus + 500$ km, $i = 97.4°$ (close to sun-synchronous) and $\Delta\Omega = 10°$, find $\dot\Omega$, the cost of a single impulse, the value of $\Delta v \cdot T$, and the cost and time for $\delta r = 50$ km. What does it cost to do it in 60 days?

d. To *increase* Ω relative to the companion, should the satellite go up or down?

Solution. The idea is to let the Earth do the work. Two orbits at slightly different heights have slightly different node rates, so their nodes slowly separate. We pay only to change the height and to change it back, and the waiting does the rest. The plan is to find how the node rate depends on height (a), how much the height change costs (a), and then to combine the two (b).

(a) *The rate.* $\dot\Omega = -K\cos i\, a^{-7/2}$, so $d\dot\Omega/da = -\frac{7}{2}(-K\cos i\, a^{-9/2}) = -\frac{7}{2}\,\dot\Omega/a$. A small change δr of the radius changes the rate by $\delta\dot\Omega = -\frac{7}{2}\dot\Omega\,\delta r/r$. Its size is $\frac{7}{2}|\dot\Omega|\,\delta r/r$, so in time T the nodes separate by

$$\Delta\Omega = \tfrac{7}{2}\,|\dot\Omega|\,\frac{\delta r}{r}\,T.$$

The cost. For a Hohmann transfer from r to $r(1+\epsilon)$ the first burn is $v(\sqrt{2(1+\epsilon)/(2+\epsilon)} - 1)$. To first order $2(1+\epsilon)/(2+\epsilon) \approx 1 + \epsilon/2$, so the burn is $v\epsilon/4$. The second burn is also $v\epsilon/4$ to first order. (Both agree with part (c) of Problem 2. To first order the Hohmann transfer costs the same as the spiral, $v_{c_1} - v_{c_2} \approx v\epsilon/2$.) Going up and coming back costs twice this,

$$\Delta v = v\,\frac{\delta r}{r}.$$

(b) Eliminate $\delta r/r$ between the two results of (a). From the first, $\delta r/r = 2\Delta\Omega/(7|\dot\Omega|T)$, so

$$\Delta v = \frac{2}{7}\,\frac{v\,\Delta\Omega}{|\dot\Omega|\,T}, \qquad \Delta v \cdot T = \frac{2}{7}\,\frac{v\,\Delta\Omega}{|\dot\Omega|}.$$

The product of fuel and time is fixed. Halving the time doubles the cost. We cannot get the separation both quickly and cheaply, but we can choose any point on this curve.

(**c**) With $r = 6878.137$ km, $v = \sqrt{\mu/r} = 7.613$ km/s and $n = \sqrt{\mu/r^3}$, the rate is $\dot\Omega = 0.985^\circ$/day. It is positive because $\cos i < 0$, and it is close to the sun-synchronous value 0.9856°/day, as intended.

A single impulse costs $2v \sin i \sin 5^\circ = 1316$ m/s. That is more than a small satellite could ever carry.

By waiting, $\Delta v \cdot T = \frac{2}{7}\,v\,\Delta\Omega/|\dot\Omega| = 22\,070$ (m/s)·day. With $\delta r = 50$ km the first-order formulas give $\Delta v = 55.3$ m/s and $T = 399$ days. The exact Hohmann costs and the exact rate difference give 55.0 m/s and 405 days, so the first-order theory is good to about 1%. To finish in 60 days instead costs about $22\,070/60 = 368$ m/s. That needs $\delta r \approx 330$ km, where the first-order formulas are only a rough guide.

The trade-off. Waiting saves a factor of more than twenty in Δv against the impulse, and the price is a year. Large constellations use exactly this to spread the satellites of one launch into several planes. There is also a cost that is not in the formula. During the wait the drifting satellite is not on its working orbit, and for a sun-synchronous orbit it also drifts in local time.

(**d**) Here $\dot\Omega > 0$ and $|\dot\Omega| \propto a^{-7/2}$ grows as a falls. A lower satellite turns its node faster in the positive sense, so to increase Ω relative to the companion, go **down**. In general, a lower orbit turns its node faster in the direction it already turns. For a prograde orbit ($\dot\Omega < 0$), going down makes Ω *decrease* relative to the companion.

5. Why the cheapest disorbit is a horizontal retro burn

Part II of the advanced-manoeuvres note finds the cheapest single impulse that takes a capsule from a circular orbit of radius r_1 to the entry radius r_2 at the entry flight-path angle γ_2. It observes from the chart that the cheapest choice of altitude always has a *horizontal* burn. Here we prove this, and then refine the answer. Use the note's notation: $m = r_1/r_2$, the post-burn state $(u, w) = (v_\theta, v_r)/v_{c_1}$, $K = m^2/\cos^2\gamma_2 - 1$, $M = 2(m-1)$, and the cost $J = (\Delta v/v_{c_1})^2 = (K+1)u^2 - 2u + 1 - M$. Write $s = |\sin\gamma_2|$ and $c = \cos\gamma_2$.

a. Where the vertex $u = 1/(K+1)$ is feasible (the tilted regime), show that the least cost is $J_v(m) = 1 - 2(m-1) - c^2/m^2$, and that J_v decreases as m grows, for every $m \geq 1$. Why does this mean that the cheapest altitude cannot lie inside the tilted regime?

b. In the horizontal regime ($w = 0$) the note shows that the cost is smallest at $m = 1 + s$. Show that at $m = 1 + s$, $K + 1 = (1+s)/(1-s)$ and $M/K = 1 - s$, and hence that the vertex is infeasible there for every entry angle. Conclude that the cheapest disorbit, over all altitudes, is a horizontal retro burn at $r_1 = (1+s)\,r_2$.

c. The note's optimum is in units of v_{c_1}, which falls as r_1 grows. For a fixed entry radius the cost in km/s is $v_{c_2}\,j(m)/\sqrt{m}$, with $j = \Delta v/v_{c_1}$. Show that the best altitude in km/s satisfies $j'(m) = j(m)/(2m)$. Show that $j(1+s) = 1 - \sqrt{1-s}$. Then, taking $j''(1+s) = \sqrt{1-s^2}/(2s(1+s)^{3/2})$ as given, show that the optimum moves up by $\delta \approx s^2/2$ for shallow entries, $m_{\text{opt}} \approx 1 + s + s^2/2$.

d. Evaluate for $\gamma_2 = -6^\circ$ and an entry interface at 100 km altitude.

Solution. The chart in the note shows the claim, but a chart can only show it for the entry angles that were drawn. To prove it for every angle we need two facts. Inside the tilted regime the cost always falls as the orbit is raised, so the minimum cannot be there (a). And the minimum of the horizontal cost lies in the horizontal regime (b). Part (c) then uses a small-shift argument, a technique that is useful whenever an optimum is perturbed by a small extra factor.

(a) The cost is a parabola in u. At its vertex $u = 1/(K+1)$,

$$J_v = \frac{K+1}{(K+1)^2} - \frac{2}{K+1} + 1 - M = 1 - M - \frac{1}{K+1} = 1 - 2(m-1) - \frac{c^2}{m^2},$$

since $K + 1 = m^2/c^2$. Its derivative is

$$\frac{dJ_v}{dm} = -2 + \frac{2c^2}{m^3} < 0,$$

because $c^2 \le 1 \le m^3$ (with equality only in the trivial case $\gamma_2 = 0$, $m = 1$). So inside the tilted regime the cost always falls when m grows. The tilted regime is defined by a strict inequality between continuous functions of m, so a little larger m is still tilted, or has just crossed the boundary, where the two costs are equal. Either way it is cheaper. So no point inside the tilted regime can be the cheapest.

(b) At $m = 1 + s$, with $c^2 = 1 - s^2 = (1-s)(1+s)$,

$$K + 1 = \frac{(1+s)^2}{(1-s)(1+s)} = \frac{1+s}{1-s}, \qquad K = \frac{2s}{1-s}, \qquad M = 2s, \qquad \frac{M}{K} = 1 - s.$$

The vertex is feasible when $1/(K+1) \ge \sqrt{M/K}$, that is when $(1-s)/(1+s) \ge \sqrt{1-s}$. Divide by $\sqrt{1-s} > 0$. The condition becomes $\sqrt{1-s} \ge 1 + s$, which is false for every $s > 0$, since the left side is below 1 and the right side above it. So at $m = 1 + s$ the vertex is infeasible and the burn is horizontal.

The global result. By (a) the cheapest altitude is not inside the tilted regime. In the horizontal regime the cost is the horizontal formula, whose smallest value over all $m > 1$ is at $m = 1 + s$. By what we just showed, $m = 1 + s$ really is in the horizontal regime, so the horizontal formula applies there. Every tilted m costs more than the boundary point beyond it, and that point is horizontal, so it costs at least as much as $m = 1 + s$. Hence the cheapest disorbit over all altitudes is the horizontal retro burn from $r_1 = (1+s)\, r_2$. (The note's 1.125 rule guarantees that the boundary point exists, since every $m > 9/8$ is horizontal.)

(c) For fixed r_2 we minimise $F(m) = j(m)/\sqrt{m}$. Its derivative is $(j'\sqrt{m} - j/(2\sqrt{m}))/m$, which is zero when

$$j'(m) = \frac{j(m)}{2m}.$$

The value at the old optimum. In the horizontal regime $j = 1 - c\sqrt{2(m-1)/(m^2-c^2)}$. At $m = 1 + s$, $m^2 - c^2 = (1+s)^2 - (1-s^2) = 2s(1+s)$, so $2(m-1)/(m^2-c^2) = 1/(1+s)$, and

$$j(1+s) = 1 - \frac{\sqrt{1-s^2}}{\sqrt{1+s}} = 1 - \sqrt{1-s}.$$

The shift. Write $m^* = 1 + s$ and $m = m^* + \delta$. Since $j'(m^*) = 0$, Taylor's theorem gives $j'(m^* + \delta) \approx j''(m^*)\,\delta$. The right side $j/(2m)$ changes much more slowly, so we evaluate it at m^*. Then

$$\delta \approx \frac{j(m^*)}{2m^*\, j''(m^*)} = \frac{(1 - \sqrt{1-s}) \cdot 2s(1+s)^{3/2}}{2(1+s)\sqrt{1-s^2}} = \frac{s\,(1 - \sqrt{1-s})}{\sqrt{1-s}}.$$

For a shallow entry $1 - \sqrt{1-s} \approx s/2$ and $\sqrt{1-s} \approx 1$, so $\delta \approx s^2/2$. The extra factor $1/\sqrt{m}$ tilts the curve slightly in favour of higher orbits. The tilt is of size $j/(2m) \approx s/4$, and the curvature $j'' \approx 1/(2s)$ is large, so the optimum moves only a little.

(d) $s = \sin 6^\circ = 0.10453$ and $r_2 = 6478.137$ km. The best altitude in units of v_{c_1} is $m = 1.10453$, so $r_1 = 7155.3$ km (altitude 777 km), where $v_{c_1} = 7.4637$ km/s. The cost is $j = 1 - \sqrt{1-s} = 0.053706$, or 401 m/s.

The best altitude in km/s is at $m_{\text{opt}} \approx 1 + s + s^2/2 = 1.110$. A direct search gives 1.111 (altitude 819 km) with a cost of 400.3 m/s.

What this means for a mission. The optimum is very flat. Moving the parking orbit up by 40 km saves less than 1 m/s. So the choice of the parking altitude is made on other grounds, such as the lifetime against drag or the ground track, and the deorbit cost hardly enters it. What the proof does tell a designer is that a tilted burn is never needed when the altitude is free.

6. What a flyby cannot change

In the patched-conic model of a flyby the speed relative to the planet, v_∞, is the same before and after. Only its direction turns. So the heliocentric orbit after the flyby depends on that direction, and it seems that almost any orbit can be reached. It cannot. Here we find the one combination of the heliocentric elements that a flyby cannot change.

Work in the plane of a planet that moves on a circle of radius a_p about the Sun with speed $V_p = \sqrt{\mu/a_p}$ and angular rate $\omega_p = V_p/a_p$ ($\mu = \mu_\odot$). The flyby happens at the planet's position and takes no time. The heliocentric velocity is $\mathbf{V}_p + \mathbf{v}_\infty$ before and after, with the same $|\mathbf{v}_\infty|$. Angular momenta are counted positive in the sense of the planet's motion.

a. Show that $C = \varepsilon - \omega_p h$, where ε and h are the heliocentric energy and angular momentum per unit mass, is the same before and after the flyby, and that $C = \frac{1}{2}v_\infty^2 - \frac{3}{2}V_p^2$.
b. Write ε and h with the semi-major axis a and eccentricity e of a prograde heliocentric orbit, and show **Tisserand's relation**

$$\frac{a_p}{a} + 2\sqrt{\frac{a(1-e^2)}{a_p}} = 3 - \frac{v_\infty^2}{V_p^2}.$$

c. Show that one flyby can put the spacecraft on an escape orbit from the Sun only if $v_\infty \geq (\sqrt{2} - 1)\,V_p$.
d. A spacecraft arrives at Jupiter on a Hohmann transfer from the Earth (circular coplanar orbits, $a_J = 5.2026$ AU). Find v_∞ and check the condition of (c). The flyby turns $\mathbf{v}_\infty$ through at most $\delta = 2\arcsin(1/(1 + r_p v_\infty^2/\mu_J))$, with the periapsis radius r_p no less than Jupiter's radius $R_J = 71\,492$ km ($\mu_J = 1.26687 \times 10^8$ km^3/s^2). Can one flyby of Jupiter send this spacecraft out of the Solar System?

Solution. The flyby can change the direction of $\mathbf{v}_\infty$ but not its size. So any quantity that depends on $|\mathbf{v}_\infty|$ alone is unchanged by the flyby. The plan is to look for such a combination of ε and h (a), and then to write it in terms of the elements (b). The weight ω_p in front of h is the one that makes the direction drop out.

(a) Write $\mathbf{v}_\infty$ in components at the planet, v_t along the planet's motion and v_r along the radius from the Sun. The heliocentric velocity has components $(V_p + v_t,\ v_r)$ and the distance from the Sun is a_p. So

$$\varepsilon = \tfrac{1}{2}\big[(V_p + v_t)^2 + v_r^2\big] - \frac{\mu}{a_p}, \qquad h = a_p\,(V_p + v_t).$$

Expand the energy, $\varepsilon = \frac{1}{2}V_p^2 + V_p v_t + \frac{1}{2}v_\infty^2 - \mu/a_p$, with $v_\infty^2 = v_t^2 + v_r^2$. The term $V_p v_t$ depends on the direction of $\mathbf{v}_\infty$. The angular momentum carries the same term, since $\omega_p h = (V_p/a_p)\,a_p(V_p + v_t) = V_p^2 + V_p v_t$. So the difference loses it,

$$C = \varepsilon - \omega_p h = \tfrac{1}{2}v_\infty^2 - \tfrac{1}{2}V_p^2 - \frac{\mu}{a_p} = \tfrac{1}{2}v_\infty^2 - \tfrac{3}{2}V_p^2,$$

using $\mu/a_p = V_p^2$. The right side depends on $|\mathbf{v}_\infty|$ only, which the flyby does not change, so C is the same before and after.

(b) For a heliocentric orbit $\varepsilon = -\mu/(2a)$ and $h = \sqrt{\mu a(1-e^2)}$. Multiply C by $-2a_p/\mu = -2/V_p^2$. The energy term becomes a_p/a. The angular-momentum term becomes

$$\frac{2a_p\omega_p}{\mu}\,h = \frac{2V_p}{\mu}\sqrt{\mu a(1-e^2)} = 2\sqrt{\frac{\mu}{a_p}}\,\frac{\sqrt{\mu a(1-e^2)}}{\mu} = 2\sqrt{\frac{a(1-e^2)}{a_p}},$$

and the right side becomes $3 - v_\infty^2/V_p^2$. This is Tisserand's relation. All the orbits that a flyby can produce from a given arrival lie on one curve in the (a, e) plane.

History. F. Tisserand (1889) used this relation to decide whether a newly found comet was an old one whose orbit Jupiter had changed. The two orbits must give the same value of the left side. The quantity C is the Jacobi constant of the restricted three-body problem with the planet's own potential left out. The full Jacobi constant, which includes it, is exactly conserved in that problem.

(c) The orbit is unbound when $\varepsilon \geq 0$. At the planet's distance this needs a heliocentric speed of at least $\sqrt{2\mu/a_p} = \sqrt{2}\,V_p$. After the flyby the heliocentric speed is $|\mathbf{V}_p + \mathbf{v}_\infty| \leq V_p + v_\infty$, with equality when $\mathbf{v}_\infty$ points along $\mathbf{V}_p$. So escape needs $V_p + v_\infty \geq \sqrt{2}\,V_p$, that is $v_\infty \geq (\sqrt{2}-1)V_p = 0.414\,V_p$.

(d) Jupiter moves at $V_J = \sqrt{\mu_\odot/a_J} = 13.06$ km/s. The Hohmann ellipse from 1 AU has its aphelion at a_J, where its speed is 7.41 km/s. So the spacecraft arrives slower than Jupiter, with $v_\infty = 5.64$ km/s pointing *backwards*, opposite to $\mathbf{V}_J$. Then $v_\infty/V_J = 0.432$, just above 0.414. As a check, the Hohmann orbit has Tisserand parameter 2.813, and $3 - 0.432^2 = 2.813$.

So the energy is available. Is the turn? If $\mathbf{v}_\infty$ makes the angle θ with $\mathbf{V}_J$ after the flyby, escape needs $V_J^2 + v_\infty^2 + 2V_J v_\infty\cos\theta \geq 2V_J^2$, so

$$\cos\theta \geq \frac{V_J^2 - v_\infty^2}{2V_J v_\infty} = 0.941, \qquad \theta \leq 19.8^\circ.$$

The arriving $\mathbf{v}_\infty$ is at $\theta = 180°$, so the flyby must turn it through at least 160.2°. A pass grazing the cloud tops has $1 + R_J v_\infty^2/\mu_J = 1.018$ and turns it through at most 158.4°. That is not quite enough. Escape would need a periapsis at $0.84\,R_J$, inside the planet. **So no**, although it fails by less than two degrees.

What this means for a mission. A spacecraft that reaches Jupiter on the slowest possible route cannot be thrown out of the Solar System by it. To leave, it must arrive with a larger v_∞, which means a faster, more energetic transfer from the Earth. Pioneer 10 reached Jupiter in under two years, against 2.7 years for the Hohmann transfer (half the period of its ellipse, $\pi\sqrt{a_t^3/\mu_\odot}$ with $a_t = 3.10$ AU), and left the Solar System after its flyby. Tisserand's relation is the first tool of a tour designer. It shows at once which orbits a sequence of flybys can reach, before any trajectory is computed.

7. Two radii for one sphere

The patched-conics note defines the sphere of influence of a planet of mass m that moves on a circle of radius a about a star of mass M by $r_{\mathrm{SOI}} = a\,(m/M)^{2/5}$. Another radius is often quoted for the same idea, the **Hill radius**. Here we derive it, compare the two, and see why they differ. Let the planet go round at the angular rate n, with $n^2 = G(M+m)/a^3 \approx GM/a^3$.

a. Use a frame that rotates with the planet at the rate n. A particle at rest in this frame on the star–planet line, a small distance x beyond the planet ($x \ll a$), feels the star's gravity, the centrifugal acceleration and the planet's pull. Show that the first two add up to $3n^2x$ away from the planet, to first order in x/a, and that the planet's pull balances them at

$$r_H = a\Big(\frac{m}{3M}\Big)^{1/3}.$$

b. Show that $r_H/r_{\mathrm{SOI}} = 3^{-1/3}(m/M)^{-1/15}$ and that the two radii are equal when $m/M = 1/243$. Which radius is larger for the planets, and which for the Moon about the Earth?

c. Evaluate both radii for the Earth about the Sun ($M/m = 332\,946$), the Moon about the Earth ($M/m = 81.30$, $a = 384\,400$ km) and Jupiter about the Sun ($M/m = 1047.35$).

d. Aditya-L1 flies round the Sun–Earth L_1 point, about 1.5 million km from the Earth towards the Sun. Is it inside the Earth's sphere of influence? What does (a) say about where L_1 is?

Solution. The two radii answer different questions. The sphere of influence asks around which body a conic makes the smaller error. The Hill radius asks where the planet's pull is strong enough to hold a particle against the star, once the motion round the star is taken into account. The second question is easiest in a frame that rotates with the planet, because there the planet and the star stand still. That is the technique of this problem.

(a) In the rotating frame a particle at rest feels the real forces and the centrifugal acceleration, n^2 times its distance from the centre of rotation. Since $m \ll M$ we take that centre at the star. At distance $a + x$ from the star,

$$\text{star: } -\frac{GM}{(a+x)^2} \approx -\frac{GM}{a^2} + \frac{2GM}{a^3}\,x, \qquad \text{centrifugal: } n^2(a+x) = \frac{GM}{a^2} + n^2x,$$

with the positive direction pointing away from the star. The two GM/a^2 terms cancel, which is just the statement that the planet itself is in orbit. What is left is

$$\frac{2GM}{a^3}\,x + n^2x = 2n^2x + n^2x = 3n^2x.$$

Two thirds of it is the **tidal** acceleration (the star pulls the near side harder than the far side) and one third is the centrifugal acceleration of the extra distance x. It pushes the particle away from the planet. The planet pulls it back with Gm/x^2. The two balance where

$$\frac{Gm}{x^3} = 3n^2 = \frac{3GM}{a^3} \quad \Longrightarrow \quad x = r_H = a\Big(\frac{m}{3M}\Big)^{1/3}.$$

The same calculation on the near side gives the same distance to first order.

(b) Divide, $r_H/r_{\mathrm{SOI}} = 3^{-1/3}(m/M)^{1/3-2/5} = 3^{-1/3}(m/M)^{-1/15}$. The two are equal when $(m/M)^{1/15} = 3^{-1/3}$, that is $m/M = 3^{-5} = 1/243$. A smaller mass ratio makes the ratio larger, so for $m/M < 1/243$ the Hill radius is the larger one. Every planet has m/M well below $1/243$ (the largest is Jupiter, $1/1047$), so for the planets $r_H > r_{\mathrm{SOI}}$. The Moon has $m/M = 1/81.3$, above $1/243$, so for the Moon $r_H < r_{\mathrm{SOI}}$.

(c)

	r_H	r_{SOI}	r_H/r_{SOI}
Earth about the Sun	1.497×10^6 km	0.925×10^6 km	1.62
Moon about the Earth	61 500 km	66 200 km	0.93
Jupiter about the Sun	5.31×10^7 km	4.82×10^7 km	1.10

(d) The point where the forces of (a) balance, on the star side, is where a particle can stay at rest in the rotating frame. That is the definition of the Lagrange point L_1, so to first order L_1 lies at the Hill radius, 1.50×10^6 km from the Earth (an exact solution of the restricted three-body problem gives 1.49×10^6 km). Aditya-L1 is therefore about 1.5×10^6 km from the Earth, well **outside** the Earth's sphere of influence of 0.925×10^6 km.

What this means for a mission. In patched-conic terms Aditya-L1 is on a heliocentric orbit. Yet it stays near the Earth, in step with it, for years. The patched-conic model cannot describe this, because the spacecraft lives exactly where the pulls of the Sun and the Earth, and the rotation, balance, and neither conic is a good approximation. Designing such an orbit needs the circular restricted three-body problem. The two radii are useful for different things. The sphere of influence decides where to switch conics on an interplanetary transfer. The Hill radius tells us how far out a planet (or the Moon) can hold a satellite at all.

8. Fire straight up from the equator

The ballistic note brings in the rotation of the Earth at the end. Here is a case that can be solved exactly. A projectile leaves a point on the equator (radius R) vertically, with speed v relative to the ground. There is no air, the burn is instantaneous at the surface, and the Earth is a sphere turning at ω (we write ω for $\omega_\oplus$ and $g = \mu/R^2$). Where does it come down?

a. Give the inertial velocity at launch, and show that the angular momentum is $h = \omega R^2$.
b. Show that the angle the projectile turns through in inertial space is $\psi = h\int dt/r^2$, taken over the flight, and that it comes down east of the launch site by the angle

$$\Delta = -\omega \int_0^{t_f} \Big(1 - \frac{R^2}{r^2}\Big)\, dt.$$

Conclude that it always lands *west* of the site, for any v below escape speed. Explain this physically.

c. Since h is tiny, the radial motion is, to first order in ω, that of a vertical flight with $h = 0$. Use the energy integral to write $\int dt/r^2 = \int dr/(r^2\dot{r})$, and show that $R^2 \int dt/r^2 = 2v/g$ exactly. Hence $\psi = 2v\omega/g$.

d. The vertical flight is a degenerate ellipse. With $r = a(1 - \cos E)$ and $t = \sqrt{a^3/\mu}\,(E - \sin E)$, show that its flight time is

$$t_f = 2\sqrt{\frac{a^3}{\mu}}\big[\arccos(1-q) + \sqrt{q(2-q)}\big], \qquad a = \frac{R}{2-q}, \quad q = \frac{v^2 R}{\mu},$$

and that $t_f \approx (2v/g)(1 + \frac{2}{3}q + \frac{2}{5}q^2)$. Hence show that the projectile lands west of the site by the distance

$$d = \omega R\big(t_f - \frac{2v}{g}\big) \approx \frac{4}{3}\,\frac{\omega v^3}{g^2}\big(1 + \frac{3}{5}\,q\big).$$

The leading term is the textbook result for the Coriolis deflection.

e. Evaluate d for $v = 2$ km/s and $v = 5$ km/s, and compare with the textbook term.

Solution. We expect a small deflection, so we want a way to get it without solving the whole conic. The idea is to follow the angular momentum. Gravity is central, so the angular momentum the projectile is given at launch stays with it. That tells us at every instant how fast the projectile turns about the Earth's centre, and we compare it with how fast the ground turns.

(**a**) The site moves east at ωR. So the inertial velocity has a radial component v and a transverse (eastward) component ωR, and $h = R \cdot \omega R = \omega R^2$. The projectile moves in the equatorial plane, because that plane contains its position and its velocity.

(**b**) The angular rate of the projectile about the centre is $\dot{\theta} = h/r^2$. So in the flight time t_f it turns through $\psi = h\int_0^{t_f} dt/r^2$. Meanwhile the launch site turns through ωt_f. The landing point is east of the site by

$$\Delta = \psi - \omega t_f = \omega \int_0^{t_f} \frac{R^2}{r^2}\,dt - \omega \int_0^{t_f} dt = -\omega \int_0^{t_f} \big(1 - \frac{R^2}{r^2}\big)dt.$$

During the flight $r > R$, so the integrand is positive and $\Delta < 0$. **It always lands west.**

Why. At launch the projectile turns with the ground, at the rate ω. Its angular momentum ωR^2 cannot change, so as it rises its angular rate $\omega R^2/r^2$ falls below ω, just as a spinning skater slows down by stretching out the arms. At every moment above the surface it falls behind the ground beneath it.

(**c**) For the vertical flight the energy integral gives $\frac{1}{2}\dot{r}^2 - \mu/r = \frac{1}{2}v^2 - \mu/R$. Let r_a be the top of the flight, where $\dot{r} = 0$, so $\mu/r_a = \mu/R - \frac{1}{2}v^2$. Then $\dot{r}^2 = 2\mu(1/r - 1/r_a)$. On the way up $dt = dr/\dot{r}$, so

$$\int_{\text{up}} \frac{dt}{r^2} = \int_R^{r_a} \frac{dr}{r^2\sqrt{2\mu(1/r - 1/r_a)}}.$$

Substitute $u = 1/r$, so that $du = -dr/r^2$,

$$\int_{1/r_a}^{1/R} \frac{du}{\sqrt{2\mu(u - 1/r_a)}} = \sqrt{\frac{2}{\mu}\big(\frac{1}{R} - \frac{1}{r_a}\big)} = \sqrt{\frac{2}{\mu}\cdot\frac{v^2}{2\mu}} = \frac{v}{\mu}.$$

The way down gives the same, so $R^2 \int dt/r^2 = 2vR^2/\mu = 2v/g$, with no approximation in the vertical flight. So

$$\psi = \frac{2v\omega}{g}.$$

Reading. The projectile travels east, in inertial space, exactly as far as the ground travels in the time $2v/g$, which is the flight time on a flat Earth with constant gravity. The actual flight time is longer, because gravity weakens with height, so the ground runs on further. That difference is the whole deflection, $d = R|\Delta| = \omega R\,(t_f - 2v/g)$.

(d) *The flight time.* With $h = 0$ the ellipse has $e = 1$ and $2a = r_a$. From $\mu/r_a = \mu/R - \frac{1}{2}v^2$ we get $r_a = R/(1 - q/2)$, so $a = R/(2 - q)$. Measure t from the centre, where $E = 0$. At the surface $R = a(1 - \cos E_R)$, so $\cos E_R = 1 - R/a = q - 1$, and the top is at $E = \pi$. The flight up takes $\sqrt{a^3/\mu}\,[(\pi - \sin\pi) - (E_R - \sin E_R)]$, and the flight down takes the same. With $\pi - E_R = \arccos(1 - q)$ and $\sin E_R = \sqrt{1 - (q-1)^2} = \sqrt{q(2-q)}$, this is the formula given.

The expansion. For small q, $\arccos(1 - q) = \sqrt{2q}\,(1 + \frac{q}{12} + \dots)$, $\sqrt{q(2-q)} = \sqrt{2q}\,(1 - \frac{q}{4} + \dots)$ and $(2-q)^{-3/2} = 2^{-3/2}(1 + \frac{3}{4}q + \dots)$. Multiplying,

$$t_f = 2\sqrt{\frac{R^3}{\mu}}\,2^{-3/2}\sqrt{2q}\,\Big(2 - \frac{q}{6}\Big)\Big(1 + \frac{3}{4}q\Big) + \dots = 2\sqrt{\frac{R^3 q}{\mu}}\Big(1 + \frac{2}{3}q + \dots\Big).$$

Since $\sqrt{R^3 q/\mu} = vR^2/\mu = v/g$, this is $t_f = (2v/g)(1 + \frac{2}{3}q + \dots)$. Carrying one more term in each series gives $\frac{2}{5}q^2$. Then

$$d = \omega R \cdot \frac{2v}{g}\Big(\frac{2}{3}q + \frac{2}{5}q^2\Big) = \frac{4}{3}\,\frac{\omega v\,Rq}{g}\Big(1 + \frac{3}{5}q\Big) = \frac{4}{3}\,\frac{\omega v^3}{g^2}\Big(1 + \frac{3}{5}q\Big),$$

using $Rq = v^2/g$.

How good is "first order in ω"? The rotation enters the radial motion only through the transverse speed ωR, whose energy is a fraction $(\omega R)^2/(gR) = 0.0035$ of the scale gR. So the vertical-flight approximation is good to a few parts in a thousand, at any speed.

(e) We use $g = 9.798$ m/s^2 and $\omega R = 0.4651$ km/s. The table collects the results.

v	q	t_f	$2v/g$	d	textbook $\frac{4}{3}\omega v^3/g^2$	exact conic
2 km/s	0.0640	426.3 s	408.2 s	8.42 km W	8.10 km	8.48 km
5 km/s	0.400	1377 s	1021 s	166 km W	127 km	167 km

The last column solves the inertial conic exactly (with the full inertial speed $\sqrt{v^2 + \omega^2 R^2}$), and agrees with d to better than 1%. The textbook term is 4% low at 2 km/s and 24% low at 5 km/s. So the weak assumption in the textbook result is not the small rotation, it is constant gravity.

At a latitude φ. The axial component of the angular momentum is conserved for the same reason, and to first order the projectile stays at its latitude, where the ground moves at $\omega R\cos\varphi$. So the westward distance is $d\cos\varphi$.

History. India's first sounding rocket, a Nike-Apache, was launched from Thumba near Thiruvananthapuram on 21 November 1963. The site was chosen because the magnetic equator passes over it. It is at 8.5°N, so for a vertical shot from Thumba $\cos\varphi = 0.989$.

9. How a geosynchronous satellite wanders

A geosynchronous satellite has a period of one sidereal day. If its orbit is circular and equatorial it stays above one point. The ground-tracks note shows that a small inclination i makes it draw a figure 8 of half-width about $i^2/4$. Here we find the width exactly, add a small eccentricity, and then combine the two. Throughout, $n = \omega_\oplus$.

a. For a circular orbit of inclination i, the longitude relative to the longitude of the node is $\Delta\lambda(u) = \arctan(\cos i \tan u) - u$, where u is the argument of latitude (for u in the first quadrant; this comes from the note's closed form). Find where $\Delta\lambda$ is extreme, and show that, exactly,

$$|\Delta\lambda|_{\max} = \arcsin\Big(\tan^2\frac{i}{2}\Big), \qquad \text{at latitude} \quad \varphi = \arcsin\Big(\sqrt{2}\sin\frac{i}{2}\Big).$$

Evaluate for the NavIC inclination $i = 29°$ and compare with $i^2/4$.

b. For an equatorial orbit of small eccentricity e, show that the longitude relative to the longitude of perigee is $\Delta\lambda = f - M$. Show that the sub-satellite point stands still and turns back where $(1 + e\cos f)^2 = (1 - e^2)^{3/2}$. To first order show that it swings by $\pm 2e$ radians, and say on which side of the mean position it is near perigee.

c. Now let both e and i be small, with i^2 negligible against e. Show that the track is $\Delta\lambda \approx 2e\sin M$, $\varphi \approx i\sin(\omega + M)$, which is an ellipse in the (λ, φ) plane. Show that it becomes a straight segment when $\omega = 0$ or 180°, and a circle of radius i when $i = 2e$ and $\omega = \pm 90°$. Which way round is the circle for $\omega = 90°$?

d. A GSAT satellite at 83°E must stay within $\pm 0.05°$ of its slot in longitude and in latitude. What eccentricity and inclination can be tolerated? Is the figure 8 a concern?

Solution. The satellite and the Earth turn at the same mean rate, so all that is left in the Earth-fixed frame is the difference between the satellite's actual motion and that uniform rate. Inclination makes the motion along the equator uneven, because the satellite's eastward speed is only partly eastward when it is away from the nodes. Eccentricity makes the motion uneven in time, by Kepler's second law. We take the two one at a time, exactly, and then add their first-order effects.

(**a**) *The extremum.* Write $c = \cos i$. Differentiate,

$$\frac{d\Delta\lambda}{du} = \frac{c\sec^2 u}{1 + c^2\tan^2 u} - 1 = \frac{c}{\cos^2 u + c^2\sin^2 u} - 1.$$

It is zero when $\cos^2 u + c^2\sin^2 u = c$, that is $1 - (1 - c^2)\sin^2 u = c$, so

$$\sin^2 u = \frac{1 - c}{1 - c^2} = \frac{1}{1 + c}, \qquad \tan^2 u = \frac{1}{c}.$$

The value. There $\tan u = 1/\sqrt{c}$, so $\arctan(c\tan u) = \arctan\sqrt{c}$ and $u = \arctan(1/\sqrt{c}) = \frac{\pi}{2} - \arctan\sqrt{c}$. Call $\beta = \arctan\sqrt{c}$. Then

$$\Delta\lambda = \beta - \Big(\frac{\pi}{2} - \beta\Big) = -\Big(\frac{\pi}{2} - 2\beta\Big), \qquad \sin|\Delta\lambda| = \cos 2\beta = \frac{1 - \tan^2\beta}{1 + \tan^2\beta} = \frac{1 - c}{1 + c} = \tan^2\frac{i}{2}.$$

So $|\Delta\lambda|_{\max} = \arcsin(\tan^2 \frac{i}{2})$. For small i, $\tan^2(i/2) \approx i^2/4$, which is the note's result.

The latitude. $\sin\varphi = \sin i \sin u = \sin i/\sqrt{1+\cos i}$. With $\sin i = 2\sin\frac{i}{2}\cos\frac{i}{2}$ and $1+\cos i = 2\cos^2\frac{i}{2}$, this is $\sqrt{2}\sin\frac{i}{2}$.

NavIC. For $i = 29°$ the exact half-width is 3.835°, reached at latitude 20.74° (and $u = 46.9°$, not 45°). The small-angle value $i^2/4$ gives 3.670°, about 4% low.

(**b**) On an equatorial orbit the longitude of the satellite is $\varpi + f - \theta(t)$, where ϖ is the longitude of perigee in inertial space and θ is the sidereal angle. The Earth turns at $\omega_\oplus = n$, so $\theta(t) = \theta_p + n(t-\tau) = \theta_p + M$, where τ is the time of perigee. So $\lambda = (\varpi - \theta_p) + f - M$. The first bracket is the fixed longitude of the perigee point on the ground, and $\Delta\lambda = f - M$.

Turning points. The point stops when $\dot f = \dot M = n$. With $\dot f = h/r^2$, $h = na^2\sqrt{1-e^2}$ and $r = a(1-e^2)/(1+e\cos f)$,

$$\dot f = n\,\frac{(1+e\cos f)^2}{(1-e^2)^{3/2}},$$

so it stops where $(1+e\cos f)^2 = (1-e^2)^{3/2}$, that is $\cos f = ((1-e^2)^{3/4}-1)/e \approx -\frac{3}{4}e$, just past 90° from perigee on either side.

The swing. To first order $f - M = 2e\sin M$ (part (a) of Problem 10 derives it), so the point swings between $\pm 2e$ radians, east of the mean position while $0 < M < \pi$. Near perigee the satellite is faster than the ground below and moves east. Near apogee it is slower and moves back west. The second-order term of $f - M$ is $\frac{5}{4}e^2\sin 2M$, which is zero at $M = 90°$ where the swing is largest, so the amplitude $2e$ is correct to second order. For $e = 0.1$ the exact swing is only 0.12% larger.

(**c**) With both effects small, add them. Eccentricity gives $\Delta\lambda = 2e\sin M$, and the inclination's contribution to the longitude is of order i^2, which we neglect. The latitude is $\sin\varphi = \sin i\sin u$ with $u = \omega + f \approx \omega + M$, so $\varphi \approx i\sin(\omega + M)$. Write $X = \Delta\lambda/(2e) = \sin M$ and $Y = \varphi/i = \sin M\cos\omega + \cos M\sin\omega$. For $\sin\omega \neq 0$, $\cos M = (Y - X\cos\omega)/\sin\omega$, and $\sin^2 M + \cos^2 M = 1$ gives

$$X^2\sin^2\omega + (Y - X\cos\omega)^2 = \sin^2\omega,$$

a positive-definite quadratic in (X, Y), which is an ellipse.

Special cases. For $\omega = 0$ or 180°, $Y = \pm X$, a straight segment. The satellite moves along a slanting line and back. For $\omega = \pm 90°$ the equation is $X^2 + Y^2 = 1$, so $(\Delta\lambda/2e)^2 + (\varphi/i)^2 = 1$, an ellipse with axes along the map. It is a circle of radius i when $2e = i$.

Direction. For $\omega = 90°$, at perigee ($M = 0$) the point is at $\varphi = i$, the northern end, and $d\Delta\lambda/dM = 2e > 0$, so it moves east. On a map with north up and east to the right it goes **clockwise**. Perigee is at the northern end of the circle, where the satellite is fastest.

(**d**) The longitude swing must satisfy $2e \le 0.05° = 8.7\times10^{-4}$ rad, so $e \le 4.4\times10^{-4}$. The latitude swing is $\pm i$, so $i \le 0.05°$. The figure 8 then has half-width $i^2/4 = 1.1\times10^{-5}$ degrees, which is negligible.

What this means for a mission. An eccentricity of 0.0005, which changes the radius by only ±21 km at the geostationary radius, already swings the satellite by ±0.057° and takes it out of a ±0.05° box. So for a geostationary satellite the eccentricity is not a detail. The operator controls it with east-west burns and the inclination with north-south burns, and both are part of the station-keeping budget.

10. Unequal seasons

The time note shows that the day has more than one length. The year is not split evenly either. The astronomical seasons begin when the Sun's ecliptic longitude $\lambda_\odot$ reaches 0° (the March equinox), 90° (the June solstice), 180° and 270°. Seen from the Earth, the Sun moves on a Kepler ellipse with $e = 0.0167086$, its longitude of perigee is $\varpi = 282.9373°$ (so the true anomaly is $f = \lambda_\odot - \varpi$), and one revolution takes the tropical year $Y = 365.242189$ days. Ignore the slow turning of the perigee.

a. Show that, to first order in e, $M = f - 2e\sin f$. (Use $dM/df = \dot{M}/\dot{f}$.)
b. Show that, to first order, the season that starts at longitude λ_k lasts

$$L_k = \frac{Y}{4} - \frac{\sqrt{2}\,eY}{\pi}\cos(\lambda_k + 45° - \varpi),$$

and that the half-year from λ_1 to $\lambda_1 + 180°$ lasts $\frac{Y}{2} + \frac{2eY}{\pi}\sin(\lambda_1 - \varpi)$.
c. Find the lengths of the four seasons, of the northern summer half-year (March to September equinox) and winter half-year, and of **Uttarayana** and **Dakshinayana**, the halves of the year in which the Sun moves north and south (taken astronomically, from solstice to solstice). Compare the first-order values with Kepler's equation solved exactly.
d. Hipparchus measured spring as $94\frac{1}{2}$ days and summer as $92\frac{1}{2}$ days, with a year of $365\frac{1}{4}$ days. Invert (b) to find the eccentricity and the longitude of perigee from his data, and compare with today's values.

Solution. The Sun moves at a uniform rate in mean anomaly, not in longitude. So equal steps of longitude take unequal times. The plan is to relate the mean anomaly to the true anomaly to first order in e (a), integrate over a quarter and a half of the orbit (b), and then read the result the other way round, from times to the orbit (d), as Hipparchus did.

(a) The mean anomaly grows at $\dot{M} = n$, the true anomaly at $\dot{f} = h/r^2$. With $h = na^2\sqrt{1-e^2}$ and $r = a(1-e^2)/(1+e\cos f)$,

$$\frac{dM}{df} = \frac{n\,r^2}{h} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} \approx 1 - 2e\cos f$$

to first order in e. Integrating from perigee, where both anomalies are zero, $M = f - 2e\sin f$. (This is the first term of the equation of the centre, $f - M \approx 2e\sin M$, read the other way round.)

(b) The time between longitudes λ_1 and λ_2 is $(M_2 - M_1)/n$, with $n = 2\pi/Y$. By (a), with $f_k = \lambda_k - \varpi$,

$$\Delta t = \frac{Y}{2\pi}\Big[(f_2 - f_1) - 2e\,(\sin f_2 - \sin f_1)\Big].$$

A season. For $f_2 = f_1 + 90°$, the identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ gives $\sin(f_1 + 90°) - \sin f_1 = 2\cos(f_1 + 45°)\sin 45° = \sqrt{2}\cos(f_1 + 45°)$. So

$$L_k = \frac{Y}{4} - \frac{\sqrt{2}\,eY}{\pi}\cos(\lambda_k + 45° - \varpi).$$

The amplitude is $\sqrt{2}\,eY/\pi = 2.747$ days about $Y/4 = 91.31$ days.

A half-year. For $f_2 = f_1 + 180°$, $\sin f_2 - \sin f_1 = -2\sin f_1$, so the half-year lasts $\frac{Y}{2} + \frac{2eY}{\pi}\sin(\lambda_1 - \varpi)$.

(c)

	starts at $\lambda_\odot$	first order (days)	exact (days)
spring	0°	92.77	92.76
summer	90°	93.64	93.65
autumn	180°	89.85	89.84
winter	270°	88.98	88.99

The first-order formula is good to about 0.01 day. The northern summer half-year lasts 186.41 days and the winter half-year 178.83 days, 7.57 days less. Uttarayana, from the December solstice to the June solstice, lasts 181.75 days, and Dakshinayana 183.49 days.

Reading. The Sun is at perigee (the Earth is closest to the Sun) in early January, when $\lambda_\odot = 283°$. Near perigee the apparent motion of the Sun is fastest, so the seasons around it (northern autumn and winter) are short, and those around apogee are long. Uttarayana contains the perigee, so it is the shorter half. Why do the two solstice halves differ by only 1.7 days, when the equinox halves differ by 7.6? A line through the perigee and the apogee would cut the year into two equal halves, by symmetry. The solstices lie only 13° from that line, so they cut the year nearly evenly. The equinoxes lie almost at right angles to it, and give the largest difference. Traditional calendars start Uttarayana at Makar Sankranti, more than three weeks after the solstice, because they count the Sun's longitude from the stars and not from the equinox.

(d) Write $d_k = Y/4 - L_k = A\cos(\lambda_k + 45° - \varpi)$ with $A = \sqrt{2}eY/\pi$. Spring ($\lambda = 0$) and summer ($\lambda = 90°$) give

$$d_1 = A\cos(45°-\varpi) = 91.3125-94.5 = -3.1875, \qquad d_2 = A\cos(135°-\varpi) = -A\sin(45°-\varpi) = 91.3125-92.5 = -1.1875.$$

So $A = \sqrt{d_1^2 + d_2^2} = 3.40$ days and $e = \pi A/(\sqrt{2}Y) = 0.0207$. For the angle, $\cos(45° - \varpi) = d_1/A < 0$ and $\sin(45° - \varpi) = -d_2/A > 0$, so $45° - \varpi = 159.6°$ lies in the second quadrant. Then $\varpi = 245.4°$ and the Sun's apogee is at 65.4°.

History. Ptolemy's *Almagest* (Book III) reports Hipparchus's season lengths of $94\frac{1}{2}$ and $92\frac{1}{2}$ days. From them Hipparchus placed the Sun on a circle off-centre from the Earth by 1/24 of its radius, with the apogee at $65\frac{1}{2}°$. An off-centre circle gives the first-order equation of the centre $2e\sin M$ when its offset is $2e$, so his 1/24 is our $2e = 1/24.2$.

Comparison. Today $e = 0.0167$ and the apogee is at 102.9°. The apogee has moved by 37.5°. The perigee advances 11.6″ a year against the stars and the equinox slides back 50.3″ a year, so against the equinox the apogee moves 61.9″ a year. Over the 2150 years since Hipparchus that is about 37°, as found. His eccentricity is too large, because the season lengths come from timing the equinoxes and solstices by eye, and an error of one day in a season changes e by about 0.006. That the method still finds the direction of the apogee so well from two numbers is remarkable.